Lectures 30-31

8. Signed measures and complex measures

In this section we discuss a generalization of the notion of a measure, to the case where the values are allowed to be outside $[0, \infty]$. The first notion is described by the following.

Definition. Suppose $\mathcal{A}$ is a σ-algebra on a non-empty set X. A function $\mu : \mathcal{A} \to [-\infty, \infty]$ is called a *signed measure on* $\mathcal{A}$, if it has the properties below.

(i) Either one of the following is true
- $\mu(A) < \infty$, $\forall\, A \in \mathcal{A}$;
- $\mu(A) > -\infty$, $\forall\, A \in \mathcal{A}$.

(ii) $\mu(\varnothing) = 0$.

(iii) For any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, one has the equality

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n). \tag{1}$$

Here we adopt the convention that if one term in the right hand side of (1) is equal to $\pm\infty$, then the entire sum is equal to $\pm\infty$. It is important to use condition (i), which avoids situations when one term is ∞ and another term is $-\infty$.

Examples 8.1. Let us agree, in this section only, to use the term "honest" measure, for a measure in the usual sense.

A. Any "honest" measure is of course a signed measure.

B. If μ is a signed measure, then $-\mu$ is again a signed measure.

C. If μ_1 and μ_2 are "honest" measures, one of which is *finite*, then $\mu_1 - \mu_2$ is a signed measure. Eventually (see Theorem 8.2) we are going to show that any signed measure can be written in this form.

One key technical result about signed measures is the following.

Theorem 8.1. *Let $\mathcal{A}$ be a σ-algebra on a non-empty set X, and let μ be a signed measure on X. Then there exist sets $L, U \in \mathcal{A}$, such that*

$$\mu(L) = \inf\big\{\mu(A) \,:\, A \in \mathcal{A}\big\}; \tag{2}$$

$$\mu(M) = \sup\big\{\mu(A) \,:\, A \in \mathcal{A}\big\}. \tag{3}$$

Proof. Since $-\mu$ is also a signed measure, it suffices to prove only the existence of M satisfying (3). Denote the right hand side of (3) by α, and choose a sequence $(\alpha_n)_{n\geq 1} \subset \mathbb{R}$, such that $\lim_{n\to\infty} \alpha_n = \alpha$, and $\alpha_n < \alpha$, $\forall\, n \geq 1$. The key construction we need is contained in the following.

Claim 1: *There exists a family of sets $\{B_k^n \,:\, k, n \in \mathbb{N},\ 1 \leq k \leq n\} \subset \mathcal{A}$, with the following properties:*

(i) *for every $n \geq 1$, one has the inclusions*

$$\begin{array}{ccccccccc} B_1^n & \subset & B_2^n & \subset & \dots & \subset & B_n^n & & \\ \cup & & \cup & & \dots & & \cup & & \\ B_1^{n+1} & \subset & B_2^{n+1} & \subset & \dots & \subset & B_n^{n+1} & \subset & B_{n+1}^{n+1} \end{array}$$

(ii) *for every $k \geq 1$ one has the inequalities*

$$\mu(B_k^n \smallsetminus B_k^{n+1}) \leq 0, \ \forall\, n \geq k.$$

(iii) $\mu(B_n^n) \geq \alpha_n$, $\forall\, n \geq 1$.

We construct this sequence inductively, one row at a time (the rows ar indexed by the upper index n). Choose $B_1^1 \in \mathcal{A}$ to be any set with $\mu(B_1^1) \geq \alpha_1$. Suppose we have constructed the first N rows, i.e. we have defined the sets B_k^n, $1 \leq k \leq n \leq m$, so that property (i) holds for all $n = 1, \dots, m-1$, property (ii) holds in the form

$$\alpha_k \leq \mu(B_k^k) \leq \mu(B_k^{k+1}) \leq \dots \leq \mu(B_k^m), \ \forall\, k = 1, \dots, m,$$

and property (iii) holdes for all $n = 1, \dots, m$. Let us explain now how the next row $B_1^{m+1} \subset B_2^{m+1} \subset \dots B_m^{m+1} \subset B_{m+1}^{m+1}$ is constructed. Define the sets $E_1, E_2, \dots, E_m \in \mathcal{A}$ by

$$E_1 = B_1^m, \text{ and } E_k = B_k^m \smallsetminus B_{k-1}^m, \ \forall\, k = 2, \dots, m.$$

The sets E_k, $k = 1, \dots, m$ are pairwise disjoint, and we have

$$B_k^m = \bigcup_{j=1}^{k} E_j, \ \forall\, k = 1, \dots, m.$$

Choose now an arbitrary set $D \in \mathcal{A}$, with $\mu(D) \geq \alpha_{m+1}$, and define, for each $j \in \{1, \dots, m\}$, the set

$$G_j = \begin{cases} E_j & \text{if } \mu(E_j \smallsetminus D) > 0 \\ E_j \cap D & \text{if } \mu(E_j \smallsetminus D) \leq 0 \end{cases}$$

Notice that we have $E_j \supset G_j$, and using the equality $\mu(E_j) = \mu(E_j \cap D) + \mu(E_j \smallsetminus D)$, we also have

$$\mu(E_j \smallsetminus G_j) \leq 0 \text{ and } \mu(G_j) \geq \mu(E_j \cap D), \ \forall\, j = 1, \dots, m. \tag{4}$$

Define also the set $G_{m+1} = D \smallsetminus B_m^m$. It is clear that the sets $G_1, G_2, \dots, G_{m+1}$ are pairwise disjoint. Construct now the $m+1$ row by taking

$$B_k^{m+1} = \bigcup_{j=1}^{k} G_j, \ \forall\, k = 1, 2, \dots, m+1.$$

It is obvious that one has the inclusions

$$B_1^{m+1} \subset B_2^{m+1} \subset \dots \subset B_{m+1}^{m+1}.$$

Since $E_k \supset G_k$, $\forall\, k = 1, \dots, m$, it is also clear that we have the vertical inclusions $B_k^m \supset B_k^{m+1}$, $\forall\, k = 1, \dots, m$. Using (4), for each $k = 1, \dots, m$, we have

$$\mu(B_k^m \smallsetminus B_k^{m+1}) = \mu\big(\bigcup_{j=1}^{k} [E_j \smallsetminus G_j]\big) = \sum_{j=1}^{k} [\mu(E_j \smallsetminus G_j) \leq 0.$$

Finally, again by (4), we have

$$\mu(B_{m+1}^{m+1}) = \mu\big(\bigcup_{k=1}^{m+1} G_k\big) = \sum_{k=1}^{m+1} \mu(G_k) \geq \mu(G_{m+1}) + \sum_{k=1}^{m} \mu(E_k \cap D) =$$

$$= \mu(G_{m+1}) + \mu\big(\bigcup_{k=1}^{m} [E_k \cap D]\big) = \mu(D \smallsetminus B_m^m) + \mu(B_m^m \cap D) = \mu(D) \geq \alpha_{m+1}.$$

Claim 2: *There exists a sequence* $(A_k)_{k=1}^{\infty} \subset \mathcal{A}$, *such that*

(i) $A_1 \subset A_2 \subset A_3 \subset \ldots$;

(ii) $\mu(A_k) \geq \alpha_k$, $\forall\, k \geq 1$.

