APPENDIX C

Ordinal numbers

In this Appendix we discuss ordinal number arithmetic. *The Axiom of Choice is assumed to be true.*

DEFINITION. Let X be a non-empty set. A *well ordering on* X is an total order relation $\prec$ on X with the following property:

(W) *every non-empty subset* $A \subset X$ *has a smallest element, i.e. there exists* $a \in A$*, such that* $a \prec x$*,* $\forall\, x \in A$*.*

In this case the pair $(X, \prec)$ is called a *well ordered set.*

NOTATIONS. Let $(W, \prec)$ be a well-ordered set. For any $a \in W$, we define

$$W(a) = \{x \in W \,:\, x \prec a \text{ and } x \neq a\}.$$

Remark that $(W(a), \prec)$ is well-ordered.

LEMMA C.1. *Let* $(W, \prec)$ *be a well ordered set. For a subset* $S \subset W$*, the following are equivalent:*

- (i) *for every* $s \in S$*, one has the inclusion* $W(s) \subset S$*;*
- (ii) *either* $S = W$*, or there exists some* $a \in W$*, such that* $S = W(a)$*.*

PROOF. $(i) \Rightarrow (ii)$. Assume $S \subsetneq W$. Take a to be the smallest element of the set $W \smallsetminus S$. If $s \in S$, then $a \neq s$, and by (i) we cannot have $a \prec s$, since this would force $a \in W(s) \subset S$. Therefore we must have $s \prec a$, i.e. $s \in W(a)$. This prove the inclusion $S \subset W(a)$. Conversely, if $s \in W(a)$, then s must belong to S. Otherwise $s \in W \smallsetminus S$ would contradict the minimality of a.

$(ii) \Rightarrow (i)$. This is trivial. □

DEFINITION. A subset S, as above, is called a *full* subset.

The key feature of well-ordered sets is the following.

LEMMA C.2 (Transfinite Induction Principle). *Let* $(W, \prec)$ *be a well-ordered set. Let* $w_1 \in W$ *be the smallest element of* W*. Assume* $A \subset W$ *is a set with the property*

(I) *If* $w \in W$ *has the property that,* $W(w) \subset A$*, then* $w \in A$*.*

Then $A = W$*.*

PROOF. Consider the set

$$S = \{s \in A \,:\, W(s) \subset A\}.$$

It is obvious that S is full, and $S \subset A$. By Lemma C.1, either $S = W$, in which case we clearly get $A = W$, or there exists $w \in W$, such that $S = W(w)$. In this case we have $W(w) \subset A$. By (I) this forces $w \in A$, so we get $w \in S$, which is impossible. $\square$

Another useful feature is

LEMMA C.3 (Recursion Principle). *Let $(W, \prec)$ be a well-ordered set, and let w_1 be the smallest element in W. Let X be a set, and assume one has a family of maps $\Phi_a : \prod_{W(a)} X \to X$, $a \in W \smallsetminus \{w_1\}$. Then for any element $x_1 \in X$, there exists a unique function $f : W \to X$, such that*

$$f(w_1) = x_1 \text{ and } f(a) = \Phi_a\big(f\big|_{W(a)}\big),\ \forall\, a \in W \smallsetminus \{w_1\}. \tag{1}$$

PROOF. For every $a \in W$ let us denote the set $W(a) \cup \{a\}$ simply by W_a, and let us define the set

$$\mathcal{F}_a = \big\{g : W_a \to X \,:\, g(w_1) = x_1 \text{ and } g(b) = \Phi_b\big(g\big|_{W(b)}\big),\ \forall\, b \in W_a \smallsetminus \{w_1\}\big\}.$$

Remark that, for any $a, b \in W$, with $a \prec b$, one has

$$f\big|_{W_a} \in \mathcal{F}_a,\ \ \forall\, f \in \mathcal{F}_b. \tag{2}$$

Claim: *For every $a \in W$, the set $\mathcal{F}_a$ is a singleton.*

We prove this statement using transfinite induction. Define

$$A = \big\{a \in W \,:\, \mathcal{F}_a \text{ is a singleton}\big\}.$$

Suppose $a \in W$ has the property $W(a) \subset A$, which means that $\mathcal{F}_b$ is a singleton, for all $b \in W(a)$. For each $b \in W(a)$, let $f_b : W_b \to X$ be the unique element in $\mathcal{F}_b$. We notice that, for any $b, c \in W(a)$, with $b \prec c$, using (2), we have

$$f_c\big|_{W_b} = f_b. \tag{3}$$

This follows immediately from the fact that $f_c\big|_{W_b}$ belongs to $\mathcal{F}_b$. Using the obvious equality

$$W(a) = \bigcup_{b \in W(a)} W_b,$$

we define $g : W(a) \to X$ as the unique function with the property that $g\big|_{W_b} = f_b$, $\forall\, b \in W(a)$. Finally, we define $f_a : W_a \to X$ by $f_a\big|_{W(a)} = g$, and $f_a(a) = \Phi_a(g)$. It is clear that $f_a \in \mathcal{F}_a$, so $\mathcal{F}_a$ has at least one element. If $h \in \mathcal{F}_a$ is another function, then for every $b \in W(a)$ we have $h\big|_{W_b} \in \mathcal{F}_b$, which forces $h\big|_{W_b} = f_b$, in particular giving $h\big|_{W(a)} = g = f_a\big|_{W(a)}$. Then $h(a) = \Phi_a(g)$, which means that we also have $h(a) = f_a(a)$, so we must have $h = f_a$.

