

LECTURES 36-37

4. Radon-Nikodym Theorems

In this section we discuss a very important property which has many important applications.

DEFINITION. Let X be a non-empty set, and let \mathcal{A} be a σ -algebra on X . Given two measures μ and ν on \mathcal{A} , we say that ν has the Radon-Nikodym property relative to μ , if there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(1) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Here we use the convention which defines the integral in the right hand side by

$$\int_A f d\mu = \begin{cases} \int_X f \chi_A d\mu & \text{if } f \chi_A \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \\ \infty & \text{if } f \chi_A \notin \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \end{cases}$$

In this case, we say that f is a density for ν relative to μ .

The Radon-Nikodym property has an equivalent useful formulation.

PROPOSITION 4.1 (Change of Variables). *Let X be a non-empty set, and let \mathcal{A} be a σ -algebra on X , let μ and ν be measures on \mathcal{A} , and let $f : X \rightarrow [0, \infty]$ be a measurable function.*

A. *The following are equivalent*

- (i) ν has the Radon-Nikodym property relative to μ , and f is a density for ν relative to μ ;
- (ii) for every measurable function $h : X \rightarrow [0, \infty]$, one has the equality¹

$$(2) \quad \int_X h d\nu = \int_X hf d\mu.$$

B. *If ν and f are as above, and \mathbb{K} is either \mathbb{R} or \mathbb{C} , then the equality (2) also holds for those measurable functions $h : X \rightarrow \mathbb{K}$ with $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.*

PROOF. A. (i) \Rightarrow (ii). Assume property (i) holds, which means that we have (1). Fix a measurable function $h : X \rightarrow [0, \infty]$, and use Theorem III.3.2, to find a sequence $(h_n)_{n=1}^{\infty} \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

- (a) $0 \leq h_1 \leq h_2 \leq \dots \leq h$;
- (b) $\lim_{n \rightarrow \infty} h_n(x) = h(x), \forall x \in X$.

Of course, we also have

$$(a') \quad 0 \leq h_1 f \leq h_2 f \leq \dots \leq hf;$$

¹ For the product hf we use the conventions $0 \cdot \infty = \infty \cdot 0 = 0$, and $t \cdot \infty = \infty \cdot t = \infty$, $\forall t \in (0, \infty]$.

$$(b') \lim_{n \rightarrow \infty} h_n(x)f(x) = h(x)f(x), \forall x \in X.$$

Using the Monotone Convergence Theorem, we then get the equalities

$$(3) \quad \int_X h \, d\nu = \lim_{n \rightarrow \infty} \int_X h_n \, d\nu \quad \text{and} \quad \int_X hf \, d\mu = \lim_{n \rightarrow \infty} \int_X h_n f \, d\nu$$

Notice that, if we fix n and we write $h_n = \sum_{k=1}^p \alpha_k \chi_{A_k}$, for some $A_1, \dots, A_p \in \mathcal{A}$, and $\alpha_1 > \dots > \alpha_p > 0$, then

$$\int_X h_n \, d\nu = \sum_{k=1}^p \alpha_k \nu(A_k) = \sum_{k=1}^p \int_X \alpha_k \chi_{A_k} f \, d\mu = \int_X h_n f \, d\mu,$$

so using (3), we immediately get (2).

The implication (ii) \Rightarrow (i) is trivial, using functions of the form $h = \chi_A$, $A \in \mathcal{A}$.

B. Suppose ν has the Radon-Nikodym property relative to μ , and f is a density for ν relative to μ , and let $h : X \rightarrow \mathbb{K}$ be a measurable function with $h \in L^1_{\mathbb{K}}(X, \mathcal{A}, \nu)$ and $hf \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. In the complex case, using the inequalities $|\operatorname{Re} h| \leq |h|$ and $|\operatorname{Im} h| \leq |h|$, it is clear that both functions $\operatorname{Re} h$ and $\operatorname{Im} h$ belong to $L^1(X, \mathcal{A}, \nu)$, and also the products $(\operatorname{Re} h)f$ and $(\operatorname{Im} h)f$ belong to $L^1(X, \mathcal{A}, \mu)$. This shows that it suffices to prove (2) under the additional hypothesis that h is real-valued. In this case we consider the functions h^\pm , defined by

$$h^+ = \max\{h, 0\} \quad \text{and} \quad h^- = \max\{-h, 0\}.$$

Since we have $0 \leq h^\pm \leq |h|$, it follows that $h^\pm \in \mathcal{L}^1_+(X, \mathcal{A}, \nu)$, as well as $h^\pm f \in \mathcal{L}^1_+(X, \mathcal{A}, \mu)$. In particular, we get the equalities

$$(4) \quad \int_X h \, d\nu = \int_X h^+ \, d\nu - \int_X h^- \, d\nu \quad \text{and} \quad \int_X hf \, d\mu = \int_X h^+ f \, d\mu - \int_X h^- f \, d\mu.$$

Since $h^\pm \geq 0$, we can use property A.(ii) above, and we have

$$\int_X h^\pm \, d\nu = \int_X h^\pm f \, d\mu,$$

and then the desired equality (2) immediately follows from (4). \square

One important issue is the uniqueness of the density. For this purpose, it will be helpful to introduce the following.

DEFINITION. Let T be one of the spaces $[-\infty, \infty]$ or \mathbb{C} , and let R be some relation on T (in our case R will be either “=,” or “ \geq ,” or “ \leq ,” on $[-\infty, \infty]$). Given a measurable space (X, \mathcal{A}, μ) , and two measurable functions $f_1, f_2 : X \rightarrow T$,

$$f_1 R f_2, \quad \mu\text{-l.a.e.}$$

if the set

$$A = \{x \in X : f_1(x) R f_2(x)\}$$

belongs to \mathcal{A} , and it has locally μ -null complement in X , i.e. $\mu([X \setminus A] \cap F) = 0$, for every set $F \in \mathcal{A}$ with $\mu(F) < \infty$. (If R is one of the relations listed above, the set A automatically belongs to \mathcal{A} , so all intersections $[X \setminus A] \cap F$, $F \in \mathcal{A}$, also belong to \mathcal{A} .) The abbreviation “ μ -l.a.e.” stands for “ μ -locally-almost everywhere.” Remark that one has the implication

$$f_1 R f_2, \quad \mu\text{-a.e.} \Rightarrow f_1 R f_2, \quad \mu\text{-l.a.e.}$$

Remark that, when μ is σ -finite, then the other implication also holds:

$$f_1 R f_2, \quad \mu\text{-l.a.e.} \Rightarrow f_1 R f_2, \quad \mu\text{-a.e.}$$

With this terminology, one has the following uniqueness result.

PROPOSITION 4.2. *Suppose \mathcal{A} is a σ algebra on some non-empty set X , and μ and ν are measures on \mathcal{A} , such that ν has the Radon-Nikodym property relative to μ . If $f, g : X \rightarrow [0, \infty]$ are densities for ν relative to μ , then*

$$f = g, \mu\text{-l.a.e.}$$

In particular, if μ is σ -finite, then

$$f = g, \mu\text{-a.e.}$$

PROOF. Consider the set $B = \{x \in X : f(x) \neq g(x)\}$, which belongs to \mathcal{A} . We need to prove that B is locally μ -null, i.e. one has $\mu(B \cap F) = 0$, for all $F \in \mathcal{A}$ with $\mu(F) < \infty$. Fix $F \in \mathcal{A}$ with $\mu(F) < \infty$, and let us write $B \cap F = D \cup E$, where

$$D = \{x \in B \cap F : f(x) < g(x)\} \text{ and } E = \{x \in B \cap F : f(x) > g(x)\}.$$

If we define, for each integer $n \geq 1$, the sets

$$D_n = \{x \in B \cap F : f(x) + \frac{1}{n} \leq g(x)\} \text{ and } E_n = \{x \in B \cap F : f(x) \geq g(x) + \frac{1}{n}\},$$

then it is clear that

$$B \cap F = D \cup E = \bigcup_{n=1}^{\infty} (D_n \cup E_n),$$

so in order to prove that $\mu(B \cap F) = 0$, it suffices to show that $\mu(D_n) = \mu(E_n) = 0$, $\forall n \geq 1$.