We fix a family $\{B_k^n \,:\, k, n \in \mathbb{N},\ 1 \leq k \leq n\}$ satisfying the properties in Claim 1. For every $k \geq 1$, we define $A_k = \bigcap_{n=k}^{\infty} B_k^n$. Notice that, using property (i) from Claim 1 (the vertical inclusions), we have

$$B_k^k = A_k \cup \big(\bigcup_{n=k}^{\infty} [B_k^n \smallsetminus B_k^{n+1}]\big),$$

and the sets $A_k, B_k^n \smallsetminus B_k^{n+1}$, $n \geq k$, are pairwise disjoint, so using property (ii) from Claim 1, we have

$$\mu(B_k^k) = \mu(A_k) + \sum_{n=k}^{\infty} \mu(B_k^n \smallsetminus B_k^{n+1}) \leq \mu(A_k).$$

Using property (iii) from Claim 1, we then get $\mu(A_k) \geq \alpha_k$. The fact that we have the inclusions $A_1 \subset A_2 \subset \ldots$ is clear, from property (i) in Claim 1 (the horizontal inclusions).

Fix now the sequence $(A_k)_{k=1}^{\infty} \subset \mathcal{A}$ as in Claim 2, and let us consider the set $M = \bigcup_{k=1}^{\infty} A_k$. If we define the sets

$$M_1 = A_1 \text{ and } M_k = A_k \smallsetminus A_{k-1},\ \ \forall\, k \geq 2,$$

then we have $M = \bigcup_{k=1}^{\infty} M_k$, and the sets $M_1, M_2, M_3, \ldots$ are pairwise disjoint. In particular, this gives

$$\mu(M) = \sum_{k=1}^{\infty} \mu(M_k) = \lim_{k\to\infty} \big[\sum_{j=1}^{k} \mu(M_j)\big] = \lim_{k\to\infty} \mu\big(\bigcup_{j=1}^{k} M_j\big).$$

Since we obviously have $\bigcup_{j=1}^{k} M_j = A_k$, $\forall\, k \geq 1$, the above equality proves that

$$\mu(M) = \lim_{k\to\infty} \mu(A_k). \tag{5}$$

Since we have $\alpha_k \leq \mu(A_k) \leq \alpha$, $\forall\, k \geq 1$, as well as $\lim_{k\to\infty} \alpha_k = \alpha$, the equality (5) forces $\mu(M) = \alpha$. □

Remark 8.1. One interesting application of the above result is the fact that, whenever μ is a signed measure on $\mathcal{A}$, such that

$$-\infty < \mu(A) < \infty,\ \ \forall\, A \in \mathcal{A}, \tag{6}$$

then

$$-\infty < \inf\{\mu(A) \,:\, A \in \mathcal{A}\} \leq \sup\{\mu(A) \,:\, A \in \mathcal{A}\} < \infty.$$

A signed measure with property (6) is called *finite*.

We are now in position to prove the statement made in Example 8.1.C.

THEOREM 8.2. *Let X be a non-empty set, let $\mathcal{A}$ be a σ-algebra on X, and let μ be a signed measure on $\mathcal{A}$. Then there exist subsets $X^+, X^- \in \mathcal{A}$, with the following properties:*

(i) *$X^+ \cap X^- = \varnothing$, and $X^+ \cup X^- = X$;*
(ii) *the maps $\mu^\pm : \mathcal{A} \to [-\infty, \infty]$, defined by*

$$\mu^\pm(A) = \pm\mu(A \cap X^\pm), \ \ \forall\, A \in \mathcal{A},$$

are "honest" measures on $\mathcal{A}$;
(iii) *one of the measures $\mu^\pm$ is finite, and one has the equality $\mu = \mu^+ - \mu^-$.*

PROOF. Without any loss of generality, we can assume that

$$\mu(A) < \infty, \ \ \forall\, A \in \mathcal{A}.$$

(Otherwise, we replace μ with $-\mu$, and the conclusion does not essentially change.) Put $\alpha = \sup\big\{\mu(A) \,:\, A \in \mathcal{A}\big\}$. By Theorem 8.1, it follows that $0 \le \alpha < \infty$, and there exists a set $X^+ \in \mathcal{A}$, such that $\mu(X^+) = \alpha$. Define $X^- = X \smallsetminus X^+$.

Claim: *The sets $X^\pm$ have the following properties:*
(a) *$0 \le \mu(A) \le \alpha$, for all $A \in \mathcal{A}$, with $A \subset X^+$;*
(b) *$0 \ge \mu(B)$, for all $B \in \mathcal{A}$, with $B \subset X^-$.*

To prove (a) start with some arbitrary subset $A \subset X^+$. First of all, by the definition of α, it is clear that $\mu(A) \le \alpha$. Second, using the equality $\mu(X^+) = \mu(A) + \mu(X^+ \smallsetminus A)$, it is clear that $\mu(A), \mu(X^+ \smallsetminus A) > -\infty$, so we have

$$\alpha \ge \mu(X^+ \smallsetminus A) = \mu(X^+) - \mu(A) = \alpha - \mu(A),$$

which clearly forces $\mu(A) \ge 0$. To prove (b), we start with some set $B \in \mathcal{A}$ with $B \subset X^-$. Using the fact that $X^+ \cap B = \varnothing$, we have

$$\alpha \ge \mu(X^+ \cup B) = \mu(X^+) + \mu(B) = \alpha + \mu(B),$$

which clearly forces $\mu(B) \le 0$.

Having proven the Claim, we define the maps $\mu^\pm : \mathcal{A} \to [-\infty, \infty]$ as in the statement of the Theorem. By the Claim, we get $\mu^\pm(A) \ge 0$, $\forall\, A \in \mathcal{A}$. It is also pretty clear that both μ^+ and μ^- are σ-additive, so they define "honest" measures. Also, by the Claim, we have $\mu^+(A) \le \alpha$, $\forall\, A \in \mathcal{A}$, so μ^+ is a finite measure. Finally, if we start with some arbitrary $A \in \mathcal{A}$, and we write it as $A = (A \cap X^+) \cup (A \cap X^-)$, then using the fact that $(A \cap X^+) \cap (A \cap X^-) = \varnothing$, we get

$$\mu(A) = \mu(A \cap X^+) + \mu(A \cap X^-) = \mu^+(A) - \mu^-(A). \quad \square$$

It will be helpful not only here, but also in some future discussions, to isolate a certain feature identified by the above result.

DEFINITION. Given a σ-algebra $\mathcal{A}$ on a non-empty set X, and two "honest" measures μ and ν on $\mathcal{A}$, we say that μ and ν are *mutually singular*, if there exists sets $M, N \in \mathcal{A}$, with $M \cup N = X$ and $M \cap N = \varnothing$, such that $\mu(N) = \nu(M) = 0$. Notice that this implies the equalities

$$\mu(A) = \mu(A \cap M) \text{ and } \nu(A) = \nu(A \cap N), \ \ \forall\, A \in \mathcal{A}.$$

If this situation occurs, we write $\mu \perp \nu$.

With this terminology, Theorem 8.2 states that any signed measure μ can be written as $\mu = \mu^+ - \mu^-$, with μ^+ and μ^- "honest" mutually singular measures, and one of them finite.

Although the sets $X^{\pm}$ may not be uniquely determined, the decomposition $\mu = \mu^+ - \mu^-$ is unique, as indicated by the following result.

THEOREM 8.3 (Minimality). *Let X be a non-empty set, let $\mathcal{A}$ be a σ-algebra on X, and let μ be a signed measure on $\mathcal{A}$. Suppose μ^+ and μ^- are mutually singular "honest" measures on $\mathcal{A}$, one of them being finite, such that $\mu = \mu^+ - \mu^-$. Suppose ν and η are two "honest" measures on $\mathcal{A}$, one of which being finite, such that $\mu = \nu - \eta$. Then one has the inequalities $\mu^+ \leq \nu$ and $\mu^- \leq \eta$.*

PROOF. Fix sets $X^+, X^- \in \mathcal{A}$, such that $X^+ \cup X^- = X$, $X^+ \cap X^- = \varnothing$, and $\mu^+(X^-) = \mu^-(X^+) = 0$.