Having proven the Claim, we now have a family of functions $f_a : W_a \to X$, $a \in W$, with $f_b\big|_{W_a} = f_a$, for all $a, b \in W$ with $a \prec b$. Using the equality

$$W = \bigcup_{a \in W} W_a,$$

we then define $f : W \to X$ to be the unique function such that $f\big|_{W_a} = f_a$, $\forall\, a \in W$.

Notice that, for each $a \in W \smallsetminus \{w_1\}$, we have $f(a) = f_a(a)$, and since $f_a \in \mathcal{F}_a$, we immediately get (1). The uniqueness of f with property (1) is also clear, since any such f will atomatically satisfy $f\big|_{W_a} \in \mathcal{F}_a$, for all $a \in W$. $\square$

Comment. The system of maps $\Phi_a : \prod_{W(a)} X \to X$, $a \in W$ is to be thought as a "recurence relation," in the sense that it is used to define the value $f(a)$ in terms of all "preceding" values $f(w)$, $w \prec a$, $w \neq a$.

Definitions. Given two well ordered sets $(W_1, \prec_1)$ and $(W_2, \prec_2)$, a map $f : (W_1, \prec_1) \to (W_2, \prec_2)$ is called an *full embedding*, if

- f is injective.
- For any two elements $x, y \in W_1$, one has

$$x \prec_1 y \Rightarrow f(x) \prec_2 f(y).$$

- $f(W_1)$ is a full subset of W_2.

If f is a full emebedding, with $f(W_1) = W_2$, then f is called an *order isomorphism*.

The properties of these types of maps are contained in the following

Proposition C.1. *A. Suppose* $(W_1, \prec_1)$ *and* $(W_2, \prec_2)$*, are well-ordered sets.*

(i) *If* $f : (W_1, \prec_1) \to (W_2, \prec_2)$ *is a full embedding, then*

$$f\big(W_1(a)\big) = W_2\big(f(a)\big), \ \forall\, a \in W_1.$$

In particular, if w_1 *is the smallest element in* W_1*, and* w_2 *is the smallest element in* W_2*, then* $f(w_1) = w_2$*.*

(ii) *If* $f : (W_1, \prec_1) \to (W_2, \prec_2)$ *is an order isomorphism, then* $f^{-1} : (W_2, \prec_2) \to (W_1, \prec_1)$ *is again an order isomorphism.*

(iii) *There exists at most one full embedding* $f : (W_1, \prec_1) \to (W_2, \prec_2)$*.*

B. Suppose $(W_1, \prec_1)$*,* $(W_2, \prec_2)$*,* $(W_3, \prec_3)$ *are well-ordered sets, and*

$$(W_1, \prec_1) \xrightarrow{f} (W_2, \prec_2) \xrightarrow{g} (W_3, \prec_3)$$

are full emebeddings.

(i) *The composition* $g \circ f : (W_1, \prec_1) \to (W_3, \prec_3)$ *is again a full emebdding.*

(ii) *The composition* $g \circ f$ *is an order isomorphism, if and only if both* f *and* g *are order isomorphisms.*

Proof. A. (i). Start first with some element $x \in W(a)$. Since $x \prec_1 a$, we have $f(x) \prec_2 f(a)$. Since f is injective, and $x \neq a$, we must have $f(x) \neq f(a)$, hence $x \in W_2\big(f(a)\big)$. Conversely, if $y \in W_2\big(f(a)\big)$, then using the fact that $f(W_2)$ is full in W_2, it follows that $y \in f(W_2)$, so there exists some $x \in W_1$, with $y = f(x)$. If $a \prec_1 x$, then we would get $f(a) \prec_2 f(x)$, which is impossible. Therefore we must have $x \prec_1 a$ and $x \neq a$, i.e. $x \in W_1(a)$, so y indeed belongs to $f\big(W_1(a)\big)$. The second assertion is now clear since we have

$$W_2\big(f(w_1)\big) = f\big(W_1(w_1)\big) = f(\varnothing) = \varnothing,$$

which clearly forces $f(w_1) = w_2$.

(ii). This is obvious.

(iii). Suppose $f, g : (W_1, \prec_1) \to (W_2, \prec_2)$ are full embeddings, and let us show that we must have $f = g$. We use transfinite induction. Define the set

$$A = \{w \in W_1 \,:\, f(w) = g(w)\}.$$

Let $w \in W_1$ be some element such that $W_1(w) \subset A$, and let us prove that $w \in A$, i.e. $f(w) = g(w)$. Denote $f(w)$ by a, and $g(w)$ by b. Using the fact that $f\big|_{W_1(w)} =$

$g\big|_{W_1(w)}$, combined with (i), we have

$$W_2(a) = W_2\big(f(w)\big) = f\big(W_1(w)\big) = g\big(W_1(w)\big) = W_2\big(g(w)\big) = W_2(b).$$

This clearly forces $a = b$. Indeed, if $a \neq b$, then either $a \prec b$, in which case $a \in W_2(b) \smallsetminus W_2(a)$, or $b \prec a$, in which case $b \in W_2(a) \smallsetminus W_2(b)$. In either case, we will get $W_2(a) \neq W_2(b)$.

