Appendix A

Zorn Lemma

In this Appendix we review basic set theoretical results, which are consequences of the following postulate:

Axiom of Choice. *Given any non-empty collection*[1] $\{X_i : i \in I\}$ *of non-empty sets, the cartesian product*

$$\prod_{i \in I} X_i$$

is non-empty.

Recall that the cartesian product is defined as

$$\prod_{i \in I} = \big\{ f : I \to \bigcup_{i \in I} X_i \,:\, f(i) \in X_i,\ \ \forall\, i \in I \big\}.$$

In order to formulate several consequences of the Axion of Choice, we need several concepts.

Definitions. Given a set X, by a *relation on* X one means simply as subset $\mathcal{R} \subset X \times X$. The standard notation for relations is:

$$x \mathcal{R} y \Longleftrightarrow (x, y) \in \mathcal{R}.$$

An *order relation on* X is a relation $\prec$ with the following properties:

- $x \prec x$, $\forall\, x \in X$;
- if $x, y, z \in X$ satisfy $x \prec y$ and $y \prec z$, then $x \prec z$;
- if $x, y \in X$ satisfy $x \prec y$ and $y \prec x$, then $x = y$.

In this case the pair $(X, \prec)$ is called an *ordered set.*

An ordered set $(X, \prec)$ is said to be *totally ordered*, if

- *for any elements* $x, y \in X$ *one has either* $x \prec y$ *or* $y \prec x$.

More generally, given an (arbitrary) ordered set $(X, \prec)$, by a *totally ordered subset of* $(X, \prec)$, one means a subset $T \subset X$, which becomes totally ordered with respect to the order relation $\prec \big|_T$.

Example. Fix a set M, and take $\mathcal{X}$ to be the collection of all subsets of M. Then $\mathcal{X}$ carries a natural order relation defined by inclusion:

$$A \prec B \Longleftrightarrow A \subset B.$$

A totally ordered subset $\mathcal{C}$ of $(\mathcal{X}, \subset)$ is called a *chain of subsets of* M. Two subset $A, B \subset M$ will be said to be *comparable*, if either $A \subset B$, or $B \subset A$, i.e. the collection $\{A, B\}$ is a chain of subsets of M.

[1] By a "collection of sets" one simply means a set whose elements are sets themselves.

Definition. Let M be a set. A collection $\mathcal{F}$ of subsets of M is said to have the *chain property*, if

(c) *whenever $\mathcal{C} \subset \mathcal{F}$ is a chain, it follows that the union $\bigcup_{C\in\mathcal{C}} C$ also belongs to $\mathcal{F}$.*

Lemma A.1. *Let M be a set, let $\mathcal{F}$ be a collection of subsets of M with the chain property. For every set $A \in \mathcal{F}$, the collection*

$$\text{comp}(A;\mathcal{F}) = \{B \in \mathcal{F} : B\text{comparable to } A\}$$

has the chain property.

Proof. Let $\mathcal{C} \subset \text{comp}(A;\mathcal{F})$ be a chain, and put $T = \bigcup_{C\in\mathcal{C}} C$. Since $\mathcal{F}$ has the chain property, we have $T \in \mathcal{F}$. To show that T is comparable with A, we consider the two pssibilities:

Case 1: *$A \supset C$, for all $C \in \mathcal{C}$.* In this case we have $A \supset \bigcup_{C\in\mathcal{C}} C = T$.

Case 2: *There exists $C_0 \in \mathcal{C}$, such that $A \subset C_0$.* In this case we have $A \subset C_0 \subset T$. □

Lemma A.2. *Let M be some non-empty set, let $\mathcal{F}$ let $\mathcal{F}$ be a non-empty collection of subsets of M, with the chain property Suppose one has a map*

$$\mathcal{F} \ni A \longmapsto x_A \in M,$$

with the property that

$$A \cup \{x_A\} \in \mathcal{F}, \ \ \forall\, A \in \mathcal{F}.$$

Then there exists $A \in \mathcal{F}$ such that $x_A \in A$.

Proof. For each $A \in \mathcal{F}$ we define $A^+ = A \cup \{x_A\}$. Call a subset $\mathcal{G} \subset \mathcal{F}$ *inductive*, if it has the chain property, and

(+) $A \in \mathcal{G} \Rightarrow A^+ \in \mathcal{G}$.

It is quite clear that if $\mathcal{G}_i$, $i \in I$ is a collection of inductive subsets of $\mathcal{F}$, then the intersection $\bigcap_{i\in I} \mathcal{G}_i$ is again an inductive subset of $\mathcal{F}$.

Fix now some subset $A_0 \in \mathcal{F}$, and define

$$\mathcal{G}_0 = \bigcap_{\substack{\mathcal{G} \text{ inductive} \\ A_0 \in \mathcal{G}}} \mathcal{G}.$$

Note that the subset $\mathcal{F}_0 = \{A \in \mathcal{F} : A \supset A_0\}$ is an inductive subset of $\mathcal{F}$, so in particular, $\mathcal{G}_0$ is non-empty, and $\mathcal{G}_0 \subset \mathcal{F}_0$, i.e.