Start with some arbitrary set $A \in \mathcal{A}$. On the one hand, since $A = (A \cap X^+) \cup (A \cap X^-)$, with $A = (A \cap X^+) \cap (A \cap X^-) = \varnothing$, we see that if λ is either one of the measures μ, μ^+ ,or μ^-, we have the equality

$$\lambda(A) = \lambda(A \cap X^+) + \lambda(A \cap X^-), \ \ \forall\, A \in \mathcal{A}. \tag{7}$$

On the other hand, since μ^+ is an "honest" measure, and $\mu^+(X^-) = 0$, the inclusion $A \cap X^- \subset X^-$ will force

$$\mu^+(A \cap X^-) = 0, \ \ \forall\, A \in \mathcal{A}.$$

Likewise, we have the equality

$$\mu^-(A \cap X^+) = 0, \ \ \forall\, A \in \mathcal{A}.$$

These equalities, combined with $\mu = \mu^+ - \mu^-$, and with (7), give the equalities

$$\mu^+(A) = \mu^+(A \cap X^+) = \mu^+(A \cap X^+) - \mu^-(A \cap X^+) = \mu(A \cap X^+), \tag{8}$$

$$\mu^-(A) = \mu^-(A \cap X^-) = -\mu^+(A \cap X^-) + \mu^-(A \cap X^-) = -\mu(A \cap X^-), \tag{9}$$

for all $A \in \mathcal{A}$. Fix now some set $A \in \mathcal{A}$. Since ν is an "honest" measure, and $\eta(A \cap X^+) \geq 0$, using (8) we get

$$\nu(A) \geq \nu(A \cap X^+) \geq \nu(A \cap X^+) - \eta(A \cap X^-) = \mu(A \cap X^+) = \mu^+(A).$$

Likewise, we have

$$\eta(A) \geq \eta(A \cap X^-) \geq \eta(A \cap X^-) - \nu(A \cap X^-) = -\mu(A \cap X^-) = \mu^-(A). \quad \square$$

COROLLARY 8.1. *Let $\mathcal{A}$ be a σ-algebra on X, let μ be a signed measure on $\mathcal{A}$, and let μ^+, μ^-, ν^+ and ν^- be "honest" measures on $\mathcal{A}$ with*

- *$\mu^+ \perp \mu^-$, and one of the measures μ^+ and μ^- is finite;*
- *$\nu^+ \perp \nu^-$, and one of the measures ν^+ and ν^- is finite;*
- *$\mu = \mu^+ - \mu^- = \nu^+ - \nu^-$.*

Then one has the equalities $\mu^+ = \nu^+$ and $\mu^- = \nu^-$.

PROOF. Apply Theorem 8.3 "both ways" to get $\mu^+ \leq \nu^+$ and $\mu^- \leq \nu^-$, as well as $\nu^+ \leq \mu^+$ and $\nu^- \leq \mu^-$. $\square$

DEFINITION. Given a signed measure μ, the decomposition $\mu = \mu^+ - \mu^-$, whose existence is shown in Theorem 8.2, and whose uniqueness is shown above, is called the *Hahn-Jordan decomposition of μ*. A pair of sets (X^+, X^-), with $X^{\pm} \in \mathcal{A}$, $X^+ \cup X^- = X$, $X^+ \cap X^- = \varnothing$, and $\mu^+(X^-) = \mu^-(X^+) = 0$, is called a *Hahn-Jordan set decomposition of X relative to μ*.

Exercise 1. Let μ be a signed measure, and let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition. Prove that the following are equivalent

(i) μ is finite, i.e. $-\infty < \mu(A) < \infty$, $\forall\, A \in \mathcal{A}$;
(ii) both "honest" measures μ^+ and μ^- are finite.

The following result characterizes mutual singularity in an approximate fashion.

LEMMA 8.1. *Let $\mathcal{A}$ be a σ-algebra on X, and let μ and ν be "honest" measures on $\mathcal{A}$. The following are equivalent*

(i) *$\mu \perp \nu$;*
(ii) *for every $\varepsilon > 0$, there exist sets $D, E \in \mathcal{A}$, such that $\mu(D) < \varepsilon$, $\nu(E) < \varepsilon$, and $D \cup E = X$.*

PROOF. The implication $(i) \Rightarrow (ii)$ is trivial.

To prove the implication $(ii) \Rightarrow (i)$ construct, for each $\varepsilon > 0$, two sequences $(D_n^\varepsilon)_{n=1}^\infty$ and $(E_n^\varepsilon)_{n=1}^\infty$ of sets in $\mathcal{A}$, such that $\mu(D_n^\varepsilon) < \varepsilon/2^n$, $\nu(E_n^\varepsilon) < \varepsilon/2^n$, and $D_n^\varepsilon \cup E_n^\varepsilon = X$. Put $A_\varepsilon = \bigcap_{n=1}^\infty D_n^\varepsilon$ and $B_\varepsilon = \bigcup_{n=1}^\infty E_n^\varepsilon$. Fix for the moment $\varepsilon > 0$. On the one hand, using the inclusion $A_\varepsilon \subset D_n^\varepsilon$, $\forall\, n \geq 1$, we get $\mu(A_\varepsilon) \leq \varepsilon/2^n$, $\forall\, n \geq 1$, which clearly forces

$$\mu(A_\varepsilon) = 0. \tag{10}$$

On the other hand, using σ-subadditivity, we have

$$\nu(B_\varepsilon) = \nu\big(\bigcup_{n=1}^\infty E_n^\varepsilon\big) \leq \sum_{n=1}^\infty \nu(E_n^\varepsilon) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon. \tag{11}$$

Finally, since we have, $X \smallsetminus D_n^\varepsilon \subset E_n^\varepsilon$, $\forall\, n \geq 1$, we get

$$X \smallsetminus A_\varepsilon = \bigcup_{n=1}^\infty (X \smallsetminus D_n^\varepsilon) \subset \bigcup_{n=1}^\infty E_n^\varepsilon = B_\varepsilon,$$

which gives

$$A_\varepsilon \supset X \smallsetminus B_\varepsilon. \tag{12}$$

Define now the sets $N = \bigcup_{n=1}^\infty A_{1/n}$ and $M = X \smallsetminus N$. On the one hand, using σ-subadditivity, combined with (10), we get $\mu(N) = 0$. On the other hand, using (12), we have

$$M = X \smallsetminus N = X \smallsetminus \big(\bigcup_{n=1}^\infty A_{1/n}\big) = \bigcap_{n=1}^\infty (X \smallsetminus A_{1/n}) \subset \bigcap_{n=1}^\infty B_{1/n} \subset B_{1/k},\ \ \forall\, k \geq 1,$$

which forces $\nu(M) = 0$. □

Although the next technical result seems a bit out of context at this point, we prove it here, and record it for future use.