B .(i). It is clear that $g \circ f$ is injective, and satisfies the second condition in the definition, so the only thing we need to prove is the fact that $(g \circ f)(W_1)$ is full. If $f(W_1) = W_2$, there is nothing to prove, since we would get $(g \circ f)(W_1) = g(W_2)$, which is full.

Assume $f(W_1) = W_2(a)$, for some $a \in W_2$. Then by (i) we have

$$(g \circ f)(W_1) = g\big(f(W_1)\big) = g\big(W_2(a)\big) = W_3\big(g(a)\big),$$

so again $(g \circ f)(W_1)$ is full.

(ii). Assume first that both f and g are order isomprphisms. Then $g \circ f : (W_1, \prec_1) \to (W_3, \prec_3)$ is a full embedding, by (i), and it is clearly surjective, hence $g \circ f$ is indeed an order isomorphism.

Conversely, assume $g \circ f : (W_1, \prec_1) \to (W_3, \prec_3)$ is an order isomorphism. This clearly forces g to be surjective, hence an order isomorphism. But then g^{-1} is an order isomorphism, and so will be $g^{-1} \circ (g \circ f) = f$. ☐

Corollary C.1. *If $(W, \prec)$ is a well-ordered set, and $a \in W$, then there is no full embedding $(W, \prec) \to (W(a), \prec)$.*

Proof. Suppose there exists a full embedding $f : (W, \prec) \to (W(a), \prec)$. Since the inclusion $\iota : (W(a), \prec) \hookrightarrow (W, \prec)$ is obviously a full embedding, the composition $\iota \circ f : (W, \prec) \to (W, \prec)$ is a full embedding. Since we also have $\mathrm{Id}_W : (W, \prec) \to (W, \prec)$ as a full embedding, this would force $\iota \circ f = \mathrm{Id}_W$, which would force ι to be surjective. But this is obviously impossible. ☐

Definitions. Two well-ordered sets $W_1, \prec_1)$ and $(W_2, \prec_2)$ are said to have the *same order type*, if there exists an order isomorphism $(W_1, \prec_1) \to (W_2, \prec_2)$. By the above considerations, this defines an equivalence relation on the class of all well-ordered sets.

An *ordinal number* is thought as an equivalence class of well-ordered sets. In other words, if we write a cardinal number as α, it is understood that α consists of all well-ordered sets of a given order type. So when we write $\mathrm{ord}(W, \prec) = \alpha$ we understand that $(W, \prec)$ belongs to this class, and for another well-ordered set $(W', \prec')$ we write $\mathrm{ord}(W', \prec') = \alpha$, exactly when $(W', \prec')$ has the same order type as $(W, \prec)$. In this case we write $\mathrm{ord}(W', \prec') = \mathrm{ord}(W, \prec)$.

We regard the empty set $\varnothing$ as a well-ordered set, with the empty relation. We write $\mathrm{ord}(\varnothing) = 0$.

Comments. If $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are well-ordered sets, then one has the obvious implication

$$\mathrm{ord}(W_1, \prec_1) = \mathrm{ord}(W_2, \prec_2) \Longrightarrow \mathrm{card}\, W_1 = \mathrm{card}\, W_2.$$

Conversely, if the well-ordered sets $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are *finite*, and $\mathrm{card}\, W_1 = \mathrm{card}\, W_2$, then $\mathrm{ord}(W_1, \prec_1) = \mathrm{ord}(W_2, \prec_2)$. Indeed, if we take $n = 1\mathrm{card}\, W_1$, then one can define recursively a finite sequence $(w_k)_{k=1}^n \subset W_1$, by taking w_1 to be the smallest element of W_1, and defining, for each $k \in \{2, 3, \dots, n\}$ the element w_k to be the smallest element of the set $W_1 \smallsetminus \{w_1, w_2, \dots, w_{k-1}\}$. The obvious bijection

$$\{1, 2, \dots, n\} \ni k \longmapsto w_k \in W_1$$

will then define an order isomorphism

$$\big(\{1, \dots, n\}, \leq\big) \to (W_1, \prec_1).$$

Likewise $(W_2, \prec_2)$ has same order type as $\big(\{1, \dots, n\}, \leq\big)$.

Using the above notations, we can then regard all non-negative integers as ordinal numbers, by identifying $\operatorname{ord}(W, \prec) = \operatorname{card}(W)$, for all finite well-ordered sets $(W, \prec)$.