Fix $n \geq 1$. It is obvious that $f(x) < \infty$, $\forall x \in D_n$, so if we define the sequence $(D_n^k)_{k=1}^{\infty} \subset \mathcal{A}$, by

$$D_n^k = \{x \in D_n : f(x) \leq k\}, \quad \forall k \geq 1,$$

we have the equality $D_n = \bigcup_{k=1}^{\infty} D_n^k$, so in order to prove that $\mu(D_n) = 0$, it suffices to show that $\mu(D_n^k) = 0$, $\forall k \geq 1$. On the one hand, since $f(x) \leq k$, $\forall k \geq 1$, using the inclusion $D_n^k \subset F$, we get

$$\nu(D_n^k) = \int_{D_n^k} f d\mu \leq \int_X k \chi_{D_n^k} d\mu = k\mu(D_n^k) \leq k\mu(F) < \infty.$$

On the other hand, since $g(x) \geq f(x) + \frac{1}{n}$, $\forall x \in D_n^k$, we get

$$\begin{aligned} \nu(D_n^k) &= \int_{D_n^k} g d\mu \geq \int_X (f \chi_{D_n^k} + \frac{1}{n} \chi_{D_n^k}) d\mu = \\ &= \int_X f \chi_{D_n^k} d\mu + \int_X \frac{1}{n} \chi_{D_n^k} d\mu = \nu(D_n^k) + \frac{1}{n} \mu(D_n^k). \end{aligned}$$

Since $\nu(D_n^k) < \infty$, the above inequality forces $\mu(D_n^k) = 0$.

The fact that $\mu(E_n) = 0$, $\forall n \geq 1$, is proven the exact same way. \square

In general, the uniqueness of the density does not hold μ -a.e., as it is seen from the following.

EXAMPLE 4.1. Take X to be some non-empty set, put $\mathcal{A} = \{\emptyset, X\}$, and define the measure μ on \mathcal{A} , by $\mu(\emptyset) = 0$ and $\mu(X) = \infty$. It is clear that μ has the Radon-Nikodym property relative to itself, but as sensitivities one can choose for instance the constant functions $f = 1$ and $g = 2$. Clearly, the equality $f = g$, μ -a.e. is not true.

REMARK 4.1. The local almost uniqueness result, given in Proposition 4.2, holds under slightly weaker assumptions. Namely, if (X, \mathcal{A}, μ) is a measure space, and if $f, g : X \rightarrow [0, \infty]$ are measurable functions for which we have the equality

$$\int_A f d\mu = \int_A g d\mu,$$

for all $A \in \mathcal{A}$ with $\mu(A) < \infty$, then we still have the equality $f = g$, μ -l.a.e. This follows actually from Proposition 4.2, applied to functions of the form $f|_A$ and $g|_A$.

Let us introduce the following.

NOTATIONS. For a measure space (X, \mathcal{A}, μ) we define

$$\begin{aligned} \mathcal{A}_0^\mu &= \{N \in \mathcal{A} : \mu(N) = 0\}; \\ \mathcal{A}_{\text{fin}}^\mu &= \{F \in \mathcal{A} : \mu(F) < \infty\}; \\ \mathcal{A}_{0,\text{loc}}^\mu &= \{A \in \mathcal{A} : \mu(A \cap F) = 0, \forall F \in \mathcal{A}_{\text{fin}}^\mu\}. \end{aligned}$$

With these notations, we have the inclusions

$$\mathcal{A}_0^\mu = \mathcal{A}_{0,\text{loc}}^\mu \cap \mathcal{A}_{\text{fin}}^\mu \subset \mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A},$$

and \mathcal{A}_0^μ and $\mathcal{A}_{0,\text{loc}}^\mu$ are in fact σ -rings.

COMMENT. The “locally-almost everywhere” terminology is actually designed to “hide some pathologies under the rug.” For instance, if (X, \mathcal{A}, μ) is a *degenerate* measure space, i.e. $\mu(A) \in \{0, \infty\}$, $\forall A \in \mathcal{A}$, then “anything happens locally almost-everywhere,” which means that we have the equality $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}$.

At the other end, there is a particular type of measure spaces on which, even in the absence of σ -finiteness, the notions of “locally-almost everywhere” and “almost everywhere” coincide, i.e. we have the equality $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$. Such spaces are described by the following.

DEFINITION. A measure space (X, \mathcal{A}, μ) is said to be *nowhere degenerate*, or with *finite subset property*, if

- (F) for every set $A \in \mathcal{A}$ with $\mu(A) > 0$, there exists some set $F \in \mathcal{A}$, with $F \subset A$, and $0 < \mu(F) < \infty$.

With this terminology, one has the following result.

PROPOSITION 4.3. For a measure space (X, \mathcal{A}, μ) , the following are equivalent:

- (i) $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$;
(ii) (X, \mathcal{A}, μ) has the finite subset property.

PROOF. (i) \Rightarrow (ii). Assume $\mathcal{A}_{0,\text{loc}}^\mu = \mathcal{A}_0^\mu$, and let us prove that (X, \mathcal{A}, μ) has the finite subset property. We argue by contradiction, so let us assume there exists some set $A \in \mathcal{A}$, with $\mu(A) = \infty$, such that $\mu(B) \in \{0, \infty\}$, for every $B \in \mathcal{A}$, with $B \subset A$. In particular, if we start with some arbitrary $F \in \mathcal{A}_{\text{fin}}^\mu$, using the fact that $\mu(A \cap F) \leq \mu(F) < \infty$, we see that we must have $\mu(A \cap F) = 0$. This proves precisely that $A \in \mathcal{A}_{0,\text{loc}}^\mu$. By assumption, it follows that $A \in \mathcal{A}_0^\mu$, i.e. $\mu(A) = 0$, which is impossible.

(ii) \Rightarrow (i). Assume that (X, \mathcal{A}, μ) has the finite subset property, and let us prove the equality (i). Since one inclusion is always true, all we need to prove is the inclusion $\mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A}_0^\mu$, which equivalent to the inclusion $\mathcal{A}_{0,\text{loc}}^\mu \subset \mathcal{A}_{\text{fin}}^\mu$. Start with some set $A \in \mathcal{A}_{0,\text{loc}}^\mu$, but assume $\mu(A) = \infty$. On the one hand, using the finite

subset property, there exists some set $F \in \mathcal{A}$ with $F \subset A$ and $\mu(F) > 0$. On the other hand, since $A \in \mathcal{A}_{0,\text{loc}}^\mu$, we have $\mu(F) = 0$, which is impossible. \square

EXAMPLE 4.2. Take X be an uncountable set, let $\mathcal{A} = \mathcal{P}(X)$, and let μ be the counting measure, i.e.

$$\mu(A) = \begin{cases} \text{Card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Then $(X, \mathcal{P}(X), \mu)$ has the finite subset property, but is not σ -finite.

When we restrict to *integrable* functions, the two notions μ -l.a.e. and μ -a.e. coincide. More precisely, we have the following.

PROPOSITION 4.4. *Let (X, \mathcal{A}, μ) be a measure space, let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let $p \in [1, \infty)$. For a function $f \in \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu)$, the following are equivalent:*

- (i) $f = 0$, μ -l.a.e.
- (ii) $f = 0$, μ -a.e.

PROOF. Of course, we only need to prove the implication (i) \Rightarrow (ii). Assume $f = 0$, μ -l.a.e. Using the function $g = |f|^p$, we can assume that $p = 1$ and $f(x) \geq 0$, $\forall x \in X$. Consider then the set $N = \{x \in X : f(x) > 0\}$, and write it as a union $N = \bigcup_{n=1}^{\infty} N_n$, where

$$N_n = \{x \in X : f(x) \geq \frac{1}{n}\}, \quad \forall n \geq 1.$$

Of course, all we need is the fact that $\mu(N_n) = 0$, $\forall n \geq 1$. Fix $n \geq 1$. On the one hand, the assumption on f , it follows that $N_n \in \mathcal{A}_{0,\text{loc}}^\mu$. On the other hand, the inequality $\frac{1}{n} \chi_{N_n} \leq f$, forces the elementary function $\frac{1}{n} \chi_{N_n}$ to be μ -integrable, i.e. $\mu(N_n) < \infty$. Consequently we have

$$N \in \mathcal{A}_{0,\text{loc}}^\mu \cap \mathcal{A}_{\text{fin}}^\mu = \mathcal{A}_0^\mu. \quad \square$$

COMMENT. In what follows we will discuss several results, which all have as conclusion the fact that one measure has the Radon-Nikodym property with respect to another one. All such results will be called “Radon-Nikodym Theorems.”

The first result is in fact quite general, in the sense that it works for finite signed or complex measures.

THEOREM 4.1 (“Easy” Radon-Nikodym Theorem). *Let (X, \mathcal{A}, μ) be a finite measure space, let \mathbb{K} denote one of the fields \mathbb{R} or \mathbb{C} , and let $C > 0$ be some constant. Suppose ν is a \mathbb{K} -valued measure on \mathcal{A} , such that*

$$|\nu(A)| \leq C\mu(A), \quad \forall A \in \mathcal{A}.$$

Then there exists some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that

$$(5) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) *Any function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, satisfying (5) has the property $|f| \leq C$, μ -a.e. If ν is an “honest” measure, then one also has the inequality $|f| \geq 0$, μ -a.e.*
- (ii) *A function satisfying (5) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (5), it follows that $f_1 = f_2$, μ -a.e.*

PROOF. The idea is to somehow make sense of $\int_X h d\nu$, for suitable measurable functions h , and to examine the properties of such a number relative to the integral $\int_X h d\mu$. The second integral is of course defined, for instance for $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, but the first integral is not, because ν is not an “honest” measure. The proof will be carried on in several steps.