LEMMA 8.2. *Let $\mathcal{A}$ be a σ-algebra on some non-empty set X, and let μ, η be signed measures on $\mathcal{A}$. Assume there is an "honest" finite measure ν on $\mathcal{A}$, with $\mu + \nu = \eta$.*

(i) *If $\mu = \mu^+ - \mu^-$ and $\eta = \eta^+ - \eta^-$ are the Hahn-Jordan decompositions of μ and η respectively, then one has the inequalities*

$$\mu^+ \leq \eta^+ \leq \mu^+ + \nu \tag{13}$$

$$\eta^- \leq \mu^- \leq \eta^- + \nu. \tag{14}$$

(ii) *If* (X^+, X^-) *is a Hahn-Jordan set decomposition of* X *relative to* μ*, and if* (Y^+, Y^-) *is a Hahn-Jordan set decomposition of* X *relative to* η*, then one has the relations* $X^+ \underset{\nu}{\subset} Y^+$ *and* $Y^- \underset{\nu}{\subset} X^-$*.*

Proof. On the one hand, the signed measure η has a decomposition

$$\eta = \mu + \nu = (\mu^+ + \nu) - \mu^-,$$

with $\mu^+ + \nu$ and μ^- "honest" measures (one of them finite). Using the minimality Theorem 8.3, we get the inequalities

$$\eta^+ \leq \mu^+ + \nu \text{ and } \eta^- \leq \mu^-. \tag{15}$$

On the other hand, we can also consider the signed measure $\mu = \eta - \nu$, which has a decomposition

$$\mu = \eta^+ - (\eta^- + \nu),$$

with η^+ and $\eta^- + \nu$ "honest" measures (one of them finite). Using again the minimality Theorem 8.3, we get the inequalities

$$\mu^+ \leq \eta^+ \text{ and } \mu^- \leq \eta^- + \nu. \tag{16}$$

Clearly the inequalities (15) and (16) cover the desired inequalities (13) and (14)

(ii). Recall (see Section 4) that the relation $A \underset{\nu}{\subset} B$ means that $\nu(A \smallsetminus B) = 0$. In our case, we have to look at the set

$$N = X^+ \smallsetminus Y^+ = Y^- \smallsetminus X^-,$$

for which we have to show that $\nu(N) = 0$. On the one hand, since $N \subset Y^-$, we get $\eta^+(N) = 0$. Using (13) this forces $\mu^+(N) = 0$. On the other hand, since $N \subset X^+$, we get $\mu^-(N) = 0$, and using (14) we also get $\eta^-(N) = 0$. In other words, we get the equalities

$$\mu(N) = \mu^+(N) - \mu^-(N) = 0,$$
$$\eta(N) = \eta^+(N) - \eta^-(N) = 0,$$

and then the equality $\eta = \mu + \nu$ clearly forces $\nu(N) = 0$. □

The Hahn-Jordan decomposition has the following interesting application to the properties of the natural order relation on "honest" measures. The result below gives the existence of a "infimum" and a "supremum" for a pair of finite "honest" measures.

Proposition 8.1 (Lattice Property). *Let* $\mathcal{A}$ *be a* σ*-algebra on a non-empty set* X*, and let* μ *and* ν *be "honest" measures on* $\mathcal{A}$*, with one of them finite.*

(i) *There exists a unique measure* $\mu \vee \nu$ *with:*
  (a) $\mu \vee \nu \geq \mu$ *and* $\mu \vee \nu \geq \nu$*;*
  (b) *whenever* ω *is an "honest" measure on* $\mathcal{A}$*, with* $\mu \leq \omega$ *and* $\nu \leq \omega$*, it follows that one has the inequality* $\mu \vee \nu \leq \omega$*.*

(ii) *There exists a unique measure* $\mu \wedge \nu$ *with:*
  (a) $\mu \geq \mu \wedge \nu$ *and* $\nu \geq \mu \wedge \nu$*;*
  (b) *whenever* λ *is an "honest" measure on* $\mathcal{A}$*, with* $\mu \geq \lambda$ *and* $\nu \geq \lambda$*, it follows that one has the inequality* $\mu \wedge \nu \geq \lambda$*.*

Proof. Since the statement of the Theorem is "symmetric," without any loss of generality we can assume that μ is finite.

Consider the signed measure $\eta = \mu - \nu$, and its Hahn-Jordan decomposition $\eta = \eta^+ - \eta^-$. Let (X^+, X^-) be a Hahn-Jordan set decomposition of X relative to η. This means that, for every $A \in \mathcal{A}$, one has

$$0 \leq \eta^+(A) = \eta(A \cap X^+) = \mu(A \cap X^+) - \nu(A \cap X^+); \tag{17}$$

$$0 \leq \eta^-(A) = -\eta(A \cap X^-) = \nu(A \cap X^-) - \mu(A \cap X^-). \tag{18}$$

In particular we get

$$\mu(A \cap X^+) \geq \nu(A \cap X^+) \text{ and } \mu(A \cap X^-) \leq \nu(A \cap X^-), \ \forall\, A \in \mathcal{A}. \tag{19}$$

(i). Define the measure $\mu \vee \nu = \mu + \eta^-$. Using (18) we have

$$(\mu \vee \nu)(A) = \mu(A \cap X^+) + \nu(A \cap X^-), \ \forall\, A \in \mathcal{A}. \tag{20}$$

Notice that, using (19), it follows that, for every $A \in \mathcal{A}$, one has the inequalities

$$(\mu \vee \nu)(A \cap X^+) = \mu(A \cap X^+) \geq \nu(A \cap X^+),$$
$$(\mu \vee \nu)(A \cap X^-) = \nu(A \cap X^-) \geq \mu(A \cap X^-),$$

In particular, this gives

$$(\mu \vee \nu)(A) = (\mu \vee \nu)(A \cap X^+) + (\mu \vee \nu)(A \cap X^-) \geq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),$$
$$(\mu \vee \nu)(A) = (\mu \vee \nu)(A \cap X^+) + (\mu \vee \nu)(A \cap X^-) \geq \nu(A \cap X^+) + \nu(A \cap X^-) = \mu(A),$$

for every $A \in \mathcal{A}$, so $\mu \vee \nu$ indeed has property (a).

To prove property (b), start with some "honest" measure ω on $\mathcal{A}$, with $\mu, \nu \leq \omega$, and let us show that $\mu \vee \nu \leq \omega$. This is quite clear, since for any $A \in \mathcal{A}$, using (20) we have

$$\omega(A) = \omega(A \cap X^+) + \omega(A \cap X^-) \geq \mu(A \cap X^+) + \nu(A \cap X^-) = (\mu \vee \nu)(A).$$

The uniqueness of $\mu \vee \nu$ is now clear from (a) and (b).

(ii). Remark that, using the Minimality Theorem 8.3, for the measure $\eta = \mu - \nu$, it follows that $\eta^+ \leq \mu$. In particular, η^+ is a finite "honest" measure, and so is the difference $\mu - \eta^+$. Put $\mu \wedge \nu = \mu - \eta^+$. Using (17) we have

$$(\mu \wedge \nu)(A) = \mu(A \cap X^-) + \nu(A \cap X^+), \ \forall\, A \in \mathcal{A}. \tag{21}$$

Notice that, using (19), it follows that, for every $A \in \mathcal{A}$, one has the inequalities

$$(\mu \wedge \nu)(A \cap X^+) = \nu(A \cap X^+) \leq \mu(A \cap X^+),$$
$$(\mu \wedge \nu)(A \cap X^-) = \mu(A \cap X^-) \geq \nu(A \cap X^-),$$

In particular, this gives

$$(\mu \wedge \nu)(A) = (\mu \wedge \nu)(A \cap X^+) + (\mu \wedge \nu)(A \cap X^-) \leq \mu(A \cap X^+) + \mu(A \cap X^-) = \mu(A),$$
$$(\mu \wedge \nu)(A) = (\mu \wedge \nu)(A \cap X^+) + (\mu \wedge \nu)(A \cap X^-) \leq \nu(A \cap X^+) + \nu(A \cap X^-) = \mu(A),$$

for every $A \in \mathcal{A}$, so $\mu \wedge \nu$ indeed has property (a).