Notation. If α is an ordinal number, say $\alpha = \operatorname{ord}(W, \prec)$, for some well-ordered set $(W, \prec)$, then the cardinal number $\operatorname{card} W$ does not depend on the particular choice of $(W, \prec)$. We will denote it by $\operatorname{card} \alpha$. As dicussed above, if

$$\operatorname{card} \alpha = \operatorname{card} \beta = \text{finite cardinal},$$

then $\alpha = \beta$. As we shall see later, this implication holds only for finite ordinal numbers.

Definitions. Let α_1 and α_2 be ordinal numbers, say $\alpha_1 = \operatorname{ord}(W_1, \prec_1)$ and $\alpha_2 = \operatorname{ord}(W_2, \prec_2)$, where $(W_1, \prec_1)$ and $(W_2, \prec_2)$ are two well-ordered sets. We write $\alpha_1 \leq \alpha_2$, if there exists a full embedding $f : (W_1, \prec_1) \to (W_2, \prec_2)$. By Proposition C.1, this definition is independent of the choices of $(W_1, \prec_1)$ and $(W_2, \prec_2)$.

We write $\alpha_1 < \alpha_2$ if $\alpha_1 \leq \alpha_2$ and $\alpha_1 \neq \alpha_2$.

Remark C.1. If α_1 and α_2 are ordinal numbers, with $\alpha_1 \leq \alpha_2$, then $\operatorname{card} \alpha_1 \leq \operatorname{card} \alpha_2$.

Proposition C.2. *The relation $\leq$ is an order relation, on any set of ordinal numbers.*

Proof. It is obvious that $\alpha \leq \alpha$, for any ordinal number α

Assume α_1 and α_2 are ordinal numbers with $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$, and let us show that this forces $\alpha_1 = \alpha_2$. Let $(W_1, \prec_1)$ and $(W_2, \prec_2)$ be well-ordered sets with $\alpha_1 = \operatorname{ord}(W_1, \prec_1)$ and $\alpha_2 = \operatorname{ord}(W_2, \prec_2)$. Since $\alpha_1 \leq \alpha_2$, there exists a full emebedding $f : (W_1, \prec_1) \to (W_2, \prec_2)$. Since $\alpha_2 \leq \alpha_1$, either there exists a full emebdding $g : (W_2, \prec_2) \to (W_1, \prec_1)$. By Proposition C.1.B, the composition $g \circ f : (W_1, \prec_1) \to (W_1, \prec_1)$ is a full emebedding. Since we already have a full emebdding $\mathrm{Id}_{W_1} : (W_1, \prec_1) \to (W_1, \prec_1)$, by Proposition C.1.A, we must have $g \circ f = \mathrm{Id}_{W_1}$. Using Proposition C.1.B this forces f (and g) to be order isomorphisms, so we indeed have $\alpha_1 = \alpha_2$.

Finally, suppose α_1, α_2 and α_3 are ordinal numbers such that $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_3$. The fact that $\alpha_1 \leq \alpha_3$ follows immediately from Proposition C.1.B. ☐

Theorem C.1 (Ordinal Comparability Theorem). *Let α_1 and α_2 be ordinal numbers. Then either $\alpha_1 \leq \alpha_2$, or $\alpha_2 \leq \alpha_1$.*

Proof. Let $(W_1, \prec_1)$ and $(W_2, \prec_2)$ be well-ordered sets with $\alpha_1 = \operatorname{ord}(W_1, \prec_1)$ and $\alpha_2 = \operatorname{ord}(W_2, \prec_2)$. For every $a \in W_1$ we denote the set $W_1(a) \cup \{a\}$ simply by W_1^a. It is clear that $(W_1^a, \prec_1)$ is well-ordered. Consider the set

$$A = \big\{a \in W_1 \ : \ \text{there exists a full embedding } (W_1^a, \prec_1) \to (W_2, \prec)\big\}.$$

By Proposition C.1.A, we know that for any $a \in A$, there exists a *unique* full embedding $(W_1^a, \prec_1) \to (W_2, \prec_2)$. We denote this full embedding by f_a.

Claim 1: *The set A is full. Moreover, for any $a, b \in A$, with $b \prec a$, we have $f_b = f_a\big|_{W_1^b}$.*

Start with some $a \in A$, and let us prove that $W_1(a) \subset A$. Fix some arbitrary $b \in W(a)$. Then the inclusion $\iota : (W_1^b, \prec_1) \hookrightarrow (W_1^a, \prec_1)$ is obviously a full embedding, since we can write

$$W_1^b = W_1(c),$$

where c is the smallest element of the set

$$D_b = \{x \in W_1 \,:\, b \prec_1 x \text{ and } b \neq x\}.$$

(The fact that $a \in D_b$ shows that $D_b \neq \varnothing$.) Then the composition

$$f_a \circ \iota : (W_1^b, \prec_1) \to (W_2, \prec_2)$$

is a full emebedding, so b indeed belongs to A. Moreover, we will have $f_b = f_a \circ \iota = f_a\big|_{W_1^b}$.

Define the map $\phi : A \to W_2$ by

$$\phi(a) = f_a(a), \;\; \forall\, a \in A.$$

Remark that

$$\phi\big|_{W_1^a} = f_a, \;\; \forall\, a \in A. \tag{4}$$

Indeed, if we take some $b \in W_1(a)$, then by Claim 1, we have $\phi(b) = f_b(b) = f_a(b)$, so we get $\phi\big|_{W_1(a)} = f_a\big|_{W_1(a)}$.

