The Stolz-Cesaro Theorem

The Theorem. *If* $(b_n)_{n=1}^\infty$ *is a sequence of positive real numbers, such that* $\sum_{n=1}^\infty b_n = \infty$, *then for any sequence* $(a_n)_{n=1}^\infty \subset \mathbb{R}$ *one has the inequalities:*

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} \leq \limsup_{n\to\infty} \frac{a_n}{b_n}; \tag{1}$$

$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} \geq \liminf_{n\to\infty} \frac{a_n}{b_n}. \tag{2}$$

In particular, if the sequence $(a_n/b_n)_{n=1}^\infty$ *has a limit, then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

Proof. . It is quite clear that we only need to prove (1), since the other inequality follows by replacing a_n with $-a_n$.

The inequality (1) is trivial, if the right-hand side is $+\infty$. Assume then that the quantity $L = \limsup_{n\to\infty}(a_n/b_n)$ is either finite or $-\infty$, and let us fix for the moment some number $\ell > L$. By the definition of lim sup, there exists some index $k \in \mathbb{N}$, such that

$$\frac{a_n}{b_n} \leq \ell, \ \ \forall\, n > k. \tag{3}$$

Using (3) we get the inequalities

$$a_1 + a_2 + \dots + a_n \leq a_1 + \dots + a_k + \ell(b_{k+1} + b_{k+2} + \dots b_n), \ \ \forall\, n > k. \tag{4}$$

If we denote for simplicity the sums $a_1 + \dots + a_n$ by A_n and $b_1 + \dots + b_n$ by B_n, the above inequality reads:

$$A_n \leq A_k + \ell(B_n - B_k), \ \ \forall\, n > k,$$

so dividing by B_n we get

$$\frac{A_n}{B_n} \leq \ell + \frac{A_k - \ell B_k}{B_n}. \tag{5}$$

Since $B_n \to \infty$, by fixing k and taking lim sup in (5), we get $\limsup_{n\to\infty}(A_n/B_n) \leq \ell$. In other words, we obtained the inequality

$$\limsup_{n\to\infty} \frac{a_1 + \dots + a_n}{b_1 + \dots + b_n} \leq \ell, \ \ \forall\, \ell \geq L,$$

which in turn forces

$$\limsup_{n\to\infty} \frac{a_1 + \dots + a_n}{b_1 + \dots + b_n} \leq L. \quad \square$$

Remark. An equivalent formulation of the above Theorem is as follows: *If* $(y_n)_{n=1}^{\infty}$ *is a strictly increasing sequence with* $\lim_{n\to\infty} y_n = \infty$*, then for any sequence* $(x_n)_{n=1}^{\infty}$*, the following inequalities hold:*

$$\limsup_{n\to\infty} \frac{x_n}{y_n} \le \limsup_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}; \tag{6}$$

$$\liminf_{n\to\infty} \frac{x_n}{y_n} \ge \liminf_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}. \tag{7}$$

In particular, if the sequence $\left(\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right)_{n=1}^{\infty}$ *has a limit, then*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

Indeed (assuming all the y_n's are positive, which happens anyway for n large enough), if we consider the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, defined by $a_1 = x_1$, $b_1 = y_1$, and $a_n = x_n - x_{n-1}$, $b_n = y_n - y_{n-1}$, $n \ge 2$, then everything is clear, since $x_n = a_1 + \cdots + a_n$ and $y_n = b_1 + \cdots + b_n$.

The Stolz-Cesaro Theorem has numerous applications in Calculus. Below are three of the most significant ones.

"Additive" Cesaro's Theorem. *For any sequence* $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ *one has the inequalities:*

$$\limsup_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \le \limsup_{n\to\infty} a_n; \tag{8}$$

$$\liminf_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} \ge \liminf_{n\to\infty} a_n. \tag{9}$$

In particular, if the sequence $(a_n)_{n=1}^{\infty}$ *has a limit, then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} a_n.$$

Proof. Particular case of Stolz-Cesaro Theorem with $b_n = 1$. □

Remark. An equivalent formulation of the above Theorem (proven using the alternative version of Stolz-Cesaro Theorem) is as follows: *For any sequence* $(x_n)_{n=1}^{\infty}$*, the following inequalities hold:*

$$\limsup_{n\to\infty} \frac{x_n}{n} \le \limsup_{n\to\infty} (x_n - x_{n-1}); \tag{10}$$

$$\liminf_{n\to\infty} \frac{x_n}{n} \ge \liminf_{n\to\infty} (x_n - x_{n-1}). \tag{11}$$