To prove property (b), start with some "honest" measure λ on $\mathcal{A}$, with $\mu, \nu \leq \omega$, and let us show that $\mu \wedge \nu \geq \lambda$. This is quite clear, since for any $A \in \mathcal{A}$, using (21) we have

$$\lambda(A) = \lambda(A \cap X^+) + \omega(A \cap X^-) \leq \nu(A \cap X^+) + \mu(A \cap X^-) = (\mu \wedge \nu)(A).$$

The uniqueness of $\mu \wedge \nu$ is now clear from (a) and (b). $\square$

We conclude with a series of results that make a connection with the theory of Radon measures discussed in Section 7.

DEFINITION. Suppose X is a locally compact space, and μ is a signed measure on $Bor(X)$. We call μ a *signed Radon measure on* X, if there exist "honest" Radon measures ν and η on X, one of which is finite, such that $\mu = \nu - \eta$.

*Exercise 2**. Let X be a locally compact space, and let μ be a signed measure on $Bor(X)$. Prove that the following are equivalent:

(i) μ is a signed Radon measure on X;
(ii) if $\mu = \mu^+ - \mu^-$ denotes the Hahn-Jordan decomposition of μ, then both μ^+ and μ^- are Radon measures on X.

HINT: To prove the implication $(i) \Rightarrow (ii)$ use the fact that $\mu^+ \leq \nu$ and $\mu^- \leq \eta$. Moreover, show that, for any $B \in Bor(X)$, one has the implications $\mu^+(B) < \infty \Rightarrow \nu(B) < \infty$ and $\mu^-(B) < \infty \Rightarrow \eta(B) < \infty$. Then use Exercise 5 from Section 7.

REMARK 8.2. Suppose X is a locally compact space. In Section 7 we discussed the *Riesz correpsondence*, which associates to each linear positive map $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$, a Radon measure μ_ϕ on X. As already suggested, this correspondence is in fact a bijection, although the proof of this fact will come later in Chapter IV. At this point we would like to analyze the Riesz correspondence in a simpler situation, namely the case when X is *compact*. In this case it is interesting to point out that Riesz correspondence can be extended beyond the positive case. The key fact (see Corollary II.5.3) is that every linear *continuous* map $\phi : C^{\mathbb{R}}(X) \to \mathbb{R}$ can be written as a difference $\phi = \phi_1 - \phi_2$, with $\phi_1, \phi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$ *positive* linear maps. (In fact ϕ_1 and ϕ_2 can be chosen such that $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. This fact will be heavily exploited a little later.) We would like then to define a finite *signed* Radon measure μ_ϕ by the formula $\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}$. There is a minor problem here: *What if we find another pair of continuous positive linear maps $\psi_1, \psi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$, such that $\phi = \psi_1 - \psi_2$? Is is true that $\mu_{\psi_1} - \mu_{\psi_2} = \mu_{\phi_1} - \mu_{\phi_2}$?* The answer is affirmative, and this is an easy consequence of Proposition 7.6, which gives the equalities

$$\mu_{\phi_1} + \mu_{\psi_2} = \mu_{\phi_1+\psi_2} = \mu_{\psi_1+\phi_2} = \mu_{\psi_1} + \mu_{\phi_2}.$$

NOTATIONS. Suppose X is a compact Hausdorff space. We define

$$\mathcal{M}^{\mathbb{R}}(X) = \big\{\phi : C^{\mathbb{R}}(X) \to \mathbb{R} \,:\, \phi \ \mathbb{R}\text{-linear continuous}\big\},$$
$$\mathfrak{R}^{\mathbb{R}}(X) = \big\{\mu \text{ signed Radon measure on } X\big\}.$$

The correspondence

$$\mathcal{M}^{\mathbb{R}}(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}^{\mathbb{R}}(X) \tag{22}$$

defined above, will still be referred to as the *extended Riesz correspondence*.

REMARK 8.3. If X is a compact Hausdorff space, then the extended Riesz correspondence (22) is a linear map. This is a consequence of Proposition 7.6.

Given $\phi \in \mathcal{M}^{\mathbb{R}}(X)$, the existence of a decomposition of ϕ, of the particular type described in Corollary II.5.3, is extremely significant, as suggested by the following result.

THEOREM 8.4. *Let X be a compact Hausdorff space, let $\phi_1, \phi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$ be positive linear maps, and let μ_{ϕ_1} and μ_{ϕ_2} be the corresponding Riesz measures. Consider the linear continuous map $\phi = \phi_1 - \phi_2$, and the finite signed measure*

$$\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}. \tag{23}$$

If $\|\phi\| = \|\phi_1\| + \|\phi_2\|$, then $\mu_{\phi_1} \perp \mu_{\phi_2}$, so (23) *represents the Hahn-Jordan decomposition of μ_ϕ.*

Proof. We are going to show that the decomposition (23) satisfies condition (ii) in Lemma 8.1. The key step in proving this fact is contained in the following.

Claim: *For every $\varepsilon > 0$, there exist functions $f_1, f_2 \in C^{\mathbb{R}}(X)$, with $f_1, f_2 \geq 0$, $f_1 + f_2 \geq 1$, and such that $\phi_1(f_2) < \varepsilon$ and $\phi_2(f_1) < \varepsilon$.*

To prove this we fix $\varepsilon > 0$, and we use the definition of the norm, to find some function $g \in C^{\mathbb{R}}(X)$, with $\|g\| \leq 1$, and $|\phi(g)| \geq \|\phi\| - \varepsilon$. Replacing g with $-g$, if necessary, we can assume that

$$\phi(g) \geq \|\phi\| - \varepsilon. \tag{24}$$

Consider the functions $g^+ = \max\{g, 0\}$ and $g^- = \max\{-g, 0\}$, so that $g = g^+ - g^-$, and we clearly have $0 \leq g^\pm \leq 1$. On the one hand, since $\|\phi_k\| = \phi_k(1)$ (see Proposition II.5.4), we have $\phi_k(g^\pm) \leq \|\phi_k$, $k = 1, 2$. On the other hand, by (24), and the positivity of ϕ_1 and ϕ_2, we know that

$$\begin{aligned}\|\phi\| - \varepsilon \leq \phi(g) = \phi_1(g) - \phi_2(g) = \phi_1(g^+) + \phi_2(g^-) - \phi_1(g^-) - \phi_2(g^+) \leq \\ \leq \phi_1(g^+) + \phi_2(g^-) \leq \|\phi_1\| + \|\phi_2\| = \|\phi\|,\end{aligned}$$

so we get

$$\begin{aligned}\varepsilon \geq \|\phi\| - \phi_1(g^+) - \phi_2(g^-) = \|\phi_1\| + \|\phi_2\| - \phi_1(g^+) - \phi_2(g^-) = \\ = \phi_1(1) + \phi_2(1) - \phi_1(g^+) - \phi_2(g^-) = \phi_1(1 - g^+) + \phi_2(1 - g^-).\end{aligned}$$

If we define $f_1 = 1 - g^-$ and $f_2 = 1 - g^+$, then it is clear that $f_1, f_2 \geq 0$. Using the fact that $g^+ + g^- = |g| \leq 1$, we get $f_1 + f_2 = 2 - |g| \geq 1$. Finally, the above estimate gives $\phi_1(f_2) + \phi_2(f_1) \leq \varepsilon$, and so the Claim immediately follows.