$$A \supset A_0, \ \ \forall\, A \in \mathcal{G}_0. \tag{1}$$

Claim: *The set $\mathcal{G}_0$ is a chain.*

What we need to prove is the fact that $\mathcal{G}_0$ is totally ordered by inclusion. Consider the set

$$\mathcal{T} = \{T \in \mathcal{G} : T \text{ is comparable with every } A \in \mathcal{G}_0\} = \bigcap_{A\in\mathcal{G}_0} \text{comp}(A;\mathcal{G}_0),$$

and we try to prove that $\mathcal{T} = \mathcal{G}_0$. By Lemma A.1 it is clear that $\mathcal{T}$ has the chain property. Using (1), it is clear that $A_0 \in \mathcal{T}$. Finally, we need to prove property (+). We prove this indirectly as follows. Fix $T \in \mathcal{T}$, consider the collection

$$\mathcal{V}_T = \text{comp}(T^+;\mathcal{G}_0) = \{A \in \mathcal{G}_0 : A \text{ comparable with } T^+\},$$

and let us prove that $\mathcal{V}_T = \mathcal{G}_0$, by showing that $\mathcal{V}_T$ is an inductive set, and contains A_0. First of all, by Lemma A.1, it follows that $\mathcal{V}_T$ has the chain property. Secondly, using (1) we have $A_0 \subset T \subset T^+$, so $A_0 \in \mathcal{V}_T$. Finally, to check property (+), we start with some $V \in \mathcal{V}_T$, and we show that $V^+ \in \mathcal{V}_T$. In the case when $T^+ \subset V$, we are done, because we have $T^+ \subset V \subset V^+$. Assume $T^+ \not\subset V$, so that we have $V \subset T$. Since T is comparable with V^+, we either have $V^+ \subset T$, in which case we are done, or we have $T \subset V^+$. In the latter case, we have

$$V \subset T \subset V^+.$$

Since $V^+ = V \cup \{x_V\}$, the above inclusions forces either $T = V$, which gives $T^+ = V^+$, or $T = V^+$. Clearly, either case gives $V^+ \in \mathcal{V}_T$. Having shown that $\mathcal{V}_T$ is inductive, the inclusion $\mathcal{V}_T \subset \mathcal{G}_0$ will force the equality $\mathcal{V}_T = \mathcal{G}_0$. In turn, the definition of $\mathcal{V}_T$ proves that $T^+ \in \mathcal{T}$, so $\mathcal{T}$ is indeed inductive. Finally, the inclusion $\mathcal{T} \subset \mathcal{G}_0$ then forces $\mathcal{T} = \mathcal{G}_0$, and by the definition of $\mathcal{T}$, it follows that $\mathcal{G}_0$ is indeed a chain.

Having proven the Claim, we now take $A = \bigcup_{G \in \mathcal{G}_0} G$. Since $\mathcal{G}_0$ has the chain property, it follows that $A \in \mathcal{G}_0$. By construction we have

$$A \supset G, \ \ \forall G \in \mathcal{G}_0.$$

In particular we have $A \supset A^+$, which clearly forces $x_A \in A$. □

Definitions. Let $(X, \prec)$ be an ordered set. By a *maximal element for* X one means an element $x \in X$ with the property:

$$\{y \in X \, : \, x \prec y\} = \{x\}.$$

In other words, this means that *there is no element* $y \in X$*, with* $x \prec y$ *and* $y \neq x$.

Given a subset $S \subset X$, an element $x \in X$ is said to be an *upper bound for* S, if

$$s \prec x, \ \ \forall s \in S.$$

If such an x exists, we say that S *has an upper bound.* (It is not assumed that x belongs to S!)

Lemma A.3 ("Easy" Zorn Lemma). *Let* M *be a set, and let* $\mathcal{F}$ *be a collection of subsets of* M*. Assume*

- *the Axiom of Choice is true;*
- $\mathcal{F}$ *has the chain property;*
- $\mathcal{F}$ *and is hereditary, in the sense that, whenever* $A \in \mathcal{F}$*, it follows that all subsets of* A *belong to* $\mathcal{F}$*.*

Then, when equipped with the inclusion relation, $(\mathcal{F}, \subset)$ *has at least one maximal element.*

Proof. The proof will be carried on by contradiction. Assume no $A \in \mathcal{F}$ is maximal. For each $A \in \mathcal{F}$, define

$$X_A = \big\{x \in M \smallsetminus A \, : \, A \cup \{x\} \in \mathcal{F}\big\}.$$

Claim: *For every* $A \in \mathcal{F}$*, the set* X_A *is non-empty.*

Indeed, since A is not maximal, there exists some $B \in \mathcal{F}$, with $A \subsetneq B$. In particular, there exists some $x \in B \smallsetminus A$, and since $A \cup \{x\} \subset B$, by the hereditary property, it follows that $x \in X_A$.

Use now the Axiom of Choice, to find a map

$$\mathcal{F} \ni A \longmapsto x_A \in M,$$

such that $x_A \in X_A$, $\forall A \in \mathcal{F}$. This means that $A \cup \{x_A\} \in \mathcal{F}$, and $x_A \notin A$, for all $A \in \mathcal{F}$. By Lemma A.2 this is however impossible. □

THEOREM A.1 (Zorn Lemma). *Assume the Axiom of Choice is true. Let* $(X, \prec)$ *be a non-empty ordered set, with the following property*

(Z) *every totally ordered subset* $A \subset X$ *has an upper bound.*

Then X *has at least one maximal element.*

PROOF. Define the collection

$$\mathcal{F} = \{A \subset X \,:\, A \text{ totally ordered subset}\}.$$

Clearly $\mathcal{F}$ is non-empty (it contains, for instance, all singletons).

It is quite clear that $\mathcal{F}$ satisfies the hypothesis of Lemma A.3. So $(\mathcal{F}, \subset)$ has a maximal element A. Take now x to be an upper bound for A, i.e. $a \prec x$, $\forall a \in A$.

Now we prove that x is maximal for $(X, \prec)$. Suppose $y \in X$ satisfies $x \prec y$. Then clearly $A \cup \{y\}$ will still be a totally ordered subset of X, i.e. $A \cup \{y\} \in \mathcal{F}$. The maximality of A in $(\mathcal{F}, \subset)$ will force $A \cup \{y\} = A$, so we get $y \in A$, hence $y \prec x$. Since we also have $x \prec y$, this forces $y = x$. □