Step 1: There exist four “honest” finite measures ν_k , $k = 1, 2, 3, 4$, and numbers α_k , $k = 1, 2, 3, 4$, such that $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 + \alpha_4\nu_4$, and

$$(6) \quad \nu_k \leq C\mu, \quad \forall k = 1, 2, 3, 4.$$

In the case $\mathbb{K} = \mathbb{R}$ we use the Hahn-Jordan decomposition $\nu = \nu^+ - \nu^-$. We also know that $\nu^\pm \leq |\nu|$, the variation measure of ν . In this case we take $\alpha_1 = 1$, $\nu_1 = \nu^+$, $\alpha_2 = -1$, $\nu_2 = \nu^-$, and we set $\nu_3 = \nu_4 = 0$, $\alpha_3 = \alpha_4 = 0$.

In the case $\mathbb{K} = \mathbb{C}$, we write $\nu = \eta + i\lambda$, with η and λ finite signed measures, and we use the Hahn-Jordan decompositions $\eta = \eta^+ - \eta^-$ and $\lambda = \lambda^+ - \lambda^-$. We also know that the variation measures of η and λ satisfy $|\eta| \leq |\nu|$ and $|\lambda| \leq |\nu|$, so we also have $\eta^\pm \leq |\nu|$ and $\lambda^\pm \leq |\nu|$. In this case we can then take $\alpha_1 = 1$, $\nu_1 = \eta^+$, $\alpha_2 = -1$, $\nu_2 = \eta^-$, $\alpha_3 = i$, $\nu_3 = \lambda^+$, $\alpha_4 = -i$, $\nu_4 = \lambda^-$.

Notice that in either case we have

$$\nu_k \leq |\nu|, \quad \forall k = 1, 2, 3, 4.$$

By Remark III.8.5 it follows that we have $|\nu| \leq C\mu$, so we immediately get the inequalities (6).

Step 2: For any measurable function $h : X \rightarrow [0, \infty]$, one has the inequality

$$(7) \quad \int_X h d\nu_k \leq C \int_X h d\mu, \quad \forall k = 1, 2, 3, 4.$$

To prove this, we choose a sequence of elementary functions $(h_n)_{n=1}^\infty \subset \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with

- $0 \leq h_1 \leq h_2 \leq \dots$ (everywhere),
- $\lim_{n \rightarrow \infty} h_n(x) = h(x)$, $\forall x \in X$,

so that by the General Monotone Convergence Theorem, we get the equalities

$$\int_X h d\mu = \lim_{n \rightarrow \infty} \int_X h_n d\mu \quad \text{and} \quad \int_X h d\nu_k = \lim_{n \rightarrow \infty} \int_X h_n d\nu_k, \quad \forall k = 1, 2, 3, 4.$$

This means that, in order to prove (7), it suffices to prove it under the extra assumption that h is elementary. In this case, we have

$$h = \beta_1 \chi_{B_1} + \dots + \beta_p \chi_{B_p},$$

with $\beta_1, \dots, \beta_p \geq 0$ and $B_1, \dots, B_p \in \mathcal{A}$. The inequality is then immediate, from (6) since we have

$$\int_X h d\nu_k = \sum_{j=1}^p \beta_j \nu_k(B_j) \leq C \sum_{j=1}^p \mu(B_j) = C \int_X h d\mu.$$

As a consequence of Step 2, we get the fact that, for every $k = 1, 2, 3, 4$, one has the inclusions

$$\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \nu_k) \quad \text{and} \quad \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu_k).$$

Taking quotients, this gives rise to correctly defined linear maps

$$(8) \quad \Phi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \ni h \longmapsto h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu_k), \quad k = 1, 2, 3, 4.$$

(Here we use the abusive notation that identifies an element in L^1 with a function in \mathfrak{L}^1 , which is defined almost uniquely.) Moreover, one has the inequality

$$\int_X |h| d\nu_k \leq C \int_X |h| d\mu, \quad \forall h \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu), \quad k = 1, 2, 3, 4,$$

in other words, the linear maps (8) are all continuous. For every $k = 1, 2, 3, 4$, let ϕ_k denote the integration map

$$\phi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \nu_k) \ni h \longmapsto \int_X h d\nu_k \in \mathbb{K}.$$

We know (see Remark 3.5) that the ϕ_k 's are continuous. In particular, the compositions $\psi_k = \phi_k \circ \Phi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, which are defined by

$$\psi_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \ni h \longmapsto \int_X h d\nu_k, \quad k = 1, 2, 3, 4,$$

are linear and continuous.

We now use Proposition 3.3 which states that one has an inclusion

$$(9) \quad \Theta : L_{\mathbb{K}}^2(X, \mathcal{A}, \mu) \hookrightarrow L_{\mathbb{K}}^1(X, \mathcal{A}, \mu),$$

which is in fact a linear continuous map. So if we consider the compositions $\theta_k = \psi_k \circ \Theta$, which are defined by

$$\theta_k : L_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \ni h \longmapsto \int_X h d\nu_k, \quad k = 1, 2, 3, 4,$$

then these compositions are linear and continuous. Apply then Riesz Theorem (in the form given in Remark 3.4), to find functions $f_1, f_2, f_3, f_4 \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, such that

$$\theta_k(h) = \langle f_k, h \rangle, \quad \forall h \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu), \quad k = 1, 2, 3, 4.$$

In particular, using functions of the form $h = \varkappa_A$, $A \in \mathcal{A}$ (which all belong to $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, due to the finiteness of μ), we get

$$\nu_k(A) = \int_X \varkappa_A d\nu_k = \int_X f_k \varkappa_A d\mu, \quad \forall A \in \mathcal{A}, \quad k = 1, 2, 3, 4.$$

Finally, if we define the function $f = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 \in L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$, then the above equalities immediately give the equality (5).

At this point we only know that f belongs to $L_{\mathbb{K}}^2(X, \mathcal{A}, \mu)$. Using the inclusion (9), it turns out that f indeed belongs to $L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.

Let us prove now the additional properties (i) and (ii).

To prove the first assertion in (i), we start off by fixing some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, which satisfies (5), and we define the set

$$A = \{x \in X : |f(x)| > C\},$$

for which we must prove that $\mu(A) = 0$. Since f is measurable, it follows that A belongs to \mathcal{A} . Consider the “rational unit sphere” $S_{\mathbb{Q}}^1$ in \mathbb{K} , defined as

$$(10) \quad S_{\mathbb{Q}}^1 = \begin{cases} \{-1, 1\} & \text{if } \mathbb{K} = \mathbb{R} \\ \{e^{2\pi it} : t \in \mathbb{Q}\} & \text{if } \mathbb{K} = \mathbb{C} \end{cases}$$

The point is that $S_{\mathbb{Q}}^1$ is dense in the unit sphere S^1 in \mathbb{K} :

$$S^1 = \{\alpha \in \mathbb{K} : |\alpha| = 1\},$$

so we immediately have the equality $A = \bigcup_{\alpha \in S_{\mathbb{Q}}^1} A_{\alpha}$, where

$$A_{\alpha} = \{x \in X : \operatorname{Re}[\alpha f(x)] > C\}.$$

Since $S_{\mathbb{Q}}^1$ is countable, in order to prove that $\mu(A) = 0$, it then suffices to show that $\mu(A_{\alpha}) = 0$, $\forall \alpha \in S_{\mathbb{Q}}^1$. Fix then $\alpha \in S_{\mathbb{Q}}^1$, and consider the \mathbb{K} -valued measure $\eta = \alpha\nu$. It is clear that we still have

$$(11) \quad |\eta(A)| = |\nu(A)| \leq C\mu(A), \quad \forall A \in \mathcal{A},$$

as well as the equality

$$(12) \quad \eta(A) = \int_A \alpha f d\mu, \quad \forall A \in \mathcal{A}.$$

For each integer $n \geq 1$, let us define the set

$$A_{\alpha}^n = \{x \in X : \operatorname{Re}[\alpha f(x)] \geq C + \frac{1}{n}\},$$

so that we obviously have the equality $A_{\alpha} = \bigcup_{n=1}^{\infty} A_{\alpha}^n$. In particular, in order to prove $\mu(A_{\alpha}) = 0$, it suffices to prove that $\mu(A_{\alpha}^n) = 0$, $\forall n \geq 1$. Fix for the moment $n \geq 1$. Using (12), it follows that

$$\operatorname{Re} \eta(A_{\alpha}^n) = \operatorname{Re} \left[\int_{A_{\alpha}^n} \alpha f d\mu \right] = \int_{A_{\alpha}^n} \operatorname{Re}[\alpha f] d\mu = \int_X \operatorname{Re}[\alpha f] \chi_{A_{\alpha}^n} d\mu.$$

Since we have $\operatorname{Re}[\alpha f] \chi_{A_{\alpha}^n} \geq (C + \frac{1}{n}) \chi_{A_{\alpha}^n}$, the above inequality can be continued with

$$\operatorname{Re} \eta(A_{\alpha}^n) \geq \int_X (C + \frac{1}{n}) \chi_{A_{\alpha}^n} d\mu = (C + \frac{1}{n}) \mu(A_{\alpha}^n).$$

Of course, this will give

$$|\eta(A_{\alpha}^n)| \geq \operatorname{Re} \eta(A_{\alpha}^n) \geq (C + \frac{1}{n}) \mu(A_{\alpha}^n).$$

Note now that, using (11), this will finally give

$$C\mu(A_{\alpha}^n) \geq (C + \frac{1}{n}) \mu(A_{\alpha}^n),$$

which clearly forces $\mu(A_{\alpha}^n) = 0$.

