

Chapter IV
Integration Theory

LECTURES 32-33

1. Construction of the integral

In this section we construct the abstract integral. As a matter of terminology, we define a *measure space* as being a triple (X, \mathcal{A}, μ) , where X is some (non-empty) set, \mathcal{A} is a σ -algebra on X , and μ is a measure on \mathcal{A} . The measure space (X, \mathcal{A}, μ) is said to be *finite*, if $\mu(X) < \infty$.

DEFINITION. Let (X, \mathcal{A}, μ) be a measure space, and let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} . A \mathbb{K} -valued elementary μ -integrable function on (X, \mathcal{A}, μ) is an function $f : X \rightarrow \mathbb{K}$, with the following properties

- the range $f(X)$ of f is a finite set;
- $f^{-1}(\{\alpha\}) \in \mathcal{A}$, and $\mu(f^{-1}(\{\alpha\})) < \infty$, for all $\alpha \in f(X) \setminus \{0\}$.

We denote by $\mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu)$ the collection of all such functions.

REMARKS 1.1. Let (X, \mathcal{A}, μ) be a measure space.

A. Every \mathbb{K} -valued elementary μ -integrable function f on (X, \mathcal{A}, μ) is measurable, as a map $f : (X, \mathcal{A}) \rightarrow (\mathbb{K}, \text{Bor}(\mathbb{K}))$. In fact, any such f can be written as

$$f = \alpha_1 \chi_{A_1} + \cdots + \alpha_n \chi_{A_n},$$

with $\alpha_k \in \mathbb{K}$, $A_k \in \mathcal{A}$ and $\mu(A_k) < \infty$, $\forall k = 1, \dots, n$. Using the notations from III.1, we have the inclusion

$$\mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu) \subset \mathcal{A}\text{-Elem}_{\mathbb{K}}(X).$$

B. If we consider the collection $\mathfrak{R} = \{A \in \mathcal{A} : \mu(A) < \infty\}$, then \mathfrak{R} is a ring, and, we have the equality

$$\mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu) = \mathfrak{R}\text{-Elem}_{\mathbb{K}}(X).$$

In particular, it follows that $\mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu)$ is a \mathbb{K} -vector space.

The following result is the first step in the construction of the integral.

THEOREM 1.1. Let (X, \mathcal{A}, μ) be a measure space, and let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} . Then there exists a unique \mathbb{K} -linear map $I_{elem}^\mu : \mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, such that

$$(1) \quad I_{elem}^\mu(\chi_A) = \mu(A),$$

for all $A \in \mathcal{A}$, with $\mu(A) < \infty$.

PROOF. For every $f \in \mathfrak{L}_{\mathbb{K},elem}^1(X, \mathcal{A}, \mu)$, we define

$$I_{elem}^\mu(f) = \sum_{\alpha \in f(X) \setminus \{0\}} \alpha \cdot \mu(f^{-1}(\{\alpha\})),$$

with the convention that, when $f(X) = \{0\}$ (which is the same as $f = 0$), we define $I_{elem}^\mu(f) = 0$. It is obvious that I_{elem}^μ satisfies the equality (1) for all $A \in \mathcal{A}$ with $\mu(A) < \infty$.

One key feature we are going to use is the following.

Claim 1: Whenever we have a finite pairwise disjoint sequence $(A_k)_{k=1}^n \subset \mathcal{A}$, with $\mu(A_k) < \infty$, $\forall k = 1, \dots, n$, one has the equality

$$I_{elem}^\mu(\alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}) = \alpha_1 \mu(A_1) + \dots + \alpha_n \mu(A_n), \quad \forall \alpha_1, \dots, \alpha_n \in \mathbb{K}.$$

It is obvious that we can assume $\alpha_j \neq 0$, $\forall j = 1, \dots, n$. To prove the above equality, we consider the elementary μ -integrable function $f = \alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}$, and we observe that $f(X) \setminus \{0\} = \{\alpha_1\} \cup \dots \cup \{\alpha_n\}$. It may be the case that some of the α 's are equal. We list $f(X) \setminus \{0\} = \{\beta_1, \dots, \beta_p\}$, with $\beta_j \neq \beta_k$, for all $j, k \in \{1, \dots, p\}$ with $j \neq k$. For each $k \in \{1, \dots, p\}$, we define the set

$$J_k = \{j \in \{1, \dots, n\} : \alpha_j = \beta_k\}.$$

It is obvious that the sets $(J_k)_{k=1}^p$ are pairwise disjoint, and we have $J_1 \cup \dots \cup J_p = \{1, \dots, n\}$. Moreover, for each $k \in \{1, \dots, p\}$, one has the equality

$$f^{-1}(\{\beta_k\}) = \bigcup_{j \in J_k} A_j,$$

so we get

$$\beta_k \mu(f^{-1}(\{\beta_k\})) = \beta_k \sum_{j \in J_k} \mu(A_j) = \sum_{j \in J_k} \alpha_j \mu(A_j), \quad \forall k \in \{1, \dots, p\}.$$

By the definition of I_{elem}^μ we then get

$$I_{elem}^\mu(f) = \sum_{k=1}^p \beta_k \mu(f^{-1}(\{\beta_k\})) = \sum_{k=1}^p \left[\sum_{j \in J_k} \alpha_j \mu(A_j) \right] = \sum_{j=1}^n \alpha_j \mu(A_j).$$

Claim 2: For every $f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$, and every $A \in \mathcal{A}$ with $\mu(A) < \infty$, one has the equality

$$(2) \quad I_{elem}^\mu(f + \alpha \chi_A) = I_{elem}^\mu(f) + \alpha \mu(A), \quad \forall \alpha \in \mathbb{K}.$$

Write $f = \alpha_1 \chi_{A_1} + \dots + \alpha_n \chi_{A_n}$, with $(A_j)_{j=1}^n \subset \mathcal{A}$ pairwise disjoint, and $\mu(A_j) < \infty$, $\forall j = 1, \dots, n$. In order to prove (2), we are going to write the function $f + \alpha \chi_A$ in a similar way, and we are going to apply Claim 1. Consider the sets $B_1, B_2, \dots, B_{2n}, B_{2n+1} \in \mathcal{A}$ defined by $B_{2n+1} = A \setminus (A_1 \cup \dots \cup A_n)$, and $B_{2k-1} = A_k \cap A$, $B_{2k} = A_k \setminus A$, $\forall k = 1, \dots, n$. It is obvious that the sets $(B_p)_{p=1}^{2n+1}$ are pairwise disjoint. Moreover, one has the equalities

$$(3) \quad B_{2k-1} \cup B_{2k} = A_k, \quad \forall k \in \{1, \dots, n\},$$

as well as the equality

$$(4) \quad A = \bigcup_{k=1}^{n+1} B_{2k-1}.$$

Using these equalities, now we have $f + \alpha \chi_A = \sum_{p=1}^{2n+1} \beta_p \chi_{B_p}$, where $\beta_{2n+1} = \alpha$, and $\beta_{2k} = \alpha_k$ and $\beta_{2k-1} = \alpha_k + \alpha$, $\forall k \in \{1, \dots, n\}$. Using these equalities,

combined with Claim 1, and (3) and (4), we now get

$$\begin{aligned}
I_{elem}^\mu(f + \alpha \chi_A) &= \sum_{p=1}^{2n+1} \beta_p \mu(B_p) = \\
&= \alpha \mu(B_{2n+1}) + \sum_{k=1}^n [(\alpha_k + \alpha) \mu(B_{2k-1}) + \alpha_k \mu(B_{2k})] = \\
&= \left[\alpha \sum_{k=1}^{n+1} \mu(B_{2k-1}) \right] + \left[\sum_{k=1}^n \alpha_k [\mu(B_{2k-1}) + \mu(B_{2k})] \right] = \\
&= \alpha \mu\left(\bigcup_{k=1}^{n+1} B_{2k-1}\right) + \sum_{k=1}^n \alpha_k \mu(B_{2k-1} \cup B_{2k}) = \\
&= \alpha \mu(A) + \sum_{k=1}^n \alpha_k \mu(A_k) = \alpha \mu(A) + I_{elem}^\mu(f),
\end{aligned}$$

and the Claim is proven.

We now prove that I_{elem}^μ is linear. The equality

$$I_{elem}^\mu(f + g) = I_{elem}^\mu(f) + I_{elem}^\mu(g), \quad \forall f, g \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$$

follows from Claim 2, using an obvious inductive argument. The equality

$$I_{elem}^\mu(\alpha f) = \alpha I_{elem}^\mu(f), \quad \forall \alpha \in \mathbb{K}, f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu).$$

is also pretty obvious, from the definition.

