Lectures 26-29

7. Measure theory on locally compact spaces

Earlier in this chapter we discussed the construction of (outer) measures, starting with more primitive objects: semiring measures. The main application was the construction of the (outer) Lebesgue measure on $\mathbb{R}^n$. In this section we describe an alternative construction, which has as its starting point another primitive object: a *regular content*. The idea is again to start with the measure defined on a "small" class of sets, extend it to an outer measure, and then use the Caratheodory construction. Among other applications, we will get an alternative construction of the (outer) Lebesgue measure on $\mathbb{R}^n$.

Definition. Let X be a locally compact space. Denote by $\mathcal{C}_X$ the collection of all compact subsets of X. A *content on* X, is a map $\omega : \mathcal{C}_X \to [0, \infty)$, with the following properties:

(i) $\omega(\varnothing) = 0$;
(ii) if $K, L \in \mathcal{C}_X$ are such that $K \subset L$, then $\omega(K) \leq \omega(L)$;
(iii) $\omega(K \cup L) \leq \omega(K) + \omega(L)$, for all $K, L \in \mathcal{C}_X$;
(iv) $\omega(K \cup L) = \omega(K) + \omega(L)$, for all $K, L \in \mathcal{C}_X$, with $K \cap L = \varnothing$.

Comments. Note that ω takes *finite values*. The collection $\mathcal{C}_X$ does not have any nice set-arithmetic properties, except for the following: (i) the union of any finite collection of sets in $\mathcal{C}_X$ is again in $\mathcal{C}_X$; (ii) an arbitrary intersection of sets in $\mathcal{C}_X$ is again in $\mathcal{C}_X$.

Examples 7.1. A. If μ is a measure on $Bor(X)$, then $\mu\big|_{\mathcal{C}_X}$ is a content.

B. Take $X = \mathbb{R}$, and for a compact subset $K \subset \mathbb{R}$, define

$$\omega(K) = \begin{cases} 1 & \text{if } 0 \in \text{Int}(K) \\ 0 & \text{if } 0 \notin \text{Int}(K) \end{cases}$$

It is obvious that ω is a content on $\mathbb{R}$. Notice however that if we consider the compact sets $K_n = [-\frac{1}{n}, \frac{1}{n}]$, then $\omega\big(\bigcap_{n=1}^{\infty} K_n\big) = 0$, but $\omega(K_n) = 1$, $\forall\, n \geq 1$. This shows that, in general, a content cannot be extended to a measure on $Bor(X)$.

One useful property, which will be invoked several times in this section, is contained in the following:

Exercise 1. Let X be a locally compact space, let $K \subset X$ be compact, and let $D_1, D_2 \subset X$ be open subsets, with $K \subset D_1 \cup D_2$. Show there exist compact sets K_1 and K_2, such that $K_1 \subset D_1$, $K_2 \subset D_2$, and $K = K_1 \cup K_2$.

As Example 7.1.B suggests, one obstruction for the extendability of a content on X, to a measure on $Bor(X)$, is its behaviour with respect to interiors. The following notion isolates an important property, which will be shown to be sufficient for the extendability property.

Definition. Let X be a locally compact space. A content ω on X is said to be *regular*, if for any $K \in \mathcal{C}_X$, one has the equality

$$\omega(K) = \inf\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ \mathrm{Int}(L) \supset K\big\}.$$

The following exercise shows how the lack of regularity can always be repaired.

Exercise 2. Let X be a locally compact space, and let ω be a content on X. Define $\breve{\omega} : \mathcal{C}_X \to [0, \infty)$, by

$$\breve{\omega}(K) = \inf\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ \mathrm{Int}(L) \supset K\big\},\ \ \forall\, K \in \mathcal{C}_X.$$

Prove that:

(i) $\breve{\omega}$ is a regular content on X;
(ii) $\breve{\omega}(K) \geq \omega(K)$, $\forall\, K \in \mathcal{C}_X$;
(iii) if η is a regular content on X, with $\eta(K) \geq \omega(K)$, $\forall\, K \in \mathcal{C}_X$, then $\eta(K) \geq \breve{\omega}(K)$, $\forall\, K \in \mathcal{C}_X$;
(iv) ω is regular, if and only if $\breve{\omega} = \omega$.