Claim 2: $\phi : (A, \prec_1) \to (W_2, \prec_2)$ *is a full embedding.*

We start by proving the first two conditions. Let $a, b \in A$ be such that $b \prec_1 a$ and $b \neq a$, and let us show that $\phi(b) \prec_2 \phi(a)$ and $\phi(b) \neq \phi(a)$. We have $b \in W_1^a$ and $\phi\big|_{W_1^a} = f_a$, so using the fact that $f_a : (W_1^a, \prec_1) \to (W_2, \prec_2)$ is a full embedding, we indeed get $\phi(b) = f_a(b) \prec f_a(a) = \phi(a)$, and $\phi(b) \neq \phi(a)$.

We now show that $\phi(A)$ is full in $(W_2, \prec_2)$. Start with some $y \in \phi(A)$, and let us show that $W_2(x) \subset \phi(A)$. On the one hand, since we obviously have

$$A = \bigcup_{a \in A} W_1^a,$$

we also have

$$\phi(A) = \phi\Big(\bigcup_{a \in A} W_1^a\Big) = \bigcup_{a \in A} \phi(W_1^a),$$

so there exists some $a \in A$, such that $y \in \phi(W_1^a) = f_a(W_1^a)$. On the other hand, since $f_a : (W_1^a, \prec_1) \to (W_2, \prec_2)$ is a full embedding, it follows that $f_a(W_1^a)$ is full, so we get $W_2(y) \subset f_a(W_1^a) = \phi(W_1^a) \subset \phi(A)$.

We now finish the proof. Since both A and $\phi(A)$ are full, there are three cases to examine

CASE 1: $A = W_1$. In this case $\phi : (W_1, \prec_1) \to (W_2, \prec_2)$ is a full embedding, so we get $\alpha_1 \leq \alpha_2$.

CASE 2: $\phi(A) = W_2$. In this case $\phi : (A, \prec_1) \to (W_2, \prec_2)$ is a an order isomorphism, so $\phi^{-1} : (W_2, \prec_2) \to (W_1, \prec_1)$ is a full embedding, and we get $\alpha_1 \leq \alpha_2$.

Case 3: *$A \subsetneq W_1$ and $\phi(A) \subsetneq W_2$.* This means there exist $a_1 \in W_1$ and $a_2 \in W_2$ such that $A = W_1(a_1)$ and $\phi(A) = W_2(a_2)$. This case turns out to be impossible. To see this, we define $\psi : W_1^{a_1} \to W_2$ by $\psi\big|_{W_1(a)} = \phi$ and $\psi(a_1) = a_2$, then $\psi : (W_1^{a_1}, \prec_1) \to (W_2, \prec_2)$ will still be an order isomorphism. Indeed, the first two conditions in the definition are clear, while the equality

$$\psi(W_1^{a_1}) = W_2^{a_2} = \{y \in W_2 \,:\, y \prec_2 a_2\},$$

proves that $\psi(W_1^{a_1})$ is full. The existence of ψ then forces $a_1 \in A$, which contradicts the equality $A = W_1(a_1)$. □

Theorem C.2. *Let α be an ordinal number. Then the class P_α of all ordinal numbers β with $\beta < \alpha$ is a set. More explicitly, if $(W, \prec)$ is a well-ordered set with $\mathrm{ord}(W, \prec) = \alpha$, then the map*

$$\phi : W \ni a \longmapsto \mathrm{ord}(W(a), \prec) \in P_\alpha$$

is a bijection. Moreover, $(P_\alpha, \leq)$ is well-ordered, and $\phi : (W, \prec) \to (P_\alpha, \leq)$ is an order isomorphism.

Proof. Let β be an ordinal number with $\beta < \alpha$. Then there exists a well-ordered set $(W_1, \prec_1)$, and a full emebedding $\phi : (W_1, \prec_1)$, such that

- $\beta = \mathrm{ord}(W_1, \prec_1)$,
- $\phi(W_1) = W(a_1)$,

for some $a_1 \in W$. This fact already proves that P_α is a set.

> *Claim*: *The element $a_1 \in W$ does not depend on the particular choice of $(W_1, \prec_1)$.*

Indeed, if $(W_2, \prec_2)$ is another well-ordered set, and $\psi : (W_2, \prec_2) \to (W, \prec)$ is another full emebdding with

- $\beta = \mathrm{ord}(W_2, \prec_2)$,
- $\psi(W_2) = W(a_2)$,

for some $a_2 \in W$, then we would get the existence of an order isomorphism $\gamma : (W(a_1), \prec) \to (W(a_2), \prec)$. We can assume (otherwise we replace γ with γ^{-1}) that $a_1 \prec a_2$. If $a_1 \neq a_2$, we would have $a_1 \in W(a_2)$, so if we work with the well-ordered set $Z = W(a_2)$ we would have an order isomorphism $(Z, \prec) \to (Z(a_1), \prec)$. By Corollary C.1 this is impossible. Therefore, we must have $a_1 = a_2$.