Having proven that $|f| \leq C$, μ -a.e., let us turn our attention now to the uniqueness property (ii). Suppose $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ are such that

$$\nu(A) = \int_A f_1 d\mu = \int_A f_2 d\mu, \quad \forall A \in \mathcal{A}.$$

Consider then the difference $f = f_1 - f_2$ and the trivial measure $\nu_0 = 0$. Obviously we have

$$|\nu_0(A)| \leq \frac{1}{n} \mu(A), \quad \forall A \in \mathcal{A},$$

for every integer $n \geq 1$, as well as

$$\nu_0(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

By the first assertion in (i), it follows that

$$|f_1 - f_2| = |f| \leq \frac{1}{n}, \quad \mu\text{-a.e.},$$

for every $n \geq 1$. So if we take the sets $(N_n)_{n=1}^{\infty} \subset \mathcal{A}$ defined by

$$N_n = \{x \in X : |f_1(x) - f_2(x)| > \frac{1}{n}\},$$

then $\mu(N_n) = 0, \forall n \geq 1$. Of course, if we put $N = \bigcup_{n=1}^{\infty} N_n$, then on the one hand we have $\mu(N) = 0$, and on the other hand, we have

$$f_1(x) - f_2(x) = 0, \quad \forall x \in X \setminus N,$$

which means that we indeed have $f_1 = f_2, \mu$ -a.e.

Finally, let us prove the second assertion in (i), which starts with the assumption that ν is an “honest” measure. Let $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (5). By the uniqueness property (ii), it follows immediately that

$$f = \operatorname{Re} f, \quad \mu\text{-a.e.},$$

so we can assume that f is already real-valued. Consider the “honest” measure $\omega = C\mu - \nu$, and notice that the function $g : X \rightarrow \mathbb{R}$ defined by

$$g(x) = C - f(x), \quad \forall x \in X,$$

clearly has the property

$$\omega(A) = \int_A g d\mu, \quad \forall A \in \mathcal{A}.$$

Since we obviously have

$$0 \leq \omega(A) \leq C\mu(A), \quad \forall A \in \mathcal{A},$$

by the first assertion of (i), applied to the measure ω and the function g , it follows that $|g| \leq C, \mu$ -a.e. In other words, we have now a combined inequality:

$$\max\{|f|, |C - f|\} \leq C, \quad \mu\text{-a.e.}$$

Of course, since f is real valued, this forces $f \geq 0, \mu$ -a.e. □

In what follows we are going to offer various generalizations of Theorem 4.1. There are several directions in which Theorem 4.1 can be generalized. The main direction, which we present here, will aim at weakening the condition $|\nu| \leq C\mu$. The following result explains that in fact the case of \mathbb{K} -valued measures can be always reduced to the case of “honest” finite ones.

PROPOSITION 4.5 (Polar Decomposition). *Let \mathcal{A} be a σ -algebra on a non-empty set X , let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let ν be a \mathbb{K} -valued measure on \mathcal{A} . Let $|\nu|$ denote the variation measure of ν . There exists some function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, such that*

$$(13) \quad \nu(A) = \int_A f d|\nu|, \quad \forall A \in \mathcal{A}.$$

Moreover

- (i) Any function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, satisfying (13) has the property $|f| = 1, |\nu|$ -a.e.
- (ii) A function satisfying (13) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$ satisfy (13), it follows that $f_1 = f_2, |\nu|$ -a.e.

PROOF. We know that

$$|\nu(A)| \leq |\nu|(A), \quad \forall A \in \mathcal{A}.$$

So if we apply Theorem 4.1 for the finite measure $\mu = |\nu|$ and $C = 1$, we immediately get the existence of $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\nu|)$, satisfying (13). Again by Theorem 4.1, the

uniqueness property (ii) is automatic, and we also have $|f| \leq 1$, $|\nu|$ -a.e. To prove the fact that we have in fact the equality $|f| = 1$, $|\nu|$ -a.e., we define the set

$$A = \{x \in X : |f(x)| < 1\},$$

which belongs to \mathcal{A} , and we prove that $|\nu|(A) = 0$. If we define the sequence of sets $(A_n)_{n=1}^\infty \subset \mathcal{A}$, by

$$A_n = \{x \in X : |f(x)| \leq 1 - \frac{1}{n}\}, \quad \forall n \geq 1,$$

then we clearly have $A = \bigcup_{n=1}^\infty A_n$, so all we have to show is the fact that $|\nu|(A_n) = 0$, $\forall n \geq 1$. Fix $n \geq 1$. For every $B \in \mathcal{A}$, with $B \subset A_n$, we have

$$|f(x)| \leq 1 - \frac{1}{n}, \quad \forall x \in B,$$

so using (13) we get

$$|\nu(B)| = \left| \int_B f d|\nu| \right| \leq \int_B |f| d|\nu| \leq \int_B (1 - \frac{1}{n}) d|\nu| = (1 - \frac{1}{n})|\nu|(B).$$

Now if we take an arbitrary pairwise disjoint sequence $(B_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty B_k = A_n$, then the above estimate will give

$$\sum_{k=1}^\infty |\nu(B_k)| \leq (1 - \frac{1}{n}) \sum_{k=1}^\infty |\nu|(B_k) = (1 - \frac{1}{n})|\nu|(A_n).$$

Taking supremum in the left hand side, and using the definition of the variation measure, the above estimate will finally give

$$|\nu|(A_n) \leq (1 - \frac{1}{n})|\nu|(A_n),$$

which clearly forces $|\nu|(A_n) = 0$. \square

REMARK 4.2. The case $\mathbb{K} = \mathbb{R}$ can be slightly generalized, to include the case of infinite signed measures. If ν is a signed measure on \mathcal{A} and if we consider the Hahn-Jordan set decomposition (X^+, X^-) , then the density f is simply the function

$$f(x) = \begin{cases} 1 & \text{if } x \in X^+ \\ -1 & \text{if } x \in X^- \end{cases}$$

The equality (13) will then hold only for those sets $A \in \mathcal{A}$ with $|\nu|(A) < \infty$. Since $|\nu|$ is allowed to be infinite, as explained in Example 4.1, the only version of uniqueness property (ii) will hold with “ $|\nu|$ -l.a.e” in place of “ $|\nu|$ -a.e” Likewise, the absolute value property (i) will have to be replaced with “ $|f| = 1$, $|\nu|$ -l.a.e”

COMMENT. Up to this point, it seems that the hypotheses from Theorem 4.1 are essential, particularly the dominance condition $|\nu| \leq C\mu$. It is worth discussing this property in a bit more detail, especially having in mind that we plan to weaken it as much as possible.

NOTATION. Suppose \mathcal{A} is a σ -algebra on some non-empty set X , and suppose μ and ν are “honest” (not necessarily finite) measures on \mathcal{A} . We shall write

$$\nu \Subset \mu,$$

if there exists some constant $C > 0$, such that

$$\nu(A) \leq C\mu(A), \quad \forall A \in \mathcal{A}.$$

A few steps in the proof of Theorem 4.1 hold even without the finiteness assumption, as indicated by the following.

*Exercise 1**. Suppose \mathcal{A} is a σ algebra on some non-empty set X , and suppose μ and ν are “honest” measures on \mathcal{A} . Prove the following.

(i) If $\nu \in \mu$, then one has the inclusions

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu) \text{ and } \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^p(X, \mathcal{A}, \nu), \quad \forall p \in [1, \infty).$$

Consequently (see the proof of Theorem 4.1) one has linear maps

$$L_{\mathbb{K}}^p(X, \mathcal{A}, \mu) \ni h \longmapsto h \in L_{\mathbb{K}}^p(X, \mathcal{A}, \nu), \quad \forall p \in [1, \infty).$$

Show that these linear maps are continuous.

(ii) Conversely, assuming one has the inclusion

$$\mathfrak{L}_{\mathbb{K}}^{p_0}(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^{p_0}(X, \mathcal{A}, \nu),$$

for *some* $p_0 \in [1, \infty)$, prove that $\nu \in \mu$.