In particular, if the sequence $(x_n - x_{n-1})_{n=1}^{\infty}$ *has a limit, then*

$$\lim_{n\to\infty} \frac{x_n}{n} = \lim_{n\to\infty} (x_n - x_{n-1}).$$

"Multiplicative" Cesaro's Theorem. *For any sequence of positive numbers* $(a_n)_{n=1}^{\infty}$ *one has the inequalities:*

$$\limsup_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \leq \limsup_{n\to\infty} a_n; \tag{12}$$

$$\liminf_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \geq \liminf_{n\to\infty} a_n. \tag{13}$$

In particular, if the sequence $(a_n)_{n=1}^{\infty}$ *has a limit, then*

$$\lim_{n\to\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = \lim_{n\to\infty} a_n.$$

Proof. Let $b_n = \ln a_n$, so that $\sqrt[n]{a_1 a_2 \dots a_n} = \exp\left(\frac{b_1 + \cdots + b_n}{n}\right)$. Everything then follows from the "Additive" Cesaro Theorem. □

Remark. An equivalent formulation of the above Theorem (proven using the alternative version of "Additive" Cesaro Theorem) is as follows: *For any sequence of positive numbers* $(x_n)_{n=1}^{\infty}$, *the following inequalities hold:*

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_n}{x_{n-1}}; \tag{14}$$

$$\liminf_{n\to\infty} \sqrt[n]{x_n} \geq \liminf_{n\to\infty} \frac{x_n}{x_{n-1}}. \tag{15}$$

In particular, if the sequence $(x_n/x_{n-1})_{n=1}^{\infty}$ *has a limit, then*

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{x_n}{x_{n-1}}.$$

L'Hopital's Rule. *Suppose* f *and* g *are differentiable on some interval that has* A *as an accumualtion point,* $\lim_{x\to A} g(x) = \infty$, *and* $\lim_{x\to A} \frac{f'(x)}{g'(x)} = L$. *Then:*

$$\lim_{x\to A} \frac{f(x)}{g(x)} = L.$$

(Comment: The notation "$\lim_{x\to A}$" can include any kind of limit: honest, one-sided, or $A = \pm\infty$. It is also implicitly assumed that $g'(x) \neq 0$ for x "near" A. Nothing is assumed about $\lim_{x\to A} f(x)$, not even its existence!)

Proof. By a simple change-of-variable argument, it suffices to consider only the case $A = \infty$. Since $g'(x) \neq 0$ on some interval (α, ∞) and $\lim_{x\to\infty} g(x) = \infty$, it follows that g is strictly increasing on (α, ∞). By a standard argument, it suffices to show that

($\star$) *Whenever* $(t_n)_{n=1}^{\infty}$ *is a strictly increasing sequence in* (α, ∞), *with* $\lim_{n\to\infty} t_n = \infty$, *it follows that:*

$$\lim_{n\to\infty} \frac{f(t_n)}{g(t_n)} = L. \tag{16}$$

To prove $(\star)$ fix a sequence $(t_n)_{n=1}^{\infty}$ as above, and let us consider the sequences $x_n = f(t_n)$ and $y_n = g(t_n)$. Using the Lagrange's Mean Value Theorem, we know that for every $n \geq 2$, there exists some s_n in (t_{n-1}, t_n), such that

$$\frac{f(t_n) - f(t_{n-1})}{g(t_n) - g(t_{n-1})} = \frac{f'(s_n)}{g'(s_n)}. \tag{17}$$

Since $(s_n)_{n=1}^{\infty}$ is obviously increasing, with $\lim_{n\to\infty} s_n = \infty$, it follows that $\lim_{n\to\infty} \dfrac{f'(s_n)}{g'(s_n)} = L$. Going back to (17), we now have

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = L,$$

with $(y_n)_{n=1}^{\infty}$ strictly increasing (because g is strictly increasing) and $\lim_{n\to\infty} y_n = \infty$. By the (alternative version of) Stolz-Cesaro Theorem, it follows that $\lim_{n\to\infty} \dfrac{x_n}{y_n} = L$, which is precisely the desired conclusion (16). ☐