Using the Claim, we then define a_β as the unique element in W, such that $\mathrm{ord}(W(a_\beta), \prec) = \beta$. Define the map $\psi : P_\alpha \ni \beta \longmapsto a_\beta \in W$. It is clear that $\phi \circ \psi = \mathrm{Id}_{P_\alpha}$.

Let us prove now that $\psi \circ \phi = \mathrm{Id}_W$. Start with some arbitrary $a \in W$, and put $\beta = \phi(a) = \mathrm{ord}(W(a), \prec)$. Since $\mathrm{ord}(W(a), \prec) = \beta$, by the Claim, we must have $a_\beta = a$, i.e. $\psi(\beta) = a$, which means that $(\psi \circ \phi)(a) = a$.

Finally, we note that, if $a, b \in W$ are elements with $a \prec b$, then the obvious full embedding $(W(a), \prec) \hookrightarrow (W(b), \prec)$ proves that $\mathrm{ord}(W(a), \prec) \leq \mathrm{ord}(W(b), \prec)$, i.e. $\phi(a) \leq \phi(b)$.

Since ϕ is bijective, it is clear that, for $a, b \in W$, we have in fact the equivalence

$$a \prec b \Longleftrightarrow \phi(a) \leq \phi(b).$$

This proves that $(P_\alpha, \leq)$ is well-ordered, and $\phi : (W, \prec) \to (P_\alpha, \leq)$ is an order isomorphism. □

Corollary C.2. *If $\mathcal{S}$ is a set of ordinal numbers, then $(\mathcal{S}, \leq)$ is well-ordered.*

Proof. By Theorem C.1, $(\mathcal{S}, \leq)$ is totally ordered. Fix some non-empty subset $\mathcal{A} \subset \mathcal{S}$, and let us show that $\mathcal{A}$ has a smallest element. Start with some arbitrary $\alpha \in \mathcal{A}$. If $\alpha \leq \beta$, $\forall \beta \in \mathcal{A}$, we are done. Otherwise, the intersection $\mathcal{A} \cap P_\alpha$ is non-empty. We then use the fact that $(P_\alpha, \leq)$ is well-ordered, to choose α_1 to be its smallest element. If we start with some arbitrary $\beta \in \mathcal{A}$, then either $\alpha \leq \beta$, in which case we immediately get $\alpha_1 < \beta$, or $\beta < \alpha$, in which case $\beta \in \mathcal{A} \cap P_\alpha$, and we again get $\alpha_1 \leq \beta$. So α_1 is in fact the smallest element of $\mathcal{A}$. □

Theorem C.3 (Well ordering Theorem). *Every non-empty set has a well ordering.*

Proof. Let

$$\mathcal{W} = \big\{(W, \prec) \,:\, (W, \prec) \text{ well-ordered, and } W \subset X\big\}.$$

For two elements $(W_1, \prec_1)$ and $(W_2, \prec_2)$, we define $(W_1, \prec_1) \sqsubset (W_2, \prec_2)$, if and only if $W_1 \subset W_2$, and the inclusion map $(W_1, \prec_1) \hookrightarrow (W_2, \prec_2)$ is a full embedding. (This is equivalent to the fact that W_1 is a full subset of $(W_2, \prec_2)$, and $\prec_1 = \prec_2 \big|_{W_1}$.)

It is obvious that $(\mathcal{W}, \sqsubset)$ is an ordered set. We want to apply Zorn Lemma to this set. We need to check the hypothesis. Start with a totally ordered subset $\mathcal{T} = \big\{(W_i, \prec_i) \,:\, i \in I\big\} \subset \mathcal{W}$, and let us show that $\mathcal{T}$ has an upper bound in $\mathcal{W}$. Define $W = \bigcup_{i \in I} W_i$. For $a, b \in W$, we define $a \prec b$, if and only if there exists $i \in I$, such that $a, b \in W_i$, and $a \prec_i b$. Let us chack that $(W, \prec)$ is a well-ordered set. First of all, we need to show that $\prec$ is an order relation on W. It is clear that $a \prec a$, $\forall a \in W$. Suppose $a, b \in W$ satisfy $a \prec b$ and $b \prec a$, and let us show that $a = b$. We know there exists $i, j \in I$ such that $a, b \in W_i$ and $a \prec_i b$, and $a, b \in W_j$ and $b \prec_j a$. Now there are two possibilities: either $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$. In the first case we get $a \prec_i b$ and $b \prec_i a$, so we would get $a = b$. In the other case, by symmetry, we again get $a = b$. Let us show now transitivity. Suppose $a, b, c \in W$ satisfy $a \prec b$ and $b \prec c$, and let us show that $a \prec c$. We know there exist $i, j \in I$, such that $a, b \in W_i$ and $a \prec_i b$, and $b, c \in W_j$ and $b \prec_j c$. As above, we have two possibilities: either $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$. In the first case we get $a, b, c \in W_j$ and $a \prec_j b \prec_j c$, so we get $a \prec_j c$. In the second case, we get $a, b, c \in W_i$ and $a \prec_i b \prec_i c$, so we get $a \prec_i c$. In either case we get $a \prec c$.