HINT: To prove (ii) show first one has the inclusion $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_+^1(X, \mathcal{A}, \nu)$. Then show that the quantity

$$C = \sup \left\{ \int_X h d\nu : h \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu), \int_X h d\mu \leq 1 \right\}$$

is finite. If $C = \infty$, there exists some sequence $(h_n)_{n=1}^{\infty} \subset \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, with

$$\int_X h_n d\mu \leq 1 \text{ and } \int_X h d\nu \geq 4^n, \quad \forall n \geq 1.$$

Consider then the series $\sum_{n=1}^{\infty} \frac{1}{2^n} h_n$, and get a contradiction. Finally prove that $\nu(A) \leq C\mu(A)$, $\forall A \in \mathcal{A}$.

It is the moment now to introduce the following relation, which is a highly non-trivial weakening of the relation \in .

DEFINITION. Let \mathcal{A} is a σ -algebra on some non-empty set X , and suppose μ and ν are “honest” (not necessarily finite) measures on \mathcal{A} . We say that ν is *absolutely continuous with respect to* μ , if for every $A \in \mathcal{A}$, one has the implication

$$(14) \quad \mu(A) = 0 \implies \nu(A) = 0.$$

In this case we are going to use the notation

$$\nu \ll \mu.$$

It is obvious that one always has the implication

$$\nu \in \mu \implies \nu \ll \mu.$$

REMARKS 4.3. Let (X, \mathcal{A}, μ) be a measure space. A. If ν is an “honest” measure on \mathcal{A} , which has the Radon-Nikodym property relative to μ , then $\nu \ll \mu$. This is pretty obvious, since if we pick $f : X \rightarrow [0, \infty]$ to be a density for ν relative to μ , then for every $A \in \mathcal{A}$ with $\mu(A) = 0$, we have $f \chi_A = 0$, μ -a.e., so we get

$$\nu(A) = \int_A f d\mu = \int_X f \chi_A d\mu = 0.$$

B. For an “honest” measure ν on \mathcal{A} , the relation $\nu \ll \mu$ is equivalent to the inclusion

$$\mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \subset \mathfrak{N}_{\mathbb{K}}(X, \mathcal{A}, \nu).$$

By Exercise 1, this already suggests that the relation \ll is much weaker than \in (see Exercise 2 below).

C. If ν is either a signed or a complex measure on \mathcal{A} , then the following are equivalent:

(i) the variation measure $|\nu|$ is absolutely continuous with respect to μ ;

(ii) for every $A \in \mathcal{A}$, one has the implication (14)

The implication (i) \Rightarrow (ii) is trivial, since one has

$$|\nu(A)| \leq |\nu|(A), \quad \forall A \in \mathcal{A}.$$

The implication (ii) \Rightarrow (i) is also clear, since if we start with some $A \in \mathcal{A}$ with $\mu(A) = 0$, then we get $|\nu(B)| = 0$, for all $B \in \mathcal{A}$ with $B \subset A$, and then arguing exactly as in the proof of Proposition 4.3, we get $|\nu|(A) = 0$.

CONVENTION. Using Remark 4.2.A, we extend the definition of absolute continuity, and the notation $\nu \ll \mu$ to include the case when ν is either a signed measure, or a complex measure on \mathcal{A} . In other words, the notation $\nu \ll \mu$ means that $|\nu| \ll \mu$.

The following technical result is key for the second Radon-Nikodym Theorem.

LEMMA 4.1. *Let (X, \mathcal{A}, μ) be a finite measure space, and let ν be an "honest" measure on \mathcal{A} , with $\nu \ll \mu$. Then there exists a sequence $(\nu_n)_{n=1}^\infty$, of "honest" measures on \mathcal{A} , such that*

- (i) $\nu_n \subseteq \mu, \forall n \geq 1$; in particular the measures $\nu_n, n \geq 1$ are all finite;
- (ii) $\nu_1 \leq \nu_2 \leq \dots$;
- (iii) $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A), \forall A \in \mathcal{A}$.

PROOF. Let us define

$$\nu_n = (n\mu) \wedge \nu, \quad \forall n \geq 1.$$

Recall (see III.8, the Lattice Property; it is essential here that one of the measures, namely $n\mu$, is finite) that by construction ν_n has the following properties:

- (a) $\nu_n \leq n\mu$ and $\nu_n \leq \nu$;
- (b) whenever ω is a measure with $\omega \leq n\mu$ and $\omega \leq \nu$, it follows that $\omega \leq \nu_n$.

Property (a) above already gives condition (i). It will be helpful to notice that property (a) also gives the inequality

$$(15) \quad \nu_n \leq \nu, \quad \forall n \geq 1.$$

The monotonicity condition is now trivial, since by (b) the inequalities $\nu_{n-1} \leq (n-1)\mu \leq n\mu$ and $\nu_{n-1} \leq \nu$, imply $\nu_{n-1} \leq (n\mu) \wedge \nu = \nu_n$.

To derive property (iii), it will be helpful to recall the actual definition of the operation \wedge . Fix for the moment $n \geq 1$. One first considers the signed measure $\lambda_n = n\mu - \nu$, and its Hahn-Jordan decomposition $\lambda_n = \lambda_n^+ - \lambda_n^-$. In our case, we get $\lambda_n^+ \leq n\mu$ and $\lambda_n^- \leq \nu$. With these notations the measures ν_n are defined by $\nu_n = n\mu - \lambda_n^+, \forall n \geq 1$. If we fix, for each $n \geq 1$, a Hahn-Jordan set decomposition (X_n^+, X_n^-) for X relative to λ_n , then we have

$$(16) \quad \nu_n(A) = \nu(A \cap X_n^+) + n\mu(A \cap X_n^-), \quad \forall A \in \mathcal{A}, n \geq 1.$$

Consider then the sets $X_\infty^+ = \bigcup_{n=1}^\infty X_n^+$ and $X_\infty^- = \bigcap_{n=1}^\infty X_n^-$. It is clear that $X_\infty^\pm \in \mathcal{A}$, and $X_\infty^- = X \setminus X_\infty^+$.

Fix now a set $A \in \mathcal{A}$, and let us prove the equality (iii). On the one hand, the obvious inclusions $X_n^- \supset X_\infty^-$, combined with (16), give the inequalities

$$(17) \quad \nu_n(A) \geq \nu(A \cap X_n^+) + n\mu(A \cap X_\infty^-), \quad \forall n \geq 1.$$

On the other hand, since $\lambda_{n+1} = \mu + \lambda_n, \forall n \geq 1$, using Lemma III.8.2, we get the relations

$$X_1^+ \subset \underset{\mu}{X_2^+} \subset \underset{\mu}{\dots}$$

(Recall that the notation $D \underset{\mu}{\subset} E$ stands for $\mu(D \setminus E) = 0$.) Since $\nu \ll \mu$, we also have the relations

$$A \cap X_1^+ \underset{\nu}{\subset} A \cap X_2^+ \underset{\nu}{\subset} \dots,$$

so using Proposition III.4.3, one gets the equality

$$\nu(A \cap X_\infty^+) = \lim_{n \rightarrow \infty} \nu(A \cap X_n^+).$$

Combining this with the inequalities (15) and (17) then yields the inequality

$$(18) \quad \nu(A) \geq \limsup_{n \rightarrow \infty} \nu_n(A) \geq \liminf_{n \rightarrow \infty} \nu_n(A) \geq \nu(A \cap X_\infty^+) + \lim_{n \rightarrow \infty} [n\mu(A \cap X_\infty^-)].$$

There are two possibilities here.

Case I: $\mu(A \cap X_\infty^-) > 0$.

In this case, the estimate (18) forces

$$\nu(A) = \limsup_{n \rightarrow \infty} \nu_n(A) = \liminf_{n \rightarrow \infty} \nu_n(A) = \infty.$$

Case II: $\mu(A \cap X_\infty^-) = 0$.

In this case, using absolute continuity, we get $\nu(A \cap X_\infty^-) = 0$, and the equality $A = (A \cap X_\infty^+) \cup (A \cap X_\infty^-)$ yields

$$\nu(A) = \nu(A \cap X_\infty^+).$$

Then (18) forces

$$\limsup_{n \rightarrow \infty} \nu_n(A) = \liminf_{n \rightarrow \infty} \nu_n(A) = \nu(A).$$

In either case, the conclusion is the same: $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A)$. \square

After the above preparation, we are now in position to prove the following.

THEOREM 4.2 (Radon-Nikodym Theorem: the finite case). *Let (X, \mathcal{A}, μ) be a finite measure space.*

A. If ν is an "honest" measure on \mathcal{A} , with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(19) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover, such a function is essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (19), it follows that $f_1 = f_2$, μ -a.e.

B. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} . If λ is a \mathbb{K} -valued measure on \mathcal{A} , with $\lambda \ll \mu$, then there exists a function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that

$$(20) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) *A function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfying (20) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (20), it follows that $f_1 = f_2$, μ -a.e.*
- (ii) *If $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is any function satisfying (20), then the variation measure $|\lambda|$ of λ is given by*

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}.$$

PROOF. A. Use Lemma 4.1 to find a sequence $(\nu_n)_{n=1}^\infty$ of “honest” measures on \mathcal{A} , such that

- $\nu_n \in \mu, \forall n \geq 1$; in particular the measures $\nu_n, n \geq 1$ are all finite;
- $\nu_1 \leq \nu_2 \leq \dots$;
- $\lim_{n \rightarrow \infty} \nu_n(A) = \nu(A), \forall A \in \mathcal{A}$.

For each $n \geq 1$, we apply the “Easy” Radon-Nikodym Theorem 4.1, to find some measurable function $f_n : X \rightarrow \mathbb{R}$, such that

$$\nu_n(A) = \int_A f_n d\mu, \quad \forall A \in \mathcal{A}.$$

Claim: The sequence $(f_n)_{n=1}^\infty$ satisfies

$$0 \leq f_n \leq f_{n+1}, \quad \mu\text{-a.e.}, \quad \forall n \geq 1.$$

Fix $n \geq 1$. On the one hand, since the ν_n 's are “honest” finite measures, and $\nu_n \in \mu$, by part (i) of Theorem 4.1, it follows that $f_n \geq 0, \mu\text{-a.e.}$ On other hand, since $\nu_{n+1} - \nu_n$ is also an “honest” finite measure with $\nu_{n+1} - \nu_n \in \mu$, and with density $f_{n+1} - f_n$, again by part (i) of Theorem 4.1, it follows that $f_{n+1} - f_n \geq 0, \mu\text{-a.e.}$

Having proven the above Claim, let us define the function $f : X \rightarrow [0, \infty]$, by

$$f(x) = \liminf_{n \rightarrow \infty} [\max\{f_n(x), 0\}] \quad \forall x \in X.$$

It is obvious that f is measurable. By the Claim, we have in fact the equality

$$f = \mu\text{-a.e.-} \lim_{n \rightarrow \infty} f_n.$$

Since we also have

$$f \chi_A = \mu\text{-a.e.-} \lim_{n \rightarrow \infty} f_n \chi_A, \quad \forall A \in \mathcal{A},$$

using the Claim and the Monotone Convergence Theorem, we get

$$\begin{aligned} \int_A f d\mu &= \int_X f \chi_A d\mu = \lim_{n \rightarrow \infty} \int_X f_n \chi_A d\mu = \lim_{n \rightarrow \infty} \int_A f_n d\mu = \\ &= \lim_{n \rightarrow \infty} \nu_n(A) = \nu(A), \quad \forall A \in \mathcal{A}. \end{aligned}$$

Having shown that f satisfies (19), let us observe that the uniqueness property stated in part A is a consequence of Proposition 4.2.

B. Let λ be a \mathbb{K} -valued. In particular, the variation measure $|\lambda|$ is finite, so by the Polar Decomposition (Proposition 4.3) there exists some measurable function $h : X \rightarrow \mathbb{K}$, such that

$$(21) \quad \lambda(A) = \int_A h d|\lambda|, \quad \forall A \in \mathcal{A},$$

and such that $|h| = 1, |\lambda|\text{-a.e.}$ Replacing h with the measurable function $h' : X \rightarrow \mathbb{K}$, defined by

$$h'(x) = \begin{cases} h(x) & \text{if } |h(x)| = 1 \\ 1 & \text{if } |h(x)| \neq 1 \end{cases}$$

we can assume that in fact we have

$$|h(x)| = 1, \quad \forall x \in X.$$

Apply then part A, to the measure $|\lambda|$, which is again absolutely continuous with respect to μ , to find some measurable function $g : X \rightarrow [0, \infty]$, such that

$$|\lambda|(A) = \int_A g d\mu, \quad \forall A \in \mathcal{A}.$$

Remark that, since

$$\int_X g d\mu = |\lambda|(X) < \infty,$$

it follows that $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Fix for the moment some set $A \in \mathcal{A}$. On the one hand, since

$$(22) \quad |h\chi_A| \leq 1,$$

and $|\lambda|$ is finite, it follows that $h\chi_A \in L_{\mathbb{K}}^1(X, \mathcal{A}, |\lambda|)$. On the other hand, since $g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, using (22) we get the fact that $h\chi_A g \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Using the Change of Variable formula (Proposition 4.1) we then get the equality

$$\int_X h\chi_A d|\lambda| = \int_X h\chi_A g d\mu,$$

which by (21) reads:

$$\lambda(A) = \int_A hg d\mu.$$

Now the function $f_0 = hg$ (which has $|f_0| = g$) belongs to $L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and clearly satisfies (20).

To prove the uniqueness property (i), we start with two functions $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ which satisfy

$$\int_A f_1 d\mu = \int_A f_2 d\mu = \lambda(A), \quad \forall A \in \mathcal{A}.$$

If we define the function $\varphi = f_1 - f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, then we clearly have

$$\int_A \varphi d\mu = \int_A 0 d\mu = \omega(A), \quad \forall A \in \mathcal{A},$$

where ω is the zero measure. Since $\omega \leq \mu$, using Theorem 4.1 it follows that $\varphi = 0$, μ -a.e.

To prove (ii) we start with some $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ that satisfies (20), and we use the uniqueness property (i) to get the equality $f = f_0$, μ -a.e., where f_0 is the function constructed above. In particular, using the construction of f_0 , the fact that $|f_0| = g$, and the fact that g is a density for $|\lambda|$ relative to μ , we get

$$\int_A |f| d\mu = \int_A |f_0| d\mu = \int_A g d\mu = |\lambda|(A), \quad \forall A \in \mathcal{A}. \quad \square$$

At this point we would like to go further, beyond the finite case. The following generalization of Theorem 4.2 is pretty straightforward.

COROLLARY 4.1 (Radon-Nikodym Theorem: the σ -finite case). *Let (X, \mathcal{A}, μ) be a σ -finite measure space.*

A. If ν is an "honest" measure on \mathcal{A} , with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(23) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover, such a function is essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (19), it follows that $f_1 = f_2$, μ -a.e.

B. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} . If λ is a \mathbb{K} -valued measure on \mathcal{A} , with $\lambda \ll \mu$, then there exists a function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, such that

$$(24) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover:

- (i) A function $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfying (20) is essentially unique, in the sense that, whenever $f_1, f_2 \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ satisfy (24), it follows that $f_1 = f_2$, μ -a.e.
- (ii) If $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$ is any function satisfying (24), then the variation measure $|\lambda|$ of λ is given by

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}.$$

PROOF. Since μ is σ -finite, there exists a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}_{\text{fin}}^{\mu}$, with $\bigcup_{n=1}^{\infty} A_n = X$. Put $X_1 = A_1$ and $X_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1})$, $\forall n \geq 2$. Then $(X_n)_{n=1}^{\infty} \subset \mathcal{A}_{\text{fin}}^{\mu}$ is pairwise disjoint, and we still have $\bigcup_{n=1}^{\infty} X_n = X$. The Corollary follows then immediately from Theorem 4.2, applied to the measure spaces $(X_n, \mathcal{A}|_{X_n}, \mu|_{X_n})$ and the measures $\nu|_{X_n}$ and $\lambda|_{X_n}$ respectively. What is used here is the fact that, if \mathbf{K} denotes one of the sets $[0, \infty]$, \mathbb{R} or \mathbb{C} , then for a function $f : X \rightarrow \mathbf{K}$ the fact that f is measurable, is equivalent to the fact that $f|_{X_n}$ is measurable for each $n \geq 1$. Moreover, given two functions $f_1, f_2 : X \rightarrow \mathbf{K}$, the condition $f_1 = f_2$, μ -a.e. is equivalent to the fact that $f_1|_{X_n} = f_2|_{X_n}$, μ -a.e., $\forall n \geq 1$. Finally, for $f : X \rightarrow \mathbb{K}$ ($= \mathbb{R}, \mathbb{C}$), the condition $f \in \mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$, is equivalent to the fact that $f|_{X_n} \in \mathcal{L}^1_{\mathbb{K}}(X_n, \mathcal{A}|_{X_n}, \mu|_{X_n})$, $\forall n \geq 1$, and

$$\sum_{n=1}^{\infty} \int_{X_n} (f|_{X_n}) d(\mu|_{X_n}) < \infty. \quad \square$$

COMMENT. The σ -finite case of the Radon-Nikodym Theorem, given above, is in fact a particular case of a more general version (Theorem 4.3 below). In order to formulate this, we need a concept which has already appeared earlier in III.5. Recall that a measure space (X, \mathcal{A}, μ) is said to be *decomposable*, if there exists a pairwise disjoint subcollection $\mathcal{F} \subset \mathcal{A}_{\text{fin}}^{\mu}$, such that

- (i) $\bigcup_{F \in \mathcal{F}} F = X$;
- (ii) for a set $A \subset X$, the condition $A \in \mathcal{A}$ is equivalent to the condition

$$A \cap F \in \mathcal{A}, \quad \forall F \in \mathcal{F};$$

- (iii) one has the equality

$$\mu(A) = \sum_{F \in \mathcal{F}} \mu(A \cap F), \quad \forall A \in \mathcal{A}_{\text{fin}}^{\mu}.$$

Such a collection \mathcal{F} is then called a *decomposition* of (X, \mathcal{A}, μ) . Condition (ii) is referred to as the *patching property*, because it characterizes measurability as follows.