Having proven the Claim, we are now in position to prove that the two measures μ_{ϕ_1} and μ_{ϕ_2} satisfy condition (ii) in Lemma 8.1. Start with some arbitrary $\varepsilon > 0$, and use the Claim to find two functions $f_1, f_2 \in C^{\mathbb{R}}(X)$ with $f_1, f_2 \geq 0$, $f_1 + f_2 \geq 1$, such that $\phi_1(f_2) \leq \varepsilon/2$ and $\phi_2(f_1) \leq \varepsilon/2$. Consider the compact subsets

$$K_1 = \big\{x \in X \,:\, f_1(x) \geq \frac{1}{2}\big\} \text{ and } K_2 = \big\{x \in X \,:\, f_2(x) \geq \frac{1}{2}\big\}.$$

Since $f_1 + f_2 \geq 1$, it follows immediately that we have $K_1 \cup K_2 = X$. By construction, we have $2f_1 \geq \varkappa_{K_1}$ and $2f_2 \geq \varkappa_{K_2}$, so using the interpolation property (see Proposition 7.5), we get

$$\begin{aligned}\mu_{\phi_1}(K_2) \leq \phi_1(2f_2) = 2\phi_1(f_2) \leq \varepsilon; \\ \mu_{\phi_2}(K_1) \leq \phi_2(2f_1) = 2\phi_2(f_1) \leq \varepsilon.\end{aligned}$$

□

The above result has several interesting consequences.

Corollary 8.2. *Suppose X is a compcat Hausdorff space. Then the extended Riesz correspondence* (22) *is injective.*

Proof. Since the correspondence (22) is linear, is suffices to prove the implication $\mu_\phi = 0 \Rightarrow \phi = 0$. Start with some linear continuous map $\phi : C^{\mathbb{R}}(X) \to \mathbb{R}$, such that $\mu_\phi = 0$. Use Corollary II.5.3 to find two positive linear maps $\phi_1, \phi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. By Theorem 8.4 the difference $\mu_{\phi_1} - \mu_{\phi_2} = \mu_\phi = 0$ is the Hahn-Jordan decomposition of the zero measure. By the uniqueness (see Corollary 8.1) it follows that $\mu_{\phi_1} = \mu_{\phi_2} = 0$. By

the interpolation property, we know that $\|\phi_k\| = \phi_k(1) = \mu_{\phi_k}(X) = 0$, $k \geq 1$, so we get $\phi_1 = \phi_2 = 0$, thus forcing $\phi = 0$. $\square$

The injectivity of the extended Riesz correspondence has as a consequence the uniqueness of the decomposition of linear continuous as differences of positive ones, of the type described in Corollary II.5.3.

COROLLARY 8.3. *Let X be a compact Hausdorff space, and let $\phi : C^{\mathbb{R}}(X) \to \mathbb{R}$ be a linear continuous map. Assume one has positive linear maps $\phi_1, \phi_2, \psi_1, \psi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$, such that*

- $\phi = \phi_1 - \phi_2 = \psi_1 - \psi_2$;
- $\|\phi_1\| + \|\phi_2\| = \|\psi_1\| + \|\psi_2\| = \|\phi\|$.

Then one has the equalities $\phi_1 = \psi_1$ and $\phi_2 = \psi_2$.

PROOF. Consider the signed measure μ_ϕ. By Theorem 8.4, the decompositions

$$\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2} = \mu_{\psi_1} - \mu_{\psi_2}$$

both represent the Hahn-Jordan decomposition of μ_ϕ. By the uniqueness (Corollary 8.1) we have $\mu_{\phi_1} = \mu_{\psi_1}$ and $\mu_{\phi_2} = \mu_{\psi_2}$. By Corollary 8.2 this forces $\phi_1 = \psi_1$ and $\phi_2 = \psi_2$. $\square$

COMMENT. If X is a compact Hausdorff space, and $\phi : C^{\mathbb{R}}(X) \to \mathbb{R}$ is a linear continuous map, then by the above result, combined with Corollary II.5.3, we know that there exist *unique* positive linear maps $\phi^{\pm} : C^{\mathbb{R}}(X) \to \mathbb{R}$ such that $\|\phi\| = \|\phi^+\| + \|\phi^-\|$, and

$$\phi = \phi^+ - \phi^-. \tag{25}$$

The decomposition (25) will be referred to as the *Hahn-Jordan decomposition of ϕ*. This noation and terminology are used for the following reason. If we take μ_ϕ the measure given by the extended Riesz correspondence, then

$$\mu_\phi = \mu_{\phi^+} - \mu_{\phi^-}$$

is precisely the Hahn-Jordan decomposition of μ_ϕ.

REMARKS 8.4. There is a version of the extended Riesz correspondence which works for general locally compact spaces. Start with a locally compact space X, and define the spaces

$$\mathcal{M}_0^{\mathbb{R}}(X) = \big\{\phi : C_0^{\mathbb{R}}(X) \to \mathbb{R} \,:\, \phi \text{ linear continuous}\big\},$$
$$\mathfrak{R}_0^{\mathbb{R}}(X) = \big\{\mu \text{ finite signed Radon measure on } X\,\big\}.$$

Since $C_0^{\mathbb{R}}(X)$ is the completion of $C_C^{\mathbb{R}}(X)$, the correspondence

$$\mathcal{M}_0^{\mathbb{R}}(X) \ni \phi \longmapsto \phi\big|_{C_c^{\mathbb{R}}(X)}$$

establishes an isometric linear isomorphism between $\mathcal{M}_0^{\mathbb{R}}(X)$ and the space of all continuous linear maps $C_c^{\mathbb{R}}(X) \to \mathbb{R}$. For every positive $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$, we denote by μ_ϕ the Riesz measure associated with the restriction $\phi^c = \phi\big|_{C_c^{\mathbb{R}}(X)}$. Since $\|\phi^c\| = \|\phi\|$, we have the equality $\mu_\phi(X) = \|\phi\|$.

We know (see Proposition II.5.10) that for every linear continuous map $\phi : C_0^{\mathbb{R}}(X) \to \mathbb{R}$, there exist linear *positive* continuous maps $\phi_1, \phi_2 : C_0^{\mathbb{R}}(X) \to \mathbb{R}$, with $\phi = \phi_1 - \phi_2$. (In fact ϕ_1 and ϕ_2 can be chosen such that $\|\phi_1\| + \|\phi_2\| = \|\phi\|$.) We use this fact to define the finite signed Radon measure $\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}$. Exactly as

in Remark 8.2, this definition is independent of the particular choice of ϕ_1 and ϕ_2. This way we have constructed a map

$$\mathcal{M}_0^{\mathbb{R}}(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}_0^{\mathbb{R}}(X) \tag{26}$$

which we will call the *extended finite Riesz correspondence.* Of course, if X is already compact, we have $C_0^{\mathbb{R}}(X) = C^{\mathbb{R}}(X)$, $\mathcal{M}_0^{\mathbb{R}}(X) = \mathcal{M}^{\mathbb{R}}(X)$, and $mathfrakR_0^{\mathbb{R}}(X) = \mathfrak{R}^{\mathbb{R}}(X)$, so (26) is the extended Riesz correspondence previously defined.

The following result generalizes the statements of Remark 8.3, Theorem 8.4, and Corollaries 8.2 and 8.3.

THEOREM 8.5. *Let X be a locally compact space.*

A. *The extended finite Riesz correspondence* (26) *is an injective linear map.*
B. *For every $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$, there exist unique positive maps $\phi^+, \phi^- \in\in \mathcal{M}_0^{\mathbb{R}}(X)$, such that $\phi = \phi^+ - \phi^-$, and $\|\phi\| = \|\phi^+\| + \|\phi^-\|$. Moreover, in this case*

$$\mu_\phi = \mu_{\phi^+} - \mu_{\phi^-}$$

is precisely the Hahn-Jordan decomposition of μ_ϕ.

PROOF. First of all, the correspondence (26) is clearly linear, again as a consequence of Proposition 7.6.

Second, we remark that the existence part in B is already known, from Proposition II.5.10. We are going to use the following version of Theorem 8.4.