Next we show that $(W, \prec)$ is totally ordered. Start with arbitrary $a, b \in W$, and let us prove that either $a \prec b$ or $b \prec a$. If we choose $i, j \in I$ such that $a \in W_i$ and $b \in W_j$, then using the two possiblities $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$ or $(W_j, \prec_j) \sqsubset (W_i, \prec_i)$ we immediately see that we can find $k \in I$ (k is either i or j), such that $a, b \in W_k$. Then using the fact that $(W_k, \prec_k)$ is totally ordered, we either have $a \prec_k b$, or $b \prec_k a$. This gives either $a \prec b$, or $b \prec a$.

In order to prove that $(W, \prec)$ is well-ordered, and $(W_i, \prec_i) \sqsubset (W, \prec)$, $\forall i \in I$, we shall use the following

Claim: *For any $i \in I$, one has the implication:*

$$a \in W_i \Longrightarrow W(a) \subset W_i.$$

Indeed, if there exists some $b \in W(a)$, but $b \notin W_i$, this would mean that there exists some $j \in I$, with $b \in W_j$, $b \prec_j a$, and $b \neq a$. This would then force $(W_i, \prec_i) \sqsubset (W_j, \prec_j)$, and $b \in W_j(a)$. But this is impossible, since the fact that W_i is full in $(W_j, \prec_j)$ would force $b \in W_j(a) \subset W_i$.

Let us show now that $(W, \prec)$ is well-ordered. Start with some arbitrary non-empty subset $A \subset W$. Choose $i \in I$, such that $A \cap W_i \neq \varnothing$, and take a to be the smallest element in $A \cap W_i$, in the well-ordered set $(W_i, \prec_i)$, i.e.

$$a \in A \cap W_i, \text{ and } a \prec_i x, \ \forall\, x \in A \cap W_i. \tag{5}$$

Let us prove that a is in fact the smallest element of A, in $(W, \prec)$. Start with some arbitrary element $b \in A$, and let us prove that $a \prec b$. Assume the opposite, which using the fact that $(W, \prec)$ is totally ordered, this means that $b \prec a$, and $b \neq a$, i.e. $b \in W(a)$. By the Claim hoewever, this will force $b \in W_i$, so we would get $b \in A \cap W_i$, and the choice of a would give $a \prec_i b$, which would then give $a \prec b$, thus contradicting the assumption on b.

We now prove $(W_i, \prec_i) \sqsubset (W, \prec)$, $\forall\, i \in I$. It is clear that the inclusion map $\iota : (W_i, \prec_i) \hookrightarrow (W, \prec)$ satsifies the first two conditions in the definition of full embeddings, so the only thing we need is the fact that W_i is full in $(W, \prec)$. But this is precisely the content of the above Claim.

Having shown that every totally orderes subset $\mathcal{T} \subset \mathcal{W}$ has an upper bound, we now invoke Zorn Lemma, to get the existence of a maximal element $(W, \prec) \in \mathcal{W}$. The proof of the Theorem will be finished onece we prove that $W = X$. We prove this equality by contardiction. Assume $W \subsetneq X$. Pick an element $x \in X \smallsetminus W$, and define the set $W_1 = W \cup \{x\}$. Equipp W_1 with the order relation $\prec_1$ defined by

$$a \prec b \Longleftrightarrow \left\langle \begin{array}{l} a, b \in W \text{ and } a \prec b, \\ \text{or } b = x \end{array} \right.$$

It is pretty obvious that $W = W_1(x)$ and $\prec = \prec_1 \big|_W$, so $(W_1, \prec_1)$ is well-ordered and $(W, \prec) \sqsubset (W_1, \prec_1)$. Since $W \subsetneq W_1$, this would contardict the maximality. ☐

Comment. An interesting consequence of the Well-Ordering Theorem is the following: *For any cardinal number* $\mathfrak{a}$*, there exists an ordinal number* α*, such that* $\operatorname{card} \alpha = \mathfrak{a}$.

Another interesting application is the following:

Corollary C.3. *If* $\mathcal{C}$ *is a set of cardinal numbers, then* $(\mathcal{C}, \leq)$ *is well-ordered.*

Proof. For any $\mathfrak{a} \in \mathcal{C}$ we choose a well-ordered set $(W_{\mathfrak{a}}, \prec_{\mathfrak{a}})$ with $\operatorname{card} W_{\mathfrak{a}} = \mathfrak{a}$. Choose any set X with

$$\mathfrak{a} < \operatorname{card} X, \ \forall\, \mathfrak{a} \in \mathcal{C}.$$

(For example, we can take $Y = \bigcup_{\mathfrak{a} \in \mathcal{C}} W_{\mathfrak{a}}$, so that $\mathfrak{a} \leq \operatorname{card} Y$, $\forall\, \mathfrak{a} \in \mathcal{C}$, and then we define $X = \{0, 1\}^Y$.) Choose a well-ordering $\prec$ on the set X. Define $\alpha = \operatorname{ord}(X, \prec)$ and $\alpha_{\mathfrak{a}} = \operatorname{ord}(W_{\mathfrak{a}}, \prec_{\mathfrak{a}})$, $\forall\, \mathfrak{a} \in \mathcal{C}$). Since

$$\operatorname{card} \alpha_{\mathfrak{a}} = \mathfrak{a} < \operatorname{card} X = \operatorname{card}(X, \prec), \ \forall\, \mathfrak{a} \in \mathcal{C},$$

it follows that we have $\alpha_{\mathfrak{a}} < \alpha$, i.e. $\alpha_{\mathfrak{a}} \in P_\alpha$, $\forall\, \mathfrak{a} \in \mathcal{C}$.