- (P) Given a measurable space (Y, \mathcal{B}) , a function $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is measurable, if and only if all restrictions $f|_F : (F, \mathcal{A}|_F) \rightarrow (Y, \mathcal{B})$, $F \in \mathcal{F}$, are measurable.

THEOREM 4.3 (Radon-Nikodym Theorem: the decomposable case). *Let (X, \mathcal{A}, μ) be a decomposable measure space. Let $\mathcal{A}_{\sigma\text{-fin}}^\mu$ be the collection of all μ - σ -finite sets in \mathcal{A} , that is,*

$$\mathcal{A}_{\sigma\text{-fin}}^\mu = \left\{ A \in \mathcal{A} : \text{there exists } (A_n)_{n=1}^\infty \subset \mathcal{A}_{\text{fin}}^\mu, \text{ with } A = \bigcup_{n=1}^\infty A_n \right\}.$$

A. If ν is an “honest” measure on \mathcal{A} , with $\nu \ll \mu$, then there exists a measurable function $f : X \rightarrow [0, \infty]$, such that

$$(25) \quad \nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Moreover, such a function is locally essentially unique, in the sense that, whenever $f_1, f_2 : X \rightarrow [0, \infty]$ are measurable functions, that satisfy (25), it follows that $f_1 = f_2$, μ -l.a.e.

B. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} . If λ is a \mathbb{K} -valued measure on \mathcal{A} , with $\lambda \ll \mu$, then there exists a function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, such that

$$(26) \quad \lambda(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Moreover:

- (i) A function $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfying (26) is essentially unique, in the sense that, whenever $f_1, f_2 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ satisfy (26), it follows that $f_1 = f_2$, μ -a.e.
- (ii) If $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is any function satisfying (26), then the variation measure $|\lambda|$, of λ , satisfies

$$|\lambda|(A) = \int_A |f| d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

PROOF. Fix \mathcal{F} to be a decomposition for (X, \mathcal{A}, μ) .

A. For every $F \in \mathcal{F}$, we apply Theorem 4.2 to the measure space $(F, \mathcal{A}|_F, \mu|_F)$ and the measure $\nu|_F$, to find some measurable function $f_F : F \rightarrow [0, \infty]$, such that

$$\nu(A) = \int_A f_F d\mu, \quad \forall A \in \mathcal{A}|_F.$$

Using the patching property, there exists a measurable function $f : X \rightarrow [0, \infty]$, such that $f|_F = f_F$, $\forall F \in \mathcal{F}$. The key feature we are going to prove is a particular case of (25).

Claim 1: $\nu(A) = \int_A f d\mu$, $\forall A \in \mathcal{A}_{\text{fin}}^\mu$.

Fix $A \in \mathcal{A}_{\text{fin}}^\mu$. On the one hand, we know that

$$\mu(A) = \sum_{F \in \mathcal{F}} \mu(A \cap F).$$

Since the sum is finite, it follows that the subcollection

$$\mathcal{F}(A) = \{F \in \mathcal{F} : \mu(A \cap F) > 0\}$$

is at most countable. We then form the set $\tilde{A} = \bigcup_{F \in \mathcal{F}(A)} [A \cap F]$, which is clearly a subset of A . The difference $D = A \setminus \tilde{A}$ has again $\mu(D) < \infty$, so its measure is also given as

$$\mu(D) = \sum_{F \in \mathcal{F}} \mu(D \cap F).$$

Notice however that we have $\mu(D \cap F) = 0, \forall F \in \mathcal{F}$. (If $F \in \mathcal{F}(A)$, we already have $D \cap F = \emptyset$, whereas if $F \in \mathcal{F} \setminus \mathcal{F}(A)$, we have $D \cap F \subset A \cap F$, with $\mu(A \cap F) = 0$.) Using then the above equality, we get $\mu(D) = 0$. By absolute continuity we also get $\nu(D) = 0$. Using the equality $A = \tilde{A} \cup D$, and σ -additivity (it is essential here that $\mathcal{F}(A)$ is countable), it follows that

$$\nu(A) = \nu(\tilde{A}) = \sum_{F \in \mathcal{F}(A)} \nu(A \cap F).$$

Using the hypothesis, we then get

$$(27) \quad \nu(A) = \sum_{F \in \mathcal{F}(A)} \int_{A \cap F} f \, d\mu.$$

Now if we list $\mathcal{F}(A) = \{F_k\}_{k=1}^{\infty}$, and if we take a partial sum, we have

$$\sum_{k=1}^n \int_{A \cap F_k} f \, d\mu = \int_{G_n} f \, d\mu = \int_X f \chi_{G_n} \, d\mu,$$

where

$$G_n = \bigcup_{k=1}^n [A \cap F_k], \quad \forall n \geq 1.$$

It is clear that we have

- $f \chi_{G_1} \leq f \chi_{G_2} \leq \dots$,
- $\lim_{n \rightarrow \infty} (f \chi_{G_n})(x) = (f \chi_{\tilde{A}})(x), \forall x \in X$,

so using the Monotone Convergence Theorem, it follows that

$$\lim_{n \rightarrow \infty} \int_X f \chi_{G_n} \, d\mu = \int_X f \chi_{\tilde{A}} \, d\mu = \int_{\tilde{A}} f \, d\mu.$$

Using (27) we then get

$$\nu(A) = \lim_{n \rightarrow \infty} \int_X f \chi_{G_n} \, d\mu = \int_{\tilde{A}} f \, d\mu.$$

On the other hand, since $\mu(A \setminus \tilde{A}) = 0$, it follows that

$$\int_A f \, d\mu = \int_{\tilde{A}} f \, d\mu,$$

so the preceding equality immediately gives the desired equality

$$\nu(A) = \int_A f \, d\mu.$$

At this point let us remark that the local almost uniqueness of f already follows from Remark 4.1.

Let us prove now the equality (25). Start with some set $A \in \mathcal{A}_{\sigma\text{-fin}}^{\mu}$, and choose a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}_{\text{fin}}^{\mu}$, such that $A = \bigcup_{n=1}^{\infty} A_n$. Define the sequence $(B_n)_{n=1}^{\infty}$ by

$$B_n = A_1 \cup \dots \cup A_n, \quad \forall n \geq 1,$$

so that we still have $B_n \in \mathcal{A}_{\text{fin}}^\mu, \forall n \geq 1$, as well as $A = \bigcup_{n=1}^\infty B_n$, but moreover we have $B_1 \subset B_2 \subset \dots$. For each $n \geq 1$, using Claim 1, we have the equality

$$\nu(B_n) = \int_{B_n} f d\mu.$$

Using these equalities, combined with

- $0 \leq f \chi_{B_1} \leq f \chi_{B_2} \leq \dots$,
- $\lim_{n \rightarrow \infty} (f \chi_{B_n})(x) = (f \chi_B)(x), \forall x \in X$,

the Monotone Convergence Theorem, combined with continuity yields

$$\int_B f d\mu = \int_X f \chi_B d\mu = \lim_{n \rightarrow \infty} \int_X f \chi_{B_n} d\mu = \lim_{n \rightarrow \infty} \int_{B_n} f d\mu = \lim_{n \rightarrow \infty} \nu(B_n) = \nu(A).$$

B. We start off by choosing a measurable function $h : X \rightarrow \mathbb{K}$, with $|h| = 1$, such that

$$\lambda(A) = \int_A h d|\lambda|, \quad \forall A \in \mathcal{A}.$$

Using part A, there exists some measurable function $g_0 : X \rightarrow [0, \infty]$, such that

$$(28) \quad |\lambda|(A) = \int_A g_0 d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

At this point, g_0 may not be integrable, but we have the freedom to perturb it (μ -l.a.e.) to try to make it integrable. This is done as follows. Consider the collection

$$\mathcal{F}_0 = \{F \in \mathcal{F} : |\lambda|(F) > 0\}.$$

Since $|\lambda|$ is *finite*, it follows that \mathcal{F}_0 is at most countable. Define then the set $X_0 = \bigcup_{F \in \mathcal{F}_0} F \in \mathcal{A}_{\sigma\text{-fin}}^\mu$. Since X_0 is μ - σ -finite, every set $A \in \mathcal{A}$ with $A \subset X_0$, is μ - σ -finite, so we have

$$|\lambda|(A) = \int_A g_0 d\mu, \quad \forall A \in \mathcal{A}|_{X_0}.$$

Applying the σ -finite version of the Radon-Nikodym Theorem to the σ -finite measure space $(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$ and the finite measure $\lambda|_{X_0}$, it follows that the density $g_0|_{X_0}$ belongs to $\mathfrak{L}_+^1(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$, which means that the function $g = g_0 \chi_{X_0}$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. With this choice of g , let us prove now that the equality (28) still holds, with g in place of g_0 . Exactly as in the proof of part A, it suffices to prove only the equality

$$(29) \quad |\lambda|(A) = \int_A g d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Claim 2: $|\lambda|(A) = |\lambda|(A \cap X_0), \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$.