Claim: *Suppose $\phi \in \mathcal{M}_0^{\mathbb{R}}(X)$ is written as a difference $\phi = \phi_1 - \phi_2$, with $\phi_1, \phi_2 \in \mathcal{M}_0^{\mathbb{R}}(X)$ positive, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. Then*

$$\mu_\phi = \mu_{\phi_1} - \mu_{\phi_2}$$

is the Hahn-Jordan decomposition of μ_ϕ.

One way to prove this is by employing the Alexandrov compactification $X^\alpha = X \sqcup \{\infty\}$. We use the identification

$$C_0^{\mathbb{R}}(X) = \{f \in C^{\mathbb{R}}(X^\alpha) \,:\, f(\infty) = 0\}.$$

We know that there exist positive linear maps $\psi_1, \psi_2 : C^{\mathbb{R}}(X) \to \mathbb{R}$, such that $\psi_k\big|_{C_0^{\mathbb{R}}(X)} = \phi_k$, and $\|\psi_k\| = \|\phi_k\|$, $k = 1, 2$. If we define $\psi : C^{\mathbb{R}}(X^\alpha) \to \mathbb{R}$ by $\psi = \psi_1 - \psi_2$, it it not hard to see that $\|\psi\| = \|\psi_1\| + \|\psi_2\|$, so if we consider the Radon measures μ_ψ, μ_{ψ_1} and μ_{ψ_2} on the compact space X^α, then using Theorem 8.4, we get the fact that

$$\mu_\psi = \mu_{\psi_1} - \mu_{\psi_2}$$

is precisely the Hahn-Jordan decomposition of μ_ψ. This means that there are sets $B_1, B_2 \in Bor(X^\alpha)$, with $B_1 \cup B_2 = X^\alpha$, $B_1 \cap B_2 = \varnothing$, and $\mu_{\psi_1}(B_2) = \mu_{\psi_2}(B_1) = 0$. We know (see Remarks 7.4) that

$$\mu_{\psi_k}(B) = \mu_{\phi_k}(B \cap X), \;\; \forall\, B \in Bor(X^\alpha), \;\; k = 1, 2,$$

so if we define $A_k = B_k \cap X$, we immediately get $A_1 \cup A_2 = X$, $A_1 \cap A_2 = \varnothing$, and $\mu_{\phi_1}(A_2) = \mu_{\phi_2}(A_1) = 0$, thus proving that $\mu_{\phi_1} \perp \mu_{\phi_2}$.

Having proven the above Claim, the proof follows line by line the proofs of Corollaries 8.3 and 8.4. □

The notion of a finite signed measure can be generalized to the complex case.

DEFINITION. Suppose $\mathcal{A}$ is a σ-algebra on a non-empty set X. A function $\mu : \mathcal{A} \to \mathbb{C}$ is called a *complex measure on* $\mathcal{A}$, if it is σ-additive in the sense that

(ADD$_\sigma$) for any pairwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, one has the equality

$$\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty \mu(A_n). \tag{27}$$

Remark that the condition $\mu(\varnothing) = 0$ is automatic in this case. Note also that a map $\mu : \mathcal{A} \to \mathbb{C}$ is a complex measure, if and only if the maps $\operatorname{Re}\mu$ and $\operatorname{Im}\mu$ are finite signed measures.

The following result describes an important construction.

THEOREM 8.6. *Let $\mathcal{A}$ be a σ-algebra, and let μ be either a signed measure, or a complex measure on $\mathcal{A}$. For every $A \in \mathcal{A}$, we define*

$$\nu(A) = \sup\Big\{\sum_{k=1}^\infty |\mu(A_k)| \,:\, (A_k)_{k=1}^\infty \subset \mathcal{A}, \text{ pairwise disjoint, } \bigcup_{k=1}^\infty A_k = A\Big\}. \tag{28}$$

The map $\nu : \mathcal{A} \to [0,\infty]$ is an "honest" measure on $\mathcal{A}$.

PROOF. The first step in the proof is contained in the following.

Claim 1: *For any pariwise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$, one has the inequality*

$$\nu\Big(\bigcup_{n=1}^\infty A_n\Big) \le \sum_{n=1}^\infty \nu(A_n). \tag{29}$$

Denote the right hand side of (29) by S, and denote the union $\bigcup_{n=1}^\infty A_n$ simply by A. Start now with some pairwise disjoint sequence $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A$. For every $k \ge 1$, we have $D_k = \bigcup_{n=1}^\infty (D_k \cap A_n)$, with $(D_k \cap A_n)_{n=1}^\infty \subset \mathcal{A}$ pairwise disjoint, so we have

$$|\mu(D_k)| = \Big|\sum_{n=1}^\infty \mu(D_k \cap A_n)\Big| \le \sum_{n=1}^\infty |\mu(D_k \cap A_n)|, \ \ \forall\, k \ge 1.$$

Summing up then yields

$$\sum_{k=1}^\infty |\mu(D_k)| \le \sum_{k=1}^\infty \Big[\sum_{n=1}^\infty |\mu(D_k \cap A_n)|\Big] = \sum_{n=1}^\infty \Big[\sum_{k=1}^\infty |\mu(D_k \cap A_n)|\Big]. \tag{30}$$

Since for each $n \ge 1$, the sequence $(D_k \cap A_n)_{k=1}^\infty \subset \mathcal{A}$ is pairwise disjoint, and satisfies $\bigcup_{k=1}^\infty (D_k \cap A_n) = A_n$, by the definition of ν, we get

$$\sum_{k=1}^\infty |\mu(D_k \cap A_n)| \le \nu(A_n), \ \ \forall\, n \ge 1.$$

Using these estimates in (30), we then get

$$\sum_{k=1}^\infty |\mu(D_k) \le \sum_{n=1}^\infty \nu(A_n).$$

Since the inequality $\sum_{k=1}^\infty |\mu(D_k)| \le S$ holds for all pairwise disjoint sequences $(D_k)_{k=1}^\infty \subset \mathcal{A}$, with $\bigcup_{k=1}^\infty D_k = A$, by the definition of ν we get $\nu(A) \le S$, and the Claim is proven.

Claim 2: *For any finite pairwise disjoint collection $(A_n)_{n=1}^N \subset \mathcal{A}$, one has the inequality*

$$\nu(A_1 \cup \cdots \cup A_N) \ge \nu(A_1) + \cdots + \nu(A_N).$$

We use induction on N, and we see immediately that it suffices only to prove the case $N = 2$. Fix for the moment a pairwise disjoint sequence $(D_k)_{k=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{k=1}^{\infty} D_k = A_1$, and denote the sum $\sum_{k=1}^{\infty} |\mu(D_k)|$ by R. Suppose we have a pairwise disjoint sequence $(E_j)_{j=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{j=1}^{\infty} E_j = A_2$. If we combine it with the D_k's, i.e. we define

$$F_p = \begin{cases} D_{p/2} & \text{if } p \text{ is even} \\ E_{(p+1)/2} & \text{if } p \text{ is odd} \end{cases}$$

then we get a new pairwise disjoint sequence $(F_p)_{p=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{p=1}^{\infty} F_p = A_1 \cup A_2$. By the definition of ν we will then get

$$\nu(A_1 \cup A_2) \geq \sum_{p=1}^{\infty} |\mu(F_p)| = \sum_{k=1}^{\infty} |\mu(D_k)| + \sum_{j=1}^{\infty} |\mu(E_j)| = R + \sum_{j=1}^{\infty} |\mu(E_j)|.$$

Taking supremum over all pairwise disjoint sequences $(E_j)_{j=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{j=1}^{\infty} E_j = A_2$, the above inequality yields $\mu(A_1 \cup A_2) \geq R + \nu(A_2)$, so now we have

$$\nu(A_1 \cup A_2) \geq \nu(A_2) + \sum_{k=1}^{\infty} |\mu(D_k)|.$$

Taking supremum over all pairwise disjoint sequences $(D_k)_{k=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{k=1}^{\infty} D_k = A_1$, the above inequality finally gives $\nu(A_1 \cup A_2) \geq \nu(A_2) + \nu(A_1)$, and the Claim is proven.