The uniqueness is also clear. \square

DEFINITION. With the notations above, the linear map

$$I_{elem}^\mu : \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$$

is called the *elementary μ -integral*.

In what follows we are going to encounter also situations when certain relations among measurable functions hold “almost everywhere.” We are going to use the following.

CONVENTION. Let T be one of the spaces $[-\infty, \infty]$ or \mathbb{C} , and let R be some relation on T (in our case R will be either “=,” or “ \geq ,” or “ \leq ,” on $[-\infty, \infty]$). Given a measurable space (X, \mathcal{A}, μ) , and two measurable functions $f_1, f_2 : X \rightarrow T$,

$$f_1 R f_2, \quad \mu\text{-a.e.}$$

if the set

$$A = \{x \in X : f_1(x) R f_2(x)\}$$

belongs to \mathcal{A} , and it has μ -null complement in X , i.e. $\mu(X \setminus A) = 0$. (If R is one of the relations listed above, the set A automatically belongs to \mathcal{A} .) The abbreviation “ μ -a.e.” stands for “ μ -almost everywhere.”

REMARK 1.2. Let (X, \mathcal{A}, μ) be a measure space, let $f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$ be such that

$$f = 0, \quad \mu\text{-a.e.}$$

Then $f \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^\mu(f) = 0$. Indeed, if we define the set

$$N = \{x \in X : f(x) \neq 0\},$$

then $N \in \mathcal{A}$ and $\mu(N) = 0$. Since $f^{-1}(\{\alpha\}) \subset N$, $\forall \alpha \in f(X) \setminus \{0\}$, it follows that $\mu(f^{-1}(\{\alpha\})) = 0$, $\forall \alpha \in f(X) \setminus \{0\}$, and then by the definition of the elementary μ -integral, we get $I_{elem}^\mu(f) = 0$.

One useful property of elementary integrable functions is the following.

PROPOSITION 1.1. *Let (X, \mathcal{A}, μ) be a measure space, let $f, g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and let $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ be such that*

$$f \leq h \leq g, \quad \mu\text{-a.e.}$$

Then $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and

$$(5) \quad I_{elem}^\mu(f) \leq I_{elem}^\mu(h) \leq I_{elem}^\mu(g).$$

PROOF. Consider the sets

$$A = \{x \in X : f(x) > h(x)\} \text{ and } B = \{x \in X : h(x) > g(x)\},$$

which both belong to \mathcal{A} , and have $\mu(A) = \mu(B) = 0$. The set $M = A \cup B$ also belongs to \mathcal{A} and has $\mu(M) = 0$. Define the functions $f_0 = f(1 - \chi_M)$, $g_0 = g(1 - \chi_M)$, and $h_0 = h(1 - \chi_M)$. It is clear that f_0 , g_0 , and h_0 are all in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$. Moreover, we have the equalities $f_0 = f$, μ -a.e., $g_0 = g$, μ -a.e., and $h_0 = h$, μ -a.e., so by Remark ??, combined with Theorem 1.1, the functions $f_0 = f + (f_0 - f)$ and $g_0 = (g_0 - g) + g$ both belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and we have the equalities

$$(6) \quad I_{elem}^\mu(f_0) = I_{elem}^\mu(f) \text{ and } I_{elem}^\mu(g_0) = I_{elem}^\mu(g).$$

Notice now that we have the (absolute) inequality

$$f_0 \leq h_0 \leq g_0.$$

Let us show that h_0 is elementary integrable. Start with some $\alpha \in h_0(X) \setminus \{0\}$. If $\alpha > 0$, then, using the inequality $h_0 \leq g_0$, we get

$$h_0^{-1}(\{\alpha\}) \subset g_0^{-1}((0, \infty)) \subset \bigcup_{\lambda \in g_0(X) \setminus \{0\}} g_0^{-1}(\{\lambda\}),$$

which proves that $\mu(h_0^{-1}(\{\alpha\})) < \infty$. Likewise, if $\alpha < 0$, then, using the inequality $h_0 \geq f_0$, we get

$$h_0^{-1}(\{\alpha\}) \subset f_0^{-1}((-\infty, 0)) \subset \bigcup_{\lambda \in f_0(X) \setminus \{0\}} f_0^{-1}(\{\lambda\}),$$

which proves again that $\mu(h_0^{-1}(\{\alpha\})) < \infty$.

Having shown that h_0 is elementary integrable, we now compare the numbers $I_{elem}^\mu(f)$, $I_{elem}^\mu(h_0)$, and $I_{elem}^\mu(g)$. Define the functions $f_1 = h_0 - f_0$, and $g_1 = g_0 - h_0$. By Theorem 1.1, we know that $f_1, g_1 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$. Since $f_1, g_1 \geq 0$, we have $f_1(X), g_1(X) \subset [0, \infty)$, so it follows immediately that $I_{elem}^\mu(f_1) \geq 0$ and $I_{elem}^\mu(g_1) \geq 0$. Now, again using Theorem 1.1, and (6), we get

$$\begin{aligned} I_{elem}^\mu(h_0) &= I_{elem}^\mu(f_0 + f_1) = I_{elem}^\mu(f_0) + I_{elem}^\mu(f_1) \geq I_{elem}^\mu(f_0) = I_{elem}^\mu(f); \\ I_{elem}^\mu(h_0) &= I_{elem}^\mu(g_0 - g_1) = I_{elem}^\mu(g_0) - I_{elem}^\mu(g_1) \leq I_{elem}^\mu(g_0) = I_{elem}^\mu(g). \end{aligned}$$

Since $h = h_0$, μ -a.e., by the above Remark it follows that $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^\mu(h) = I_{elem}^\mu(h_0)$, so the desired inequality (5) follows immediately. \square

We now define another type of integral.

DEFINITION. Let (X, \mathcal{A}, μ) be a measure space. A measurable function $f : X \rightarrow [0, \infty]$ is said to be μ -integrable, if

- (a) every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$, is elementary μ -integrable;
- (b) $\sup \{I_{elem}^{\mu}(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), 0 \leq h \leq f\} < \infty$.

If this is the case, the above supremum is denoted by $I_{+}^{\mu}(f)$. The space of all such functions is denoted by $\mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$. The map

$$I_{+}^{\mu} : \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu) \rightarrow [0, \infty)$$

is called the *positive μ -integral*.

The first (legitimate) question is whether there is an overlap between the two definitions. This is answered by the following.

PROPOSITION 1.2. *Let (X, \mathcal{A}, μ) be a measure space, and let $f \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ be a function with $f \geq 0$. The following are equivalent*

- (i) $f \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$;
- (ii) $f \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$.

Moreover, if f is as above, then $I_{elem}^{\mu}(f) = I_{+}^{\mu}(f)$.

PROOF. The implication (i) \Rightarrow (ii) is trivial.

To prove the implication (ii) \Rightarrow (i) we start with an arbitrary elementary $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$. Using Proposition 1.1, we clearly get

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{elem}^{\mu}(h) \leq I_{elem}^{\mu}(f)$.

Using these two facts, it follows that $f \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, as well as the equality

$$\sup \{I_{elem}^{\mu}(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), h \leq f\} = I_{elem}^{\mu}(f),$$

which gives $I_{+}^{\mu}(f) = I_{elem}^{\mu}(f)$. \square

We now examine properties of the positive integral, which are similar to those of the elementary integral. The following is an analogue of Proposition 1.1.

PROPOSITION 1.3. *Let (X, \mathcal{A}, μ) be a measure space, let $f \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, and let $g : X \rightarrow [0, \infty]$ be a measurable function, such that $g \leq f$, μ -a.e., then $g \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, and $I_{+}^{\mu}(g) \leq I_{+}^{\mu}(f)$.*

PROOF. Start with some elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq g$. Consider the sets

$$M = \{x \in X : h(x) > f(x)\} \text{ and } N = \{x \in X : g(x) > f(x)\},$$

which obviously belong to \mathcal{A} . Since $N \subset N$, and $\mu(N) = 0$, we have $\mu(M) = 0$. If we define the elementary function $h_0 = h(1 - \chi_M)$, then we have $h = h_0$, μ -a.e., and $0 \leq h_0 \leq f$, so it follows that $h_0 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^{\mu}(h_0) \leq I_{elem}^{\mu}(f)$. Since $h = h_0$, μ -a.e., by Proposition 1.1., it follows that $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and $I_{elem}^{\mu}(h) = I_{elem}^{\mu}(h_0) \leq I_{+}^{\mu}(f)$. By definition, this gives $g \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$ and $I_{+}^{\mu}(g) \leq I_{+}^{\mu}(f)$. \square

REMARK 1.3. Let (X, \mathcal{A}, μ) be a measure space, and let $f \in \mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$. Although f is allowed to take the value ∞ , it turns out that this is inessential. More precisely one has

$$\mu(f^{-1}(\{\infty\})) = 0.$$

This is in fact a consequence of the equality

$$(7) \quad \lim_{t \rightarrow \infty} \mu(f^{-1}([t, \infty])) = 0.$$

Indeed, if we define, for each $t \in (0, \infty)$, the set $A_t = f^{-1}([t, \infty]) \in \mathcal{A}$, then we have $0 \leq t\chi_{A_t} \leq f$. This forces the functions $t\chi_{A_t}$, $t \in (0, \infty)$ to be elementary integrable, and

$$\mu(A_t) \leq \frac{I_+^\mu(f)}{t}, \quad \forall t \in (0, \infty).$$

This forces $\lim_{t \rightarrow \infty} \mu(A_t) = 0$.