Since (use the fact that $|\lambda|$ is finite) the equality is equivalent to

$$|\lambda|(A \setminus X_0) = 0, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu,$$

it suffices to prove it only for $A \in \mathcal{A}_{\text{fin}}^\mu$. If $A \in \mathcal{A}_{\text{fin}}^\mu$, using the properties of the decomposition \mathcal{F} , we have

$$\begin{aligned} |\lambda|(A) &= \sum_{F \in \mathcal{F}} |\lambda|(A \cap F) = \sum_{F \in \mathcal{F}_0} |\lambda|(A \cap F) + \sum_{F \in \mathcal{F} \setminus \mathcal{F}_0} |\lambda|(A \cap F) = \\ &= |\lambda|\left(\bigcup_{F \in \mathcal{F}_0} [A \cap F]\right) + \sum_{F \in \mathcal{F} \setminus \mathcal{F}_0} |\lambda|(A \cap F) = \\ &= |\lambda|(A \cap X_0) + \sum_{F \in \mathcal{F}'(A)} |\lambda|(A \cap F). \end{aligned}$$

Notice now that, for $F \in \mathcal{F} \setminus \mathcal{F}_0$, we have $|\lambda|(F) = 0$, which gives $|\lambda|(A \cap F) = 0$, so the Claim follows immediately from the above computation.

Having proven the above Claim, let us prove now (29). Fix $A \in \mathcal{A}_{\text{fin}}^\mu$. The desired equality is now immediate from Claim 2, combined with (28):

$$\begin{aligned} |\lambda|(A) &= |\lambda|(A \cap X_0) = \int_{A \cap X_0} g_0 \, d\mu = \int_X g_0 \chi_{A \cap X_0} \, d\mu = \\ &= \int_X g_0 \chi_{X_0} \chi_A \, d\mu = \int_X g \chi_A \, d\mu = \int_A g \, \mu. \end{aligned}$$

Define now the function $f_0 = hg$. Since $|f_0| = g \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, it follows that $f_0 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Let us prove that f_0 satisfies the equality (26). Start with some $A \in \mathcal{A}_{\sigma\text{-fin}}^\mu$. On the one hand, using Claim 2, we have

$$|\lambda|(A \setminus X_0) \leq |\lambda|(A \setminus X_0) = 0,$$

so we get $\lambda(A) = \lambda(A \cap X_0)$. Using the σ -finite version of the Radon-Nikodym Theorem for $(X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})$ and $\lambda|_{X_0}$, we then have

$$\begin{aligned} \lambda(A) &= \lambda(A \cap X_0) = \int_{A \cap X_0} hg_0 \, d\mu = \int_X hg_0 \chi_{A \cap X_0} \, d\mu = \\ &= \int_X hg_0 \chi_{X_0} \chi_A \, d\mu = \int_X hg \chi_A \, d\mu = \int_A hg \, d\mu = \int_A f_0 \, d\mu. \end{aligned}$$

We now prove the uniqueness property (i) of f (μ -a.e.). Assume $f \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ is another function, such that

$$\lambda(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^\mu.$$

Claim 3: $f = f_0$, μ -l.a.e.

What we need to show here is the fact that

$$f \chi_B = f_0 \chi_B, \quad \mu\text{-a.e.}, \quad \forall B \in \mathcal{A}_{\text{fin}}^\mu.$$

But this follows immediately from the uniqueness from part B of Theorem 4.2, applied to the finite measure space $(B, \mathcal{A}|_B, \mu|_B)$ and the measure $\lambda|_B$, which has both $f|_B$ and $f_0|_B$ as densities.

Using Claim 3, we now have $f - f_0 \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, with $f - f_0 = 0$, μ -l.a.e., so we can apply Proposition 4.4, which forces $f - f_0 = 0$, μ -a.e., so we indeed get $f = f_0$, μ -a.e.

Property (ii) is obvious, since by (i), any function $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, that satisfies (26), automatically satisfies $|f| = |f_0| = g$, μ -a.e. \square

COMMENT. One should be aware of the (severe) limitations of Theorem 4.3, notably the fact that the equalities (25) and (26) hold *only for* $A \in \mathcal{A}_{\sigma\text{-fin}}^{\mu}$. For example, if one considers the measure space $(X, \mathcal{P}(X), \mu)$, with X uncountable, and μ defined by

$$\mu(A) = \begin{cases} \infty & \text{if } A \text{ is uncountable} \\ 0 & \text{if } A \text{ is countable} \end{cases}$$

This measure space is decomposable, with a decomposition consisting of singletons: $\mathcal{F} = \{\{x\} : x \in X\}$. For a measure ν on $\mathcal{P}(X)$, the condition $\nu \ll \mu$ means precisely that $\nu(A) = 0$ for all countable subsets $A \subset X$. In this case the equality (25) says practically nothing, since it is restricted solely to countable sets $A \subset X$, when both sides are zero.

In this example, it is also instructive to analyze the case when ν is finite (see part B in Theorem 4.3). If we follow the proof of the Theorem, we see that at some point we have constructed a certain set $X_0 = \bigcup_{F \in \mathcal{F}_0} F$, where $\mathcal{F}_0 = \{F \in \mathcal{F} : \nu(F) > 0\}$. In our situation however it turns out that $X_0 = \emptyset$. This example brings up a very interesting question, which turns out to sit at the very foundation of set theory.

Question: Does there exist an uncountable set X , and a finite measure ν on $\mathcal{P}(X)$, such that $\nu(X) > 0$, but $\nu(A) = 0$, for every countable subset $A \subset X$?

(The above vanishing condition is of course equivalent to the fact that $\nu(\{x\}) = 0$, $\forall x \in X$.) It turns out that, not only that the answer of this question is unknown, but in fact several mathematicians are seriously thinking of proposing it as an axiom to be added to the current system of axioms used in set theory!

The limitations of Theorem 4.3 also force limitations in the Change of Variables property (see Proposition 4.1), which in this case has the following statement.

PROPOSITION 4.6 (Local Change of Variables). *Let (X, \mathcal{A}, μ) be a measure space, and let ν be a measure on \mathcal{A} , and let $f : X \rightarrow [0, \infty]$ be a measurable function.*

A. *The following are equivalent:*

(i) *one has*

$$\nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}_{\sigma\text{-fin}}^{\mu};$$

(ii) *for every measurable function $h : X \rightarrow [0, \infty]$, with the property that the set $E_h = \{x \in X : h(x) \neq 0\}$ belongs to $\mathcal{A}_{\sigma\text{-fin}}^{\mu}$, one has the equality*

$$(30) \quad \int_X h d\nu = \int_X hf d\mu.$$

B. *If ν and f are as above, and \mathbb{K} is either \mathbb{R} or \mathbb{C} , then the equality (30) also holds for those measurable functions $h : X \rightarrow \mathbb{K}$ with $E_h \in \mathcal{A}_{\sigma\text{-fin}}^{\mu}$, for which $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$.*

PROOF. A. (i) \Rightarrow (ii). Assume (i) holds. Start with some measurable function $h : X \rightarrow [0, \infty]$, such that the set $E_h = \{x \in X : h(x) \neq 0\}$ belongs to $\mathcal{A}_{\sigma\text{-fin}}^{\mu}$. The equality (30) is then immediate from Proposition 4.1, applied to the measure space $(E_h, \mathcal{A}|_{E_h}, \mu|_{E_h})$, and the measure $\nu|_{E_h}$, which has density $f|_{E_h}$.

(ii) \Rightarrow (i). Assume (ii) holds. If we start with some $A \in \mathcal{A}_{\sigma\text{-fin}}^{\mu}$, then obviously the measurable function $h = \chi_A$ will have $E_h = A$, so by (ii) we immediately get

$$\nu(A) = \int_X \chi_A d\nu = \int_X \chi_A f d\mu = \int_A f d\mu.$$

B. Assume now ν and f satisfy the equivalent conditions (i) and (ii). Suppose $h : X \rightarrow \mathbb{K}$ is measurable, with $E_h \in \mathcal{A}_{\sigma\text{-fin}}^{\mu}$, such that $h \in L_{\mathbb{K}}^1(X, \mathcal{A}, \nu)$ and $hf \in L_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. Then the equality (30) follows again from Proposition 4.1, applied to the measure space $(E_h, \mathcal{A}|_{E_h}, \mu|_{E_h})$, and the measure $\nu|_{E_h}$, which has density $f|_{E_h}$. \square