Definition. With the notations from Exercise 2, the regular content $\breve{\omega}$ is called the *regularization of* ω.

Theorem 7.1. *Let X be a locally compact space, and let ω be a content on X. Denote by $\mathcal{T}_X$ the collection of all open subsets of X. Define the map $\hat{\omega} : \mathcal{T}_X \to [0, \infty]$ by*

$$\hat{\omega}(D) = \sup\big\{\omega(K) \,:\, K \in \mathcal{C}_X,\ K \subset D\big\},\ \ \forall\, D \in \mathcal{T}_X,$$

and define the map $\omega^* : \mathcal{P}(X) \to [0, \infty]$, by

$$\omega^*(A) = \inf\big\{\hat{\omega}(D) \,:\, D \in \mathcal{T}_X,\ D \supset A\big\},\ \ \forall\, A \subset X.$$

Then ω^* is an outer measure on X.

Proof. We begin by collecting the useful properties of the map $\hat{\omega}$.

Claim: *The map $\hat{\omega}$ has the following properties*

(i) $\hat{\omega}(\varnothing) = 0$;
(ii) *$\hat{\omega}$ is monotone, i.e. whenever $D, E \in \mathcal{T}_X$ satisfy $D \subset E$, it follows that $\hat{\omega}(D) \leq \hat{\omega}(E)$;*
(iii) *$\hat{\omega}$ is σ-sub-additive, i.e., for any sequence $(D_n)_{n=1}^\infty \subset \mathcal{T}_X$, one has the inequality $\hat{\omega}\big(\bigcup_{n=1}^\infty D_n\big) \leq \sum_{n=1}^\infty \hat{\omega}(D_n)$.*

Properties (i) and (ii) are trivial.

To prove property (iii), let us start with some sequence $(D_n)_{n=1}^\infty$ of open sets, and let us denote for simplicity the union $\bigcup_{n=1}^\infty D_n$ by D. Start with some arbitrary compact set $K \subset D$. Using compactness, there exists some index $p \geq 1$, such that $K \subset D_1 \cup D_2 \cup \cdots \cup D_p$. Use Exercise 1 (and induction) to find compact sets $K_1 \subset D_1$, $K_2 \subset D_2$, $\dots$, $K_p \subset D_p$, such that $K = K_1 \cup K_2 \cup \cdots \cup K_p$. We then clearly have the inequalities

$$\omega(K) \leq \sum_{n=1}^p \omega(K_n) \leq \sum_{n=1}^p \hat{\omega}(D_n) \leq \sum_{n=1}^\infty \hat{\omega}(D_n).$$

Since we have

$$\omega(K) \leq \sum_{n=1}^\infty \hat{\omega}(D_n), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset D,$$

by the definition of $\hat{\omega}$, we immediately get

$$\hat{\omega}(D) \le \sum_{n=1}^{\infty} \hat{\omega}(D_n).$$

Having proven the Claim, we now check the conditions in the definition of an outer measure. It is clear that $\omega^*(\varnothing) = 0$. It is also clear, from the definition, and property (ii) from the Claim, that

$$A \subset B \Longrightarrow \omega^*(A) \le \omega^*(B).$$

Finally, we need to show σ-sub-additivity, i.e.

$$\omega^*\big(\bigcup_{n=1}^{\infty} A_n\big) \le \sum_{n=1}^{\infty} \omega^*(A_n). \tag{1}$$

Start with some sequence $(A_n)_{n=1}^{\infty}$ of subsets of X. Of course, if one of the terms in the right hand side of (1) is infinite, there is nothing to prove. Assume that $\omega^*(A_n) < \infty$, $\forall\, n \ge 1$. Fix some $\varepsilon > 0$, and choose, for each $n \ge 1$, an open set $D_n \supset A_n$, such that $\hat{\omega}(D_n) \le \omega^*(A_n) + \frac{\varepsilon}{2^n}$. Put $D = \bigcup_{n=1}^{\infty} D_n$. Using part (iii) of the Claim, we have

$$\hat{\omega}(D) \le \sum_{n=1}^{\infty} \hat{\omega}(D_n) \le \sum_{n=1}^{\infty} \big[\omega^*(A_n) + \frac{\varepsilon}{2^n}\big] = \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n).$$