The next result explains the fact that positive integrability is a “decomposable” property.

PROPOSITION 1.4. *Let (X, \mathcal{A}, μ) be a measure space. Suppose $(A_k)_{k=1}^n \subset \mathcal{A}$ is a pairwise disjoint finite sequence, with $A_1 \cup \cdots \cup A_n = X$. For a measurable function $f : X \rightarrow [0, \infty]$, the following are equivalent.*

- (i) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$;
- (ii) $f\chi_{A_k} \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, $\forall k = 1, \dots, n$.

Moreover, if f satisfies these equivalent conditions, one has

$$I_+^\mu(f) = \sum_{k=1}^n I_+^\mu(f\chi_{A_k}).$$

PROOF. The implication (i) \Rightarrow (ii) is trivial, since we have $0 \leq f\chi_{A_k} \leq f$, so we can apply Proposition 1.3.

To prove the implication (ii) \Rightarrow (i), start by assuming that f satisfies condition (ii). We first observe that every elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f$, has the properties:

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{elem}^\mu(h) \leq \sum_{k=1}^n I_+^\mu(f\chi_{A_k})$.

This is immediate from the fact that we have the equality $h = \sum_{k=1}^n h\chi_{A_k}$, and all function $h\chi_{A_k}$ are elementary, and satisfy $0 \leq h\chi_{A_k} \leq f\chi_{A_k}$, and then everything follows from Theorem 1.1 and the definition of the positive integral which gives $I_{elem}^\mu(h\chi_{A_k}) \leq I_+^\mu(f\chi_{A_k})$.

Of course, the properties (a) and (b) above prove that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f) \leq \sum_{k=1}^n I_+^\mu(f\chi_{A_k}).$$

To prove that we have in fact equality, we start with some $\varepsilon > 0$, and we choose, for each $k \in \{1, \dots, n\}$, a function $h_k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, such that $0 \leq h_k \leq f\chi_{A_k}$, and $I_{elem}^\mu(h_k) \geq I_+^\mu(f\chi_{A_k}) - \frac{\varepsilon}{n}$. By Theorem 1.1, the function $h = h_1 + \cdots + h_n$ belongs to $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and has

$$(8) \quad I_{elem}^\mu(h) = \sum_{k=1}^n I_{elem}^\mu(h_k) \geq \left(\sum_{k=1}^n I_+^\mu(f\chi_{A_k}) \right) - \varepsilon.$$

We obviously have

$$h = \sum_{k=1}^n h_k \leq \sum_{k=1}^n f\chi_{A_k} = f,$$

so we get $I_{elem}^\mu(h) \leq I^\mu(f)$, thus the inequality (8) gives

$$I^\mu(f) \geq \left(\sum_{k=1}^n I_+^\mu(f \chi_{A_k}) \right) - \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we get $I^\mu(f) \geq \sum_{k=1}^n I_+^\mu(f \chi_{A_k})$, and we are done. \square

REMARK 1.4. Let (X, \mathcal{A}, μ) be a measure space, and let $S \in \mathcal{A}$. We can

$$\mathcal{A}|_S = \{A \cap S : A \in \mathcal{A}\} = \{A \in \mathcal{A} : A \subset S\},$$

so that $\mathcal{A}|_S \subset \mathcal{A}$ is a σ -algebra on S . The restriction of μ to $\mathcal{A}|_S$ will be denoted by $\mu|_S$. With these notations, $(S, \mathcal{A}|_S, \mu|_S)$ is a measure space. It is not hard to see that for a measurable function $f : X \rightarrow [0, \infty]$, the conditions

- $f \chi_S \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$,
- $f|_S \in \mathfrak{L}_+^1(S, \mathcal{A}|_S, \mu|_S)$

are equivalent. Moreover, in this case one has the equality

$$I_+^\mu(f \chi_S) = I_+^{\mu|_S}(f|_S).$$

This is a consequence of the fact that these two conditions are equivalent if f is elementary, combined with the fact that the restriction map $h \mapsto h|_S$ establishes a bijection between the sets

$$\begin{aligned} & \{h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X) : 0 \leq h \leq f \chi_S\}, \\ & \{k \in \mathcal{A}|_S\text{-Elem}_{\mathbb{R}}(S) : 0 \leq k \leq f|_S\}. \end{aligned}$$

The next result gives an alternative definition of the positive integral, for functions that are dominated by elementary integrable ones.

PROPOSITION 1.5. *Let (X, \mathcal{A}, μ) be a measure space, let $f : X \rightarrow [0, \infty]$ be a measurable function. Assume there exists $h_0 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $h_0 \geq f$. Then $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality*

$$(9) \quad I_+^\mu(f) = \inf \{I_{elem}^\mu(h) : h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), h \geq f\}.$$

PROOF. Since $h_0 \geq 0$, by Proposition 1.2, we know that $h_0 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. The fact that $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ then follows from Proposition 1.3, combined with the inequality $h_0 \geq f$. More generally, again by Propositions 1.2 and 1.3, we know that for any $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $h \geq f$, we have $h \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f) \leq I_+^\mu(h) = I_{elem}^\mu(h).$$

So, if we denote the right hand side of (9) by $J(f)$, we have $I_+^\mu(f) \leq J(f) \leq I_{elem}^\mu(h_0)$.

We now prove the other inequality $I_+^\mu(f) \geq J(f)$. If $h_0 = 0$, there is nothing to prove. Assume h_0 is not identically zero. Without any loss of generality, we can assume that $h_0 = \beta \chi_B$, for some $\beta \in (0, \infty)$ and $B \in \mathcal{A}$ with $\mu(B) < \infty$. (If we define $B = h_0^{-1}((0, \infty)) = \bigcup_{\alpha \in h_0(X) \setminus \{0\}} h_0^{-1}(\{\alpha\})$, and if we set $\beta = \max h_0(X)$, then we clearly have $\mu(B) < \infty$, and $h_0 \leq \beta \chi_B$.)

For every integer $n \geq 1$, we define the sets $A_1^n, \dots, A_n^n \in \mathcal{A}$ by

$$A_k^n = f^{-1}\left(\left(\frac{(k-1)\beta}{n}, \frac{k\beta}{n}\right]\right), \quad \forall k = 1, \dots, n,$$

and we define the elementary functions

$$g_n = \sum_{k=1}^n \frac{(k-1)\beta}{n} \chi_{A_k^n} \text{ and } h_n = \sum_{k=1}^n \frac{k\beta}{n} \chi_{A_k^n}.$$

The main features of these constructions are collected in the following.