We are now in position to prove that ν is a measure on $\mathcal{A}$. The equality $\nu(\varnothing) = 0$ is trivial. To prove σ-additivity, we start with some pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, and we must prove the equality

$$\nu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \nu(A_n).$$

On the one hand, using Claim 1, we know that we have the inequality $\nu\big(\bigcup_{n=1}^{\infty} A_n\big) \leq \sum_{n=1}^{\infty} \nu(A_n)$. On the other hand, if we denote the union $\bigcup_{n=1}^{\infty} A_n$ simply by A, then using Claim 2, we see that

$$\nu(A) \geq \nu(A \smallsetminus [A_1 \cup \cdots \cup A_N]) + \nu(A_1) + \ldots \nu(A_N) \geq \nu(A_1) + \ldots \nu(A_N), \ \ \forall\, N \geq 1,$$

which immedaitely gives the other inequality $\nu(A) \geq \sum_{n=1}^{\infty} \nu(A_n)$. $\square$

Definition. With the notations above, and under the hypothesis of Theorem 8.6, the "honest" measure ν, defined by (28), is called the *variation measure of* μ, and will be denoted by $|\mu|$. By construction, we have the inequality

$$|\mu(A)| \leq |\mu|(A), \ \ \forall\, A \in \mathcal{A}.$$

Remark 8.5. Let μ be either a signed measure, or a complex measure on the σ-algebra $\mathcal{A}$. Exactly as with numbers (or functions), the measure $|\mu|$ has a minimality property, which can be stated as follows. *Whenever ν is an "honest" measure on $\mathcal{A}$ with*

$$|\mu(A)| \leq \nu(A), \ \ \forall\, A \in \mathcal{A},$$

it follows that we have

$$|\mu|(A) \leq \nu(A), \ \ \forall\, A \in \mathcal{A}.$$

This is quite clear, because for any pairwise disjoint sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, with $\bigcup_{n=1}^{\infty} A_n = A$, one has the inequality

$$\sum_{n=1}^{\infty} |\mu(A_n)| \leq \sum_{n=1}^{\infty} \nu(A_n) = \nu(A),$$

and then the desired inequality follows by taking the supremum in the left hand side.

In the case of signed measures, the variation measure is also given by the following.

PROPOSITION 8.2. *Let μ be a signed measure on the σ-algebra $\mathcal{A}$. Then one has the equality*

$$|\mu| = \mu^+ + \mu^-,$$

where $\mu = \mu^+ - \mu^-$ is the Hahn-Jordan decomposition of μ.

PROOF. Denote the measure $\mu^+ + \mu^-$ simply by ν. Remark that we obviously have

$$-\nu(A) = -\mu^+(A) - \mu^-(A) \leq \mu^+(A) - \mu^-(A) \leq \mu^+(A) + mu^-(A) = \nu(A), \ \ \forall\, A \in \mathcal{A},$$

which gives

$$|\mu(A)| \leq \nu(A), \ \ \forall\, A \in \mathcal{A}.$$

By Remark 8.5, this forces the inequality $|\mu| \leq \nu$.

To prove the other inequality, we start by fixing sets $X^+, X^- \in \mathcal{A}$ as in Theorem 8.2. We decompose each set $A \in \mathcal{A}$ as $A = A^+ \cup A^-$, where $A^{\pm} = A \cap X^{\pm}$, so that we have

$$\nu(A) = \nu(A^+) + \nu(A^-) = \mu^+(A^+) + \mu^+(A^-) + \mu^-(A^+) + \mu^-(A^-) = \mu^+(A^+) + \mu^-(A^-).$$

Notice now that $\mu(A^+) = \mu^+(A^+) \geq 0$, and $-\mu(A^-) = \mu^-(A^-) \geq 0$, which means that we have the equalities $\mu^+(A^+) = |\mu(A^+)|$ and $\mu^-(A^-) = |\mu(A^-)|$, so the above equality reads

$$\nu(A) = |\mu(A^+)| + |\mu(A^-)|,$$

and by the definition of $|\mu|$ we then immediately get $\nu(A) \leq |\mu|(A)$. □

An interesting consequence is the following.

COROLLARY 8.4. *Let μ be either a finite signed measure, or a comlex measure on the σ-algebra $\mathcal{A}$. Then the variation measure $|\mu|$ is finite.*

PROOF. The signed measure case is clear from the above result.

In the complex case, we write $\mu = \nu + i\eta$, with ν and η finite signed measures on $\mathcal{A}$. We apply the signed case, to get the fac that both $|\nu|$ and $|\eta|$ are finite. Notice that we have

$$|\mu(A)| = |\nu(A) + i\eta(A)| \leq |\nu(A)| + |\eta(A)| \leq |\nu|(A) + |\eta|(A), \ \ \forall\, A \in \mathcal{A},$$

so by Remark 8.5 we get $|\mu| \leq |\nu| + |\eta|$, and then the finiteness of $|\mu|$ is a consequence of the finiteness of $|\nu|$ and $|\eta|$. □

Exercise 3. Let $\mathcal{A}$ be a σ-algebra, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For the purpose of this exercise, let us agree to use the term $\mathbb{K}$-measure for designating either a finite signed measure (when $\mathbb{K} = \mathbb{R}$), or a complex measure (when $\mathbb{K} = \mathbb{C}$). Prove the following.

(i) The collection of all $\mathbb{K}$-measures on $\mathcal{A}$ is a vector space.

(ii) For anu two $\mathbb{K}$-measures if μ and ν, one has the inequality

$$|\mu + \nu| \leq |\mu| + |\nu|.$$

(iii) For any $\mathbb{K}$-measure μ and any $\alpha \in \mathbb{K}$, one has the equality

$$|\alpha\mu| = |\alpha| \cdot |\mu|.$$

Proposition 8.2 has another interesting consequence, which is relevant for the study of the extended finite Riesz correspondence.

Corollary 8.5. *Let X be a locally compact space. Then the extended finite Riesz correspondence* (26) *has the property*

$$|\mu_\phi|(X) = \|\phi\|, \ \ \forall\, \phi \in \mathcal{M}_0^{\mathbb{R}}(X). \tag{31}$$

Proof. From Proposition 8.1 and Theorem 8.5, we know that $|\mu_\phi| = \mu_{\phi^+} + \mu_{\phi^-}$. Using Remark 8.4, and Theorem 8.5 again, we have

$$|\mu_\phi|(X) = \mu_{\phi^+}(X) + \mu_{\phi^-}(X) = \|\phi^+\| + \|\phi^-\| = \|\phi\|. \quad \square$$

Comments. Given a locally compact space X, we can define a *complex Radon measure on X* as being a complex measure on X, whose real and imaginary part are both (finite) signed Radon measures. The extended finite Riesz correspondence can be then defined also over the complex numbers, as a map

$$\mathcal{M}_0(X) \ni \phi \longmapsto \mu_\phi \in \mathfrak{R}_0(X),$$

where

$$\mathcal{M}_0(X) = \big\{\phi : C_0(X) \to \mathbb{C} \,:\, \phi \text{ linear constinuous}\big\},$$
$$\mathfrak{R}_0(X) = \big\{\mu \text{ complex Radon measure on } X\big\}.$$

This correspondence is again linear. One will still have the equality (31), but the proof of this fact will appear later in Chapter IV.