Since we obviously have the inclusion $D \supset \bigcup_{n=1}^{\infty} A_n$, the above inequality gives

$$\omega^*\big(\bigcup_{n=1}^{\infty} A_n\big) \le \hat{\omega}(D) \le \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n).$$

Now we have

$$\omega^*\big(\bigcup_{n=1}^{\infty} A_n\big) \le \varepsilon + \sum_{n=1}^{\infty} \omega^*(A_n),$$

for all $\varepsilon > 0$, so the inequality (1) follows. ☐

Definition. Let X be a locally compact space, and let ω be a content on X. The outer measure ω^* on X, defined in Theorem 7.1, is called the *outer measure induced by* ω.

Remarks 7.1. Let X be a locally compact space, let ω be a content on X, and let ω^* be the outer measure induced by ω.

A. The map $\hat{\omega} : \mathcal{T}_X \to [0, \infty]$, defined in the statement of Theorem 7.1, is given by $\hat{\omega} = \omega^*\big|_{\mathcal{T}_X}$. To see that this is the case, start with some open set D. On the one hand, by the definition of ω^*, we know that

$$\omega^*(D) = \inf\big\{\hat{\omega}(E) \,:\, E \in \mathcal{T}_X,\ E \supset D\big\},$$

which (using $E = D$) immediately gives the inequality $\omega^*(D) \le \hat{\omega}(D)$. On the other hand, using property (ii) from the Claim stated in the proof, we also know that

$$\hat{\omega}(E) \ge \hat{\omega}(D), \text{ for all } E \in \mathcal{T}_X \text{ with } E \supset D,$$

which gives the reverse inequality, $\omega^*(D) \ge \hat{\omega}(D)$.

B. As a consequence of the above remark, we get the fact that ω^* is *regular from above, with respect to the collection* $\mathcal{T}_X$ *of all open sets in* X, i.e.

$$\omega^*(A) = \inf\big\{\omega^*(D) \,:\, D \in \mathcal{T}_X,\ D \supset A\big\},\ \ \forall\, A \subset X.$$

C. If one denotes by $\breve{\omega}$ the regularization of ω (see Exercise 2), then $\breve{\omega}^* = \omega^*$. In fact, using the notations from Theorem 7.1, we have the equality $\hat{\omega} = \hat{\breve{\omega}}$. Indeed, on the one hand, since we have the inequality

$$\breve{\omega}(K) \geq \omega(K),\ \ \forall\, K \in \mathcal{C}_X,$$

it follows immediately by the definitions, that

$$\hat{\breve{\omega}}(D) \geq \hat{\omega}(D),\ \ \forall\, D \in \mathcal{T}_X.$$

To prove the other inequality, fix some open set $D \subset X$. Suppose $K \subset D$ is some compact subset. Using the well-known properties of locally compact spaces, there exists some compact set L, with

$$K \subset \text{Int}(L) \subset L \subset D,$$

so by the definitions of $\hat{\omega}$ and $\breve{\omega}$, we get

$$\hat{\omega}(D) \geq \omega(L) \geq \breve{\omega}(K).$$

Since we have the inequality

$$\hat{\omega}(D) \geq \breve{\omega}(K), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset D,$$

taking supremum in the right hand side yields

$$\hat{\omega}(D) \geq \sup\big\{\breve{\omega}(K) \,:\, K \in \mathcal{C}_X,\ K \subset D\big\} = \hat{\breve{\omega}}(D).$$

PROPOSITION 7.1. *Let X be a locally compact space, let ω be a content on X, and let ω^* be the outer measure induced by ω. If we denote by $\breve{\omega}$ the regularization of ω, then one has the equality*

$$\omega^*\big|_{\mathcal{C}_X} = \breve{\omega}.$$

PROOF. Using Remark 7.1.C, we can assume that ω is regular, and in this case we need to prove that $\omega^*\big|_{\mathcal{C}_X} = \omega$. Start with some compact set $K \subset X$. By the definition of ω^*, using the notations from Theorem 7.1, we know that

$$\omega^*(K) = \inf\big\{\hat{\omega}(D) \,:\, D \in \mathcal{T}_X,\ D \supset K\big\}. \tag{2}$$