Claim: For every $n \geq 1$, the functions g_n and h_n are elementary integrable, and satisfy the inequalities $0 \leq g_n \leq f \leq h_n \leq h_0$, as well as

$$I_{elem}^\mu(h_n) \leq I_{elem}^\mu(g_n) + \frac{\beta\mu(B)}{n}.$$

To prove this fact, we fix $n \geq 1$, and we first remark that the sets $(A_k^n)_{k=1}^n$ are pairwise disjoint. Since $0 \leq f \leq h_0 = \beta\chi_B$, we have

$$A_1^n \cup \dots \cup A_n^n = f^{-1}((0, \beta]) \subset B.$$

In particular, if we define $A_n = A_1^n \cup \dots \cup A_n^n \subset B$, we have

$$h_n = \sum_{k=1}^n \frac{k\beta}{n} \chi_{A_k^n} \leq \beta \sum_{k=1}^n \chi_{A_k^n} = \beta\chi_{A_n} \leq \beta\chi_B.$$

Let us prove the inequalities $g_n \leq f \leq h_n$. Start with some arbitrary point $x \in X$, and let us show that $g_n(x) \leq f(x) \leq h_n(x)$. If $f(x) = 0$, there is nothing to prove, because this forces $\chi_{A_k^n}(x) = 0$, $\forall k = 1, \dots, n$. Assume now $f(x) > 0$. Since $f \leq \beta\chi_B$, we now that $f(x) \in (0, \beta]$, so there exists a unique $k \in \{1, \dots, n\}$, such that $\frac{(k-1)\beta}{n} < f(x) \leq \frac{k\beta}{n}$, i.e. $x \in A_k^n$. We then obviously have

$$g_n(x) = \frac{(k-1)\beta}{n} \chi_{A_k^n}(x) = \frac{(k-1)\beta}{n} < f(x) \leq \frac{k\beta}{n} = \frac{k\beta}{n} \chi_{A_k^n}(x) = h_n(x),$$

and we are done. Finally, let us observe that since $g_n \leq h_n \leq h_0$, it follows that g_n and h_n are in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so g_n and h_n are elementary integrable. Notice that

$$h_n - g_n = \frac{\beta}{n} \sum_{k=1}^n \chi_{A_k^n} = \frac{\beta}{n} \chi_{A_n} \leq \frac{\beta}{n} \chi_B,$$

so we have $I_{elem}^\mu(h_n - g_n) \leq I_{elem}^\mu(\frac{\beta}{n} \chi_B) = \frac{\beta\mu(B)}{n}$, so using Theorem 1.1, we get

$$I_{elem}^\mu(h_n) = I_{elem}^\mu(g_n) + I_{elem}^\mu(h_n - g_n) \leq I_{elem}^\mu(g_n) + \frac{\beta\mu(B)}{n}.$$

Having proven the Claim, we immediately see that by the definition of the positive integral, we have

$$J(f) \leq I_{elem}^\mu(h_n) \leq I_{elem}^\mu(g_n) + \frac{\beta\mu(B)}{n} \leq I_+^\mu(f) + \frac{\beta\mu(B)}{n}.$$

Since the inequality $J(f) \leq I_+^\mu(f) + \frac{\beta\mu(B)}{n}$ holds for all $n \geq 1$, it will clearly force $J(f) \leq I_+^\mu(f)$. \square

Our next goal is to prove an analogue of Theorem 1.1, for the positive integral (Theorem 1.2 below). We discuss first a weaker version.

LEMMA 1.1. Let (X, \mathcal{A}, μ) be a measure space.

- (i) If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g + f \geq 0$, then $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g + f) = I_{elem}^\mu(g) + I_+^\mu(f)$.
- (ii) If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g - f \geq 0$, then $g - f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g - f) = I_{elem}^\mu(g) - I_+^\mu(f)$.

PROOF. (i). We start with a weaker version.

Claim: If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, are such that $g + f \geq 0$, then $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(g + f) \leq I_{elem}^\mu(g) + I_+^\mu(f)$.

What we need to prove is the fact that, for every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq g + f$, we have:

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$;
- (b) $I_{elem}^\mu(h) \leq I_{elem}^\mu(g) + I_+^\mu(f)$.

Consider the elementary function $h_1 = \max\{h - g, 0\}$. It is obvious that $0 \leq h_1 \leq f$, so by Proposition 1.3, it follows that $h_1 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I_+^\mu(h_1) \leq I_+^\mu(f)$. By Proposition 1.2, this gives $h_1 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and

$$(10) \quad I_{elem}^\mu(h_1) = I_+^\mu(h_1) \leq I_+^\mu(f).$$

Using the obvious inequality $-g \leq h - g \leq h_1$, again by Proposition 1.2, it follows that $h - g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, and

$$(11) \quad I_{elem}^\mu(h - g) \leq I_{elem}^\mu(h_1).$$

Of course, by Theorem 1.1, this gives the fact that $h = (h - g) + g$ is elementary μ -integrable, as well as the equality

$$I_{elem}^\mu(h) = I_{elem}^\mu(h - g) + I_{elem}^\mu(g).$$

Combining this with (11) and (10) immediately gives

$$I_{elem}^\mu(h) \leq I_{elem}^\mu(h_1) + I_{elem}^\mu(g) \leq I_+^\mu(f) + I_{elem}^\mu(g),$$

and the Claim is proven.

Having proven the above Claim, we now proceed with the proof of (i). If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ are such that $g + f \geq 0$, then by the Claim, we already know that $g + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $I^\mu(g + f) \leq I_{elem}^\mu(g) + I_+^\mu(f)$. We apply now again the Claim to the functions $f_1 = g + f$ and $g_1 = -g$, to get

$$I_+^\mu(f) = I_+^\mu(g_1 + f_1) \leq I_{elem}^\mu(g_1) + I_+^\mu(f_1) = -I_+^\mu(g) + I_+^\mu(g + f),$$

which gives the other inequality $I_{elem}^\mu(g) + I_+^\mu(f) \leq I_+^\mu(g + f)$.

(ii). Start with $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $g \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $g - f \geq 0$. First of all, since $0 \leq g - f \leq g$, by Proposition 1.5, it follows that $g - f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$(12) \quad I_+^\mu(g - f) = \inf \{ I_{elem}^\mu(k) : k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), k \geq g - f \}.$$

Second, remark that, whenever $k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is such that $g - f \leq k$, it follows that $k + f \geq g$, so using part (i) combined with Proposition 1.3, we see that $k + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$I_{elem}^\mu(g) = I_+^\mu(g) \leq I_+^\mu(k + f) = I_{elem}^\mu(k) + I_+^\mu(f).$$

This means that we have

$$I_{elem}^\mu(k) \geq I_{elem}^\mu(g) - I_+^\mu(f),$$

for all $k \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $k \geq g - f$, and then by (12), we immediately get

$$I_+^\mu(g - f) \geq I_{elem}^\mu(g) - I_+^\mu(f).$$

To prove the other inequality, we use the definition of the positive integral, which gives

$$(13) \quad I_+^\mu(g - f) = \sup \{ I_{elem}^\mu(h) : h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu), 0 \leq h \leq g - f \}.$$

Remark that, whenever $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is such that $0 \leq h \leq g - f$, it follows that $0 \leq h + f \leq g$, so using part (i) combined with Proposition 1.3, we see that $h + f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and

$$I_{elem}^\mu(g) = I_+^\mu(g) \geq I_+^\mu(h + f) = I_{elem}^\mu(h) + I_+^\mu(f).$$

This means that we have

$$I_{elem}^\mu(h) \leq I_{elem}^\mu(g) - I_+^\mu(f),$$

for all $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, with $0 \leq h \leq g - f$, and then by (13), we immediately get $I_+^\mu(g - f) \leq I_{elem}^\mu(g) - I_+^\mu(f)$. \square

We are now in position to prove the following result (compare with Theorem 1.1).

THEOREM 1.2. *Let (X, \mathcal{A}, μ) be a measure space.*

- (i) *If $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, then $f_1 + f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality $I_+^\mu(f_1 + f_2) = I_+^\mu(f_1) + I_+^\mu(f_2)$.*
- (ii) *If $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and $\alpha \in [0, \infty)$, then $\alpha f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and one has the equality $I_+^\mu(\alpha f) = \alpha I_+^\mu(f)$.*

PROOF. (i). Fix $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

Claim 1: Whenever $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ satisfies $0 \leq h \leq f_1 + f_2$, it follows that

- (a) $h \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$,
- (b) $I_{elem}^\mu(h) \leq I_+^\mu(f_1) + I_+^\mu(f_2)$.

Fix an elementary function $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f_1 + f_2$, and let us first show that h is elementary integrable. Fix some $\alpha \in h(X) \setminus \{0\}$, and let us prove that $\mu(h^{-1}(\{\alpha\})) < \infty$. If we define the sets $A_j = f_j^{-1}([\alpha/2, \infty]) \in \mathcal{A}$, $j = 1, 2$, then the elementary functions $h_j = \frac{\alpha}{2} \chi_{A_j}$ satisfy $0 \leq h_j \leq f_j$, $j = 1, 2$. In particular, it follows that $h_1, h_2 \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$, which forces $\mu(A_1) < \infty$ and $\mu(A_2) < \infty$. Notice however that, for every $x \in h^{-1}(\{\alpha\})$, we have $f_1(x) + f_2(x) \geq h(x) = \alpha$, which forces either $f_1(x) \geq \frac{\alpha}{2}$ or $f_2(x) \geq \frac{\alpha}{2}$. This argument shows that we have the inclusion $h^{-1}(\{\alpha\}) \subset A_1 \cup A_2$, so it follows that we indeed have $\mu(h^{-1}(\{\alpha\})) < \infty$.