Apply now the fact that the ordinal set $(P_\alpha, \leq)$ is a well-ordered, to find some $\mathfrak{a}_0 \in \mathcal{C}$, such that

$$\alpha_{\mathfrak{a}_0} \leq \alpha_{\mathfrak{a}}, \ \forall\, \mathfrak{a} \in \mathcal{C}.$$

This will clearly imply

$$\mathfrak{a}_0 = \operatorname{card} \alpha_{\mathfrak{a}_0} \leq \operatorname{card} \alpha_{\mathfrak{a}} = \mathfrak{a}, \ \forall\, \mathfrak{a} \in \mathcal{C}. \quad \square$$

EXAMPLES. As previously discussed, for every *finite* cardinal number $n \geq 0$, there exists exactly one ordinal number with n as its cardinality.

The next interesting case is the class

$$A = \{\alpha : \alpha \text{ ordinal number with } \operatorname{card} \alpha \leq \aleph_0\},$$

then A is a set. Indeed if we choose an ordinal number γ_1 with $\operatorname{card} \gamma_1 = \mathfrak{c}$, then A is a subset of P_{γ_1}. Moreover, if we choose an ordinal number γ_2 with $\operatorname{card} \gamma_2 = 2^{\mathfrak{c}}$, then we see that $\gamma_1 \in P_{\gamma_2} \smallsetminus A$. We can then take Ω to be the smallest element of the non-empty set $P_{\gamma_2} \smallsetminus A$, and we have

$$A = P_\Omega.$$

The inclusion $A \subset P_\Omega$ is clear. To prove the other inclusion, we start with some ordinal number $\alpha < \Omega$ and we see that this forces $\alpha \in P_{\gamma_2}$, so it will be impossible to have $\aleph_0 < \operatorname{card} \alpha$, because this would give $\alpha \in P_{\gamma_2} \smallsetminus A$, contradicting the minimality of Ω.

The ordinal number Ω is called the *smallest uncountable ordinal number.*

Fact 1: *The set P_Ω is uncountable.*

This follows from the fact that

$$\Omega = \operatorname{ord}(P_\Omega, \leq),$$

which gives $\aleph_0 < \operatorname{card} \Omega = \operatorname{card} P_\Omega$.

Fact 2: *The cardinal number $\aleph_1 = \operatorname{card} \Omega = \operatorname{card} P_\Omega$ is the smallest uncountable cardinal number.*

Indeed, if one starts with some cardinal number $\mathfrak{a} < \aleph_1$, then if we choose a well-ordered set $(W, \prec)$ with $\operatorname{card} W = \mathfrak{a}$, then, since we have $\operatorname{card} W < \operatorname{card} \Omega$ we must have $\operatorname{ord}(W, \prec) < \Omega$, which then forces $\mathfrak{a} \leq \aleph_0$.

Fact 3: *Any countable subset $A \subset P_\Omega$ has a strict upper bound in P_Ω, that is, there exists $\beta \in P_\Omega$, such that $\alpha < \beta$, $\forall\, \alpha \in A$.*

We prove this by contradiction. Assume A has no strict upper bound in P_Ω, which means that for every $\beta \in P_\Omega$, there exists some $\alpha \in A$ such that $\beta \leq \alpha$. This gives

$$P_\Omega = \bigcup_{\alpha \in A} (P_\alpha \cup \{\alpha\}). \tag{6}$$

But for every $\alpha \in P_\Omega$ we have $\operatorname{ord} P_\alpha = \alpha$, which forces $\operatorname{card} P_\alpha \leq \aleph_0$. Then the fact that A is countable, combined with (6) will force P_Ω to be countable, which is impossible.

The above construction can be generalized to arbitrary cardinal numbers, giving the following

Fact 4: *Given any cardinal number $\mathfrak{a}$, there exist a smallest ordinal number $\Omega_{\mathfrak{a}}$ with $\mathfrak{a} < \operatorname{card} \Omega_{\mathfrak{a}}$, and the cardinal number $\mathfrak{a}' = \operatorname{card} \Omega_{\mathfrak{a}}$ is the smallest cardinal number with $\mathfrak{a} < \mathfrak{a}'$. Any set $A \subset P_{\Omega_{\mathfrak{a}}}$, with $\operatorname{card} A \leq \mathfrak{a}$, has a strict upper bound in $P_{\Omega_{\mathfrak{a}}}$.*