Having shown property (a), let us prove property (b). Define the sets

$$B = \{x \in X : h(x) \geq f_1(x)\} \text{ and } D = X \setminus B.$$

It is obvious that $B, D \in \mathcal{A}$ are pairwise disjoint, and $B \cup D = X$. Define the elementary functions $h' = h \chi_B$, and $h'' = h - h' = h \chi_D$. On the one hand, we have

$$f_1 \chi_B \leq h' \leq f_1 \chi_B + f_2 \chi_B,$$

which gives

$$0 \leq h' - f_1 \chi_B \leq f_2 \chi_B.$$

¹ Here we use the convention that when $\alpha = 0$, we take $\alpha f = 0$.

By Lemma 1.1.(ii), combined with Proposition 1.4, it follows that $h' - f_1\chi_B \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ and $I_{elem}^\mu(h') - I_+^\mu(f_1\chi_B) = I_+^\mu(h' - f_1\chi_B) \leq I_+^\mu(f_2\chi_B)$, so we get

$$(14) \quad I_{elem}^\mu(h') \leq I_+^\mu(f_1\chi_B) + I_+^\mu(f_2\chi_B).$$

On the other hand, we have

$$h'' = h\chi_D \leq f_1\chi_D,$$

which gives

$$(15) \quad I_{elem}^\mu(h'') \leq I_+^\mu(f_1\chi_D) \leq I_+^\mu(f_1\chi_D) + I_+^\mu(f_2\chi_D).$$

Since $h = h' + h''$, with h' and h'' elementary integrable, using Theorem 1.1 combined with Proposition 1.4, by adding the inequalities (14) and (15) we get

$$\begin{aligned} I_{elem}^\mu(h) &= I_{elem}^\mu(h') + I_{elem}^\mu(h'') \leq \\ &\leq I_+^\mu(f_1\chi_B) + I_+^\mu(f_2\chi_B) + I_+^\mu(f_1\chi_D) + I_+^\mu(f_2\chi_D) = I_+^\mu(f_1) + I_+^\mu(f_2), \end{aligned}$$

and the Claim is proven.

Claim 1 obviously implies the fact that $f_1 + f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, as well as the inequality

$$I_+^\mu(f_1 + f_2) \leq I_+^\mu(f_1) + I_+^\mu(f_2).$$

To prove the other inequality, we use the following.

Claim 2: For every $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, with $0 \leq h \leq f_1$, one has the inequality

$$I_{elem}^\mu(h) \leq I_+^\mu(f_1 + f_2) - I_+^\mu(f_2).$$

Indeed, if h is as above, then h is in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, hence elementary integrable, and we obviously have $0 \leq h + f_2 \leq f_1 + f_2$. Then by Lemma 1.1.(i), combined with Proposition 1.3, we get

$$I_{elem}^\mu(h) + I_+^\mu(f_2) = I_+^\mu(h + f_2) \leq I_+^\mu(f_1 + f_2),$$

and the Claim follows.

Using Claim 2, and the definition of the positive integral, we get

$$I_+^\mu(f_1) = \sup \{ I_{elem}^\mu(h) : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), 0 \leq h \leq f_1 \} \leq I_+^\mu(f_1 + f_2) - I_+^\mu(f_2),$$

which then gives

$$I_+^\mu(f_1) + I_+^\mu(f_2) \leq I_+^\mu(f_1 + f_2).$$

(ii). This part is obvious. \square

DEFINITIONS. Let (X, \mathcal{A}, μ) be a measure space. Denote the extended real line $[-\infty, \infty]$ by $\overline{\mathbb{R}}$. A measurable function $f : X \rightarrow \overline{\mathbb{R}}$ is said to be μ -integrable, if there exist functions $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, such that

$$(16) \quad f(x) = f_1(x) - f_2(x), \quad \forall x \in X \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})].$$

By Remark 1.3, we know that the sets $f_k^{-1}(\{\infty\})$, $k = 1, 2$, have measure zero. The equality (16) gives then the fact $f = f_1 - f_2$, μ -a.e. We define

$$\mathfrak{L}_{\overline{\mathbb{R}}}^1(X, \mathcal{A}, \mu) = \{ f : X \rightarrow \overline{\mathbb{R}} : f \text{ } \mu\text{-integrable} \}.$$

We also define the space of ‘‘honest’’ real-valued μ -integrable functions, as

$$\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) = \{ f \in \mathfrak{L}_{\overline{\mathbb{R}}}^1(X, \mathcal{A}, \mu) : f - \infty < f(x) < \infty, \forall x \in X \}.$$

Finally, we define the space of complex-valued μ -integrable functions as

$$\mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu) = \{ f : X \rightarrow \mathbb{C} : \operatorname{Re} f, \operatorname{Im} f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \}.$$

The next result collects the basic properties of $\mathfrak{L}_{\mathbb{R}}^1$. Among other things, it states that it is an “almost” vector space.

THEOREM 1.3. *Let (X, \mathcal{A}, μ) be a measure space.*

- (i) *For a measurable function $f : X \rightarrow \bar{\mathbb{R}}$, the following are equivalent:*
 (a) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
 (b) $f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.
 (ii) *If $f, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and if $h : X \rightarrow \bar{\mathbb{R}}$ is a measurable function, such that*

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus [f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\})],$$

then $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

- (iii) *If $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $\alpha \in \mathbb{R}$, and if $g : X \rightarrow \bar{\mathbb{R}}$ is a measurable function, such that*

$$g(x) = \alpha f(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

then $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

- (iv) *One has the inclusion*

$$\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu) \cup \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu).$$

PROOF. (i). Consider the functions measurable functions $f^\pm : X \rightarrow [0, \infty]$ defined as

$$f^+ = \max\{f, 0\} \text{ and } f^- = \max\{-f, 0\}.$$

To prove the implication (a) \Rightarrow (b), assume $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, which means there exist $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, such that

$$f(x) = f_1(x) - f_2(x), \quad \forall x \in X \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})].$$

Notice that we have the inequalities

$$(17) \quad f^+ \leq f_1, \quad \mu\text{-a.e.},$$

$$(18) \quad f^- \leq f_2, \quad \mu\text{-a.e.}.$$

Indeed, if we put $N = f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})$, then $\mu(N) = 0$, and if we start with some $x \in X \setminus N$, we either have $f_1(x) \geq f_2(x) \geq 0$, in which case we get

$$f^+(x) = f(x) = f_1(x) - f_2(x) \leq f_1(x),$$

$$f^-(x) = 0 \leq f_2(x),$$

or we have $f_1(x) \leq f_2(x)$, in which case we get

$$f^+(x) = 0 \leq f_1(x),$$

$$f^-(x) = -f(x) = f_2(x) - f_1(x) \leq f_2(x).$$

In other words, we have

$$f^+(x) \leq f_1(x) \text{ and } f^-(x) \leq f_2(x), \quad \forall x \in X \setminus N,$$

so we indeed get (17) and (18). Using these inequalities, and Proposition 1.3, it follows that $f^\pm \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so by Theorem 1.2, it follows that $f^+ + f^- = |f|$ also belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

To prove the implication (b) \Rightarrow (a), start by assuming that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Then, since we obviously have the inequalities $0 \leq f^\pm \leq |f|$, again by Proposition 1.3, it follows that $f^\pm \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Since we obviously have

$$f(x) = f^+(x) - f^-(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

it follows that f indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(ii). Assume f, g , and h are as in (ii). By (i), both functions $|f|$ and $|g|$ are in $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, it follows that the function $k = |f| + |g|$ also belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Notice that we have the equality

$$f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\}) = k^{-1}(\{\infty\}),$$

so the hypothesis on h reads

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus k^{-1}(\{\infty\}),$$

which then gives

$$|h(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)|, \quad \forall x \in X \setminus k^{-1}(\{\infty\}).$$

Of course, since $\mu(k^{-1}(\{\infty\})) = 0$, this gives

$$|h| \leq k, \quad \mu\text{-a.e.},$$

and using (i) it follows that h indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(iii). Assume f, α , and g are as in (iii). Exactly as above, we have $|g| = |\alpha| \cdot |f|$, μ -a.e., and then by Theorem 1.2 it follows that $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

(iv). The inclusion $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ is trivial. To prove the inclusion $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, we use parts (ii) and (iii) to reduce this to the fact that $\varkappa_A \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, for all $A \in \mathcal{A}$, with $\mu(A) < \infty$. But this fact is now obvious, because any such function belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. \square

COROLLARY 1.1. *Let (X, \mathcal{A}, μ) be a measure space, and let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} .*

- (i) *For a \mathbb{K} -valued measurable function $f : X \rightarrow \mathbb{K}$, the following are equivalent:*
- (a) $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$;
 - (b) $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.
- (ii) *When equipped with the pointwise addition and scalar multiplication, the space $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ becomes a \mathbb{K} -vector space.*

PROOF. (i). The case $\mathbb{K} = \mathbb{R}$ is immediate from Theorem 1.3

In the case when $\mathbb{K} = \mathbb{C}$, we use the obvious inequalities

$$(19) \quad \max \{ |\operatorname{Re} f|, |\operatorname{Im} f| \} \leq |f| \leq |\operatorname{Re} f| + |\operatorname{Im} f|.$$

If $f \in \mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu)$, then both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, so by Theorem 1.3, both $|\operatorname{Re} f|$ and $|\operatorname{Im} f|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, the function $g = |\operatorname{Re} f| + |\operatorname{Im} f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and then using the second inequality in (19), it follows that $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

Conversely, if $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, then using the first inequality in (19), it follows that both $|\operatorname{Re} f|$ and $|\operatorname{Im} f|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so by Theorem 1.3, both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, i.e. f belongs to $\mathfrak{L}_{\mathbb{C}}^1(X, \mathcal{A}, \mu)$.

(ii). This part is pretty clear. If $f, g \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, then by (i) both $|f|$ and $|g|$ belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and by Theorem 1.2, the function $|f| + |g|$ will

also belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Since $|f + g| \leq |f| + |g|$, it follows that $|f + g|$ itself belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so using (i) again, it follows that $f + g$ indeed belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. If $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$ and $\alpha \in \mathbb{K}$, then $|f|$ belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so $|\alpha f| = |\alpha| \cdot |f|$ again belongs to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, which by (i) gives the fact that αf belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$. \square

REMARK 1.5. Let (X, \mathcal{A}, μ) be a measure space. Then one has the equalities

$$(20) \quad \mathfrak{L}_+^1(X, \mathcal{A}, \mu) = \{f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) : f(X) \subset [0, \infty]\};$$

$$(21) \quad \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) = \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{K}}(X).$$

Indeed, by Theorem 1.3 that we have the inclusion

$$\mathfrak{L}_+^1(X, \mathcal{A}, \mu) \subset \{f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) : f(X) \subset [0, \infty]\}.$$

The inclusion in the other direction follows again from Theorem 1.3, since any function that belongs to the right hand side of (20) satisfies $f = |f|$. The inclusion

$$\mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$$

is again contained in Theorem 1.3. To prove the inclusion in the other direction, it suffices to consider the case $\mathbb{K} = \mathbb{R}$. Start with $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \cap \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, which gives $|h| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. The function $|h|$ is obviously in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, so we get $|h| \in \mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$. Since $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ is a vector space, it will also contain the function $-|h|$. The fact that h itself belongs to $\mathfrak{L}_{\mathbb{R}, elem}^1(X, \mathcal{A}, \mu)$ then follows from Proposition 1.1, combined with the obvious inequalities

$$-|h| \leq h \leq |h|.$$

The following result deals with the construction of the integral.

THEOREM 1.4. *Let (X, \mathcal{A}, μ) be a measure space. There exists a unique map $I_{\mathbb{R}}^{\mu}(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$, with the following properties:*

(i) *Whenever $f, g, h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ are such that*

$$h(x) = f(x) + g(x), \quad \forall x \in X \setminus [f^{-1}(\{-\infty, \infty\}) \cup g^{-1}(\{-\infty, \infty\})],$$

it follows that $I_{\mathbb{R}}^{\mu}(h) = I_{\mathbb{R}}^{\mu}(f) + I_{\mathbb{R}}^{\mu}(g)$.

(ii) *Whenever $f, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ and $\alpha \in \mathbb{R}$ are such that*

$$g(x) = \alpha f(x), \quad \forall x \in X \setminus f^{-1}(\{-\infty, \infty\}),$$

it follows that $I_{\mathbb{R}}^{\mu}(g) = \alpha I_{\mathbb{R}}^{\mu}(f)$.

(iii) $I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f)$, $\forall f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.

PROOF. Let us first show the existence. Start with some $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and define the functions $f^{\pm} : X \rightarrow [0, \infty]$ by $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ so that $f = f^+ - f^-$, and $f^+, f^- \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. We then define

$$I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

It is obvious that $I_{\mathbb{R}}^{\mu}$ satisfies condition (iii).

The key fact that we need is contained in the following.

Claim: Whenever $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $f_1, f_2 \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$ are such that

$$f(x) = f^+(x) - f^-(x), \quad \forall x \in X. \setminus [\neg^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})],$$

it follows that we have the equality

$$I_{\mathbb{R}}^{\mu}(f) = I_+^{\mu}(f_1) - I_+^{\mu}(f_2).$$

Indeed, since we have $f = f^+ - f^-$, it follows immediately that we have the equality

$$f_2(x) + f^+(x) = f_1(x) + f^-(x), \quad \forall x \in X. \setminus [f_1^{-1}(\{\infty\}) \cup f_2^{-1}(\{\infty\})],$$

which gives

$$f_2 + f^+ = f_1 + f^-, \quad \mu\text{-a.e.}$$

By Theorem 1.2, this immediately gives

$$I_+^{\mu}(f_2) + I_+^{\mu}(f^+) = I_+^{\mu}(f_1) + I_+^{\mu}(f^-),$$

which then gives

$$I_+^{\mu}(f_1) - I_+^{\mu}(f_2) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-) = I_{\mathbb{R}}^{\mu}(f).$$

Having prove the above Claim, let us show now that $I_{\mathbb{R}}^{\mu}$ has properties (i) and (ii). Assume f, g and h are as in (i). Notice that if we define $h_1 = f^+ + g^+$ and $h_2 = f^- + g^-$, then we clearly have $0 \leq h_1 \leq |f| + |g|$ and $0 \leq h_2 \leq |f| + |g|$, so h_1 and h_2 both belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. By Theorem 1.2, we then have

$$(22) \quad I_+^{\mu}(h_1) = I_+^{\mu}(f^+) + I_+^{\mu}(g^+) \text{ and } I_+^{\mu}(h_2) = I_+^{\mu}(f^-) + I_+^{\mu}(g^-).$$

Notice also that, because of the equalities

$$h_1^{-1}(\{\infty\}) = f^{-1}(\{\infty\}) \cup g^{-1}(\{\infty\}) \text{ and } h_2^{-1}(\{\infty\}) = f^{-1}(\{-\infty\}) \cup g^{-1}(\{-\infty\}),$$

we have

$$h = h_1(x) - h_2(x), \quad \forall x \in X. \setminus [h_1^{-1}(\{\infty\}) \cup h_2^{-1}(\{\infty\})],$$

so by the above Claim, combined with (22), we get

$$I_{\mathbb{R}}^{\mu}(h) = I_+^{\mu}(h_1) - I_+^{\mu}(h_2) = I_+^{\mu}(f^+) + I_+^{\mu}(g^+) - I_+^{\mu}(f^-) - I_+^{\mu}(g^-) = I_{\mathbb{R}}^{\mu}(f) + I_{\mathbb{R}}^{\mu}(g).$$

Property (ii) is pretty obvious.

The uniqueness is also obvious. If we start with a map $J : \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \rightarrow \mathbb{R}$ with properties (i)-(iii), then for every $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, we must have

$$J(f) = J(f^+) - J(f^-) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

(For the second equality we use condition (iii), combined with the fact that both f^+ and f^- belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$.) \square

COROLLARY 1.2. *Let (X, \mathcal{A}, μ) be a measure space, and let \mathbb{K} be either \mathbb{R} or \mathbb{C} . There exists a unique linear map $I_{\mathbb{K}}^{\mu}(X, \mathcal{A}, \mu) \rightarrow \mathbb{K}$, such that*

$$I_{\mathbb{K}}^{\mu}(f) = I_+^{\mu}(f), \quad \forall f \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu) \cap \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu).$$

PROOF. Let us start with the case $\mathbb{K} = \mathbb{R}$. In this case, we have the inclusion

$$\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu) \subset \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu),$$

so we can define $I_{\mathbb{R}}^{\mu}$ as the restriction of $I_{\mathbb{K}}^{\mu}$ to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. The uniqueness is again clear, because of the equalities

$$I_{\mathbb{R}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(f^+) - I_{\mathbb{R}}^{\mu}(f^-) = I_+^{\mu}(f^+) - I_+^{\mu}(f^-).$$

In the case $\mathbb{K} = \mathbb{C}$, we define

$$I_{\mathbb{C}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(\operatorname{Re} f) + iI_{\mathbb{R}}^{\mu}(\operatorname{Im} f).$$

The linearity is obvious. The uniqueness is also clear, because the restriction of $I_{\mathbb{C}}^{\mu}$ to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$ must agree with $I_{\mathbb{R}}^{\mu}$. \square

DEFINITION. Let (X, \mathcal{A}, μ) be a measure space, and let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} . For any $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, the number $I_{\mathbf{K}}^{\mu}(f)$ (which is real, if $\mathbf{K} = \bar{\mathbb{R}}$ or \mathbb{R} , and is complex if $\mathbf{K} = \mathbb{C}$) will be denoted by

$$\int_X f d\mu,$$

and is called the μ -integral of f . This notation is unambiguous, because if $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, then we have $I_{\mathbb{R}}^{\mu}(f) = I_{\mathbb{C}}^{\mu}(f) = I_{\mathbb{R}}^{\mu}(f)$.

REMARK 1.6. If (X, \mathcal{A}, μ) is a measure space, then for every $A \in \mathcal{A}$, with $\mu(A) < \infty$, using the above Corollary, we get

$$\int_X \chi_A d\mu = I_{+}^{\mu}(\chi_A) = \mu(A).$$

By linearity, if $\mathbb{K} = \mathbb{R}, \mathbb{C}$, one has then the equality

$$\int_X h d\mu = I_{elem}^{\mu}(h), \quad \forall h \in \mathfrak{L}_{\mathbb{K}, elem}^1(X, \mathcal{A}, \mu).$$

To make the exposition a bit easier, it will adopt the following.

CONVENTION. If (X, \mathcal{A}, μ) is a measure space, and if $f : X \rightarrow [0, \infty]$ is a measurable function, which *does not belong to* $\mathfrak{L}_{+}^1(X, \mathcal{A}, \mu)$, then we *define*

$$\int_X f d\mu = \infty.$$

REMARKS 1.7. Let (X, \mathcal{A}, μ) be a measure space.

A. Using the above convention, when $h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$ is a function with $h(X) \subset [0, \infty)$, the condition $\int_X h d\mu = \infty$ is equivalent to the existence of some $\alpha \in h(X) \setminus \{0\}$, with $\mu(h^{-1}(\{\alpha\})) = \infty$.

B. Using the above convention, for every measurable function $f : X \rightarrow [0, \infty]$, one has the equality

$$\int_X f d\mu = \sup \left\{ \int_X h d\mu : h \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X), 0 \leq h \leq f \right\}.$$

C. If $f, g : X \rightarrow [0, \infty]$ are measurable, then one has the equalities

$$\begin{aligned} \int_X (f + g) d\mu &= \int_X f d\mu + \int_X g d\mu, \\ \int_X (\alpha f) d\mu &= \alpha \int_X f d\mu, \quad \forall \alpha \in [0, \infty), \end{aligned}$$

even in the case when some term is infinite. (We use the convention $\infty + t = \infty$, $\forall t \in [0, \infty]$, as well as $\alpha \cdot \infty = \infty$, $\forall \alpha \in (0, \infty)$, and $0 \cdot \infty = 0$.)

D. If $f, g : X \rightarrow [0, \infty]$ are measurable, and $f \leq g$, μ -a.e., then (using B) one has the inequality

$$\int_X f d\mu \leq \int_X g d\mu,$$

even if one side (or both) is infinite.

E. Let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} , and let $f : X \rightarrow \mathbf{K}$ be a measurable function. Then the function $|f| : X \rightarrow [0, \infty]$ is measurable. Using the above convention, the condition that f belongs to $\mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$ is equivalent to the inequality $\int_X |f| d\mu < \infty$.

In the remainder of this section we discuss several properties of integration that are analogous to those of the positive/elementary integration.

We begin with a useful estimate

PROPOSITION 1.6. *Let (X, \mathcal{A}, μ) be a measure space, and let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} . For every function $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, one has the inequality*

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

PROOF. Let us first examine the case when $\mathbf{K} = \bar{\mathbb{R}}, \mathbb{R}$. In this case we define $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$, so we have $f = f^+ - f^-$, as well as $|f| = f^+ + f^-$. Using the inequalities $I_+^\mu(f^\pm) \geq 0$, we have

$$\begin{aligned} \int_X f d\mu &= I_+^\mu(f^+) - I_+^\mu(f^-) \leq I_+^\mu(f^+) + I_+^\mu(f^-) = \int_X |f| d\mu; \\ - \int_X f d\mu &= -I_+^\mu(f^+) + I_+^\mu(f^-) \leq I_+^\mu(f^+) + I_+^\mu(f^-) = \int_X |f| d\mu. \end{aligned}$$

In other words, we have

$$\pm \int_X f d\mu \leq \int_X |f| d\mu,$$

and the desired inequality immediately follows.

Let us consider now the case $\mathbf{K} = \mathbb{C}$. Consider the number $\lambda = \int_X f d\mu$, and let us choose some complex number $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, and $\alpha\lambda = |\lambda|$. (If $\lambda \neq 0$, we take $\alpha = \lambda^{-1}|\lambda|$; otherwise we take $\alpha = 1$.) Consider the measurable function $g = \alpha f$. Notice now that

$$\left(\int_X \operatorname{Re} g d\mu \right) + i \left(\int_X \operatorname{Im} g d\mu \right) = \int_X g d\mu = \alpha \int_X f d\mu = \alpha\lambda = |\lambda| \geq 0,$$

so in particular we get

$$|\lambda| = \int_X \operatorname{Re} g d\mu.$$

If we apply the real case, we then get

$$(23) \quad |\lambda| \leq \int_X |\operatorname{Re} g| d\mu.$$

Notice now that, we have the inequality $|\operatorname{Re} g| \leq |g| = |f|$, which gives

$$I_+^\mu(|\operatorname{Re} g|) \leq I^\mu(|f|) = \int_X |f| d\mu,$$

so the inequality (23) immediately gives

$$\left| \int_X f d\mu \right| = |\lambda| \leq \int_X |f| d\mu. \quad \square$$

COROLLARY 1.3. *Let (X, \mathcal{A}, μ) be a measure space, and let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} . If a measurable function $f : X \rightarrow \mathbf{K}$ satisfies $f = 0$, μ -a.e., then $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, and $\int_X f d\mu = 0$.*

PROOF. Consider the measurable function $|f| : X \rightarrow [0, \infty]$, which satisfies $|f| = 0$, μ -a.e. By Proposition 1.3, it follows that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, hence $f \in \mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and $\int_X |f| d\mu = 0$. Of course, the last equality forces $\int_X f d\mu = 0$. \square

COROLLARY 1.4. *Let \mathbb{K} be either \mathbb{R} or \mathbb{C} . If (X, \mathcal{A}, μ) is a finite measure space, then every bounded measurable function $f : X \rightarrow \mathbb{K}$ belongs to $\mathfrak{L}_{\mathbb{K}}^1(X, \mathcal{A}, \mu)$, and satisfies*

$$\left| \int_X f d\mu \right| \leq \mu(X) \cdot \sup_{x \in X} |f(x)|.$$

PROOF. If we put $\beta = \sup_{x \in X} |f(x)|$, then we clearly have $|f| \leq \beta \chi_X$, which shows that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, and also gives $\int_X |f| d\mu \leq \int_X \beta \chi_X d\mu = \mu(X) \cdot \beta$. Then everything follows from Proposition 1.6. \square

COMMENT. The introduction of the space $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, of extended real-valued μ -integrable functions, is useful mostly for technical reasons. In effect, everything can be reduced to the case when only “honest” real-valued functions are involved. The following result clarifies this matter.

LEMMA 1.2. *Let (X, \mathcal{A}, μ) be a measure space, and let $f : X \rightarrow \bar{\mathbb{R}}$ be a measurable function. The following are equivalent*

- (i) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
- (ii) there exists $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $g = f$, μ -a.e.

Moreover, if f satisfies these equivalent conditions, then any function g , satisfying (ii), also has the property

$$\int_X f d\mu = \int_X g d\mu.$$

PROOF. Consider the set $F = \{x \in X : -\infty < f(x) < \infty\}$, which belongs to \mathcal{A} . We obviously have the equality $X \setminus F = |f|^{-1}(\{\infty\})$.

(i) \Rightarrow (ii). Assume $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, which means that $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. In particular, we get $\mu(X \setminus F) = 0$. Define the measurable function $g = f \chi_F$. On the one hand, it is clear, by construction, that we have $-\infty < g(x) < \infty$, $\forall x \in X$. On the other hand, it is clear that $g|_F = f|_F$, so using $\mu(X \setminus F) = 0$, we get the fact that $f = g$, μ -a.e. Finally, the inequality $0 \leq |g| \leq |f|$, combined with Proposition 1.3, gives $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so g indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

(ii) \Rightarrow (i). Suppose there exists $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, with $f = g$, μ -a.e., and let us prove that

- (a) $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$;
- (b) $\int_X f d\mu = \int_X g d\mu$.

The first assertion is clear, because by Proposition 1.3, the equality $|f| = |g|$, μ -a.e., combined with $|g| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, forces $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, i.e. $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. To prove (b), we consider the difference $h = f - g$, which is a measurable function $h : X \rightarrow \bar{\mathbb{R}}$, and satisfies $h = 0$, μ -a.e. By Corollary 1.3, we know that $h \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and $\int_X h d\mu = 0$. By Theorem 1.3, we get

$$\int_X f d\mu = \int_X g d\mu + \int_X h d\mu = \int_X g d\mu. \quad \square$$

The following result is an analogue of Proposition 1.1 (see also Proposition 1.3).

PROPOSITION 1.7. Let (X, \mathcal{A}, μ) , and let $f_1, f_2 \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$. Suppose $f : X \rightarrow \bar{\mathbb{R}}$ is a measurable function, such that $f_1 \leq f \leq f_2$, μ -a.e. Then $f \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and one has the inequality

$$\int_X f_1 d\mu \leq \int_X f d\mu \leq \int_X f_2 d\mu.$$

PROOF. First of all, since f_1 and f_2 belong to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, it follows that $|f_1|$ and $|f_2|$, hence also $|f_1| + |f_2|$, belong to $\mathfrak{L}_+^1(X, \mathcal{A}, \mu)$. Second, since we have $f_2 \leq |f_2| \leq |f_1| + |f_2|$ and $f_1 \geq -|f_1| \geq -|f_1| - |f_2|$ (everywhere!), the inequalities $f_1 \leq f \leq f_2$, μ -a.e., give

$$-|f_1| - |f_2| \leq f \leq |f_1| + |f_2|, \quad \mu\text{-a.e.},$$

which reads

$$|f| \leq |f_1| + |f_2|, \quad \mu\text{-a.e.}$$

Since $|f_1| + |f_2| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, by Proposition 1.3., we get $|f| \in \mathfrak{L}_+^1(X, \mathcal{A}, \mu)$, so f indeed belongs to $\mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$.

To prove the inequality for integrals, we use Lemma 1.2, to find functions $g_1, g_2, g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $f_1 = g_1$, μ -a.e., $f_2 = g_2$, μ -a.e., and $f = g$, μ -a.e. Lemma 1.2 also gives the equalities $\int_X f_1 d\mu = \int_X g_1 d\mu$, $\int_X f_2 d\mu = \int_X g_2 d\mu$, and $\int_X f d\mu = \int_X g d\mu$, so what we need to prove are the inequalities

$$(24) \quad \int_X g_1 d\mu \leq \int_X g d\mu \leq \int_X g_2 d\mu.$$

Of course, we have

$$g_1 \leq g \leq g_2, \quad \mu\text{-a.e.}$$

To prove the first inequality in (24), we consider the function $h = g - g_1 \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, and we prove that $\int_X h d\mu \geq 0$. But this is quite clear, because we have $h \geq 0$, μ -a.e., which means that $h = |h|$, μ -a.e., so by Lemma 1.2, we get

$$\int_X h d\mu = \int_X |h| d\mu = I_+^{\mu}(|h|) \geq 0.$$

The second inequality in (24) is prove the exact same way. \square

The next result is an analogue of Proposition 1.4.

PROPOSITION 1.8. Let (X, \mathcal{A}, μ) be a measure space, and let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} . Suppose $(A_k)_{k=1}^n \subset \mathcal{A}$ is a pairwise disjoint finite sequence, with $A_1 \cup \dots \cup A_n = X$. For a measurable function $f : X \rightarrow \mathbf{K}$, the following are equivalent.

- (i) $f \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$;
- (ii) $f \chi_{A_k} \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu)$, $\forall k = 1, \dots, n$.

Moreover, if f satisfies these equivalent conditions, one has

$$(25) \quad \int_X f d\mu = \sum_{k=1}^n \int_X f \chi_{A_k} d\mu.$$

PROOF. It is fairly obvious that $|f \chi_{A_k}| = |f| \chi_{A_k}$. Then the equivalence (i) \Leftrightarrow (ii) follows from Proposition 1.4 applied to the function $|f| : X \rightarrow [0, \infty]$. In the cases when $\mathbf{K} = \mathbb{R}, \mathbb{C}$, the equality (25) follows immediately from linearity, and the obvious equality $f = \sum_{k=1}^n f \chi_{A_k}$. In the case when $\mathbf{K} = \bar{\mathbb{R}}$, we take $g \in \mathfrak{L}_{\mathbb{R}}^1(X, \mathcal{A}, \mu)$, such that $f = g$, μ -a.e. Then we obviously have $f \chi_{A_k} = g \chi_{A_k}$,

μ -a.e., for all $k = 1, \dots, n$, and the equality (25) follows from the corresponding equality that holds for g . \square

REMARK 1.8. The equality (25) also holds for arbitrary measurable functions $f : X \rightarrow [0, \infty]$, if we use the convention that preceded Remarks 1.7. This is an immediate consequence of Proposition 1.4, because the left hand side is infinite, if and only if one of the terms in the right hand side is infinite.

The following is an obvious extension of Remark 1.4.

REMARK 1.9. Let \mathbf{K} be one of the symbols $\bar{\mathbb{R}}$, \mathbb{R} , or \mathbb{C} , let (X, \mathcal{A}, μ) be a measure space. For a set $S \in \mathcal{A}$, and a measurable function $f : X \rightarrow \mathbf{K}$, one has the equivalence

$$f \chi_S \in \mathfrak{L}_{\mathbf{K}}^1(X, \mathcal{A}, \mu) \iff f|_S \in \mathfrak{L}_{\mathbf{K}}^1(S, \mathcal{A}|_S, \mu|_S).$$

If this is the case, one has the equality

$$(26) \quad \int_X f \chi_S d\mu = \int_S f|_S d\mu|_S.$$

The above equality also holds for arbitrary measurable functions $f : X \rightarrow [0, \infty]$, again using the convention that preceded Remarks 1.7.

NOTATION. The above remark states that, whenever the quantities in (26) are defined, they are equal. (This only requires the fact that $f|_S$ is measurable, and either $f|_S \in \mathfrak{L}_{\mathbf{K}}^1(S, \mathcal{A}|_S, \mu|_S)$, or $f(S) \subset [0, \infty]$.) In this case, the equal quantities in (26) will be simply denoted by $\int_S f d\mu$.

Exercise 1. Let I be some non-empty set. Consider the σ -algebra $\mathcal{P}(I)$, of all subsets of I , equipped with the counting measure

$$\mu(A) = \begin{cases} \text{Card } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

Prove that $\mathfrak{L}_{\bar{\mathbb{R}}}^1(I, \mathcal{P}(I), \mu) = \mathfrak{L}_{\mathbb{R}}^1(I, \mathcal{P}(I), \mu)$. Prove that, if \mathbb{K} is either \mathbb{R} or \mathbb{C} , then

$$\mathfrak{L}_{\mathbb{K}}^1(I, \mathcal{P}(I), \mu) = \ell_{\mathbb{K}}^1(I),$$

the Banach space discussed in II.2 and II.3.

Exercise 2. There is an instance when the entire theory developed here is essentially vacuous. Let X be a non-empty set, and let \mathcal{A} be a σ -algebra on X . For a measure μ on \mathcal{A} , prove that the following are equivalent

- (i) $\mathfrak{L}_+^1(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ measurable, and } f = 0, \mu\text{-a.e.}\}$;
- (ii) for every $A \in \mathcal{A}$, one has $\mu(A) \in \{0, \infty\}$.

A measure space (X, \mathcal{A}, μ) , with property (ii), is said to be *degenerate*.

Exercise 3 \diamond . Let (X, \mathcal{A}, μ) be a measure space, and let $f : X \rightarrow [0, \infty]$ be a measurable function, with $\int_X f d\mu = 0$. Prove that $f = 0, \mu$ -a.e.

HINT: Define the measurable sets $A_n = \{x \in X : f(x) \geq \frac{1}{n}\}$, and analyze the relationship between f and χ_{A_n} .