It is clear that, for every open set $D \supset K$, we have the inequality

$$\hat{\omega}(D) \geq \omega(K),$$

so taking infimum in the left hand side, and using (2), immediately gives the inequality

$$\omega^*(K) \geq \omega(K).$$

To prove the reverse inequality, we start by fixing $\varepsilon > 0$, and we use regularity to find some compact set L with $K \subset \text{Int}(L)$, and $\omega(L) \leq \omega(K) + \varepsilon$. Consider the open set $D = \text{Int}(L)$. On the one hand, for every compact set $F \subset D$, we have the onbious inclusion $F \subset L$, which gives $\omega(F) \leq \omega(L)$. Taking supremum over all copact sets $F \subset D$ then gives $\hat{\omega}(D) \leq \omega(L)$. By the choice of L, by the definition of ω^*, and using the inclusion $D \supset K$, we then get

$$\omega^*(K) \leq \hat{\omega}(D) \leq \omega(L) \leq \omega(K) + \varepsilon.$$

Since the inequality

$$\omega^*(K) \leq \omega(K) + \varepsilon,$$

holds for all $\varepsilon > 0$, we then must have $\omega^*(K) \leq \omega(K)$. ☐

The above result gives a nice characterization for the regularity of a content, in terms of the induced outer measure.

COROLLARY 7.1. *Let X be a locally comoact space. A content ω is regular, if and only if $\omega^*\big|_{\mathcal{C}_X} = \omega$.*

PROOF. Immediate from Proposition 7.1 and exercise 2. □

THEOREM 7.2. *Let X be a locally compact space, let ω be a content on X, and let ω^* be the outer measure induced by ω. Then every open set $D \subset X$ is ω^*-measurable.*

PROOF. Fix an open set $D \subset X$. We need to prove (see Section 5) that D "sharply cuts" every subset of X, which is equivalent to the fact that, for every $A \subset X$, one has the inequality:

$$\omega^*(A) \geq \omega^*(A \cap D) + \omega^*(A \smallsetminus D). \tag{3}$$

This will be shown in several steps.

Claim 1: *For any open set $E \subset X$, and any compact set $K \subset E$, one has the inequality*

$$\omega^*(E) \geq \omega(K) + \omega^*(E \smallsetminus K).$$

To prove this inequality, we first note that, since both E and $E \smallsetminus K$ are open, by Remark 7.1.A, we have the equalities $\omega^*(E) = \hat{\omega}(E)$ and $\omega^*(E \smallsetminus K) = \hat{\omega}(E \smallsetminus K)$, where $\hat{\omega} : \mathcal{T}_X \to [0, \infty]$ is the map defined in the statement of Theorem 7.1. If $L \subset E \smallsetminus K$ is an arbitrary compact set, then we obviously have $K \cap L = \varnothing$, so using the inclusion $K \cup L \subset E$, we get

$$\omega(K) + \omega(L) = \omega(K \cup L) \leq \hat{\omega}(E) = \omega^*(E),$$

which then gives

$$\omega^*(E) - \omega(K) \geq \omega(L), \text{ for all } L \in \mathcal{C}_X \text{ with } L \subset E \smallsetminus K.$$

Taking supremum in the right hand side then gives

$$\omega^*(E) - \omega(K) \geq \sup\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ L \subset E \smallsetminus K\big\} = \hat{\omega}(E \smallsetminus K) = \omega^*(E \smallsetminus K),$$

and the Claim follows.

Claim 2: *The inequality* (3) *holds for all open subsets $A \subset X$.*

Assume A is open. If the left hand side of (3) is infinite, there is nothing to prove. Assume $\omega^*(A) < \infty$, so both $\omega^*(A \cap D)$ and $\omega^*(A \smallsetminus D)$ are also finite. Since $A \cap D$ is open, we have

$$\omega^*(A \cap D) = \hat{\omega}(A \cap D) = \sup\big\{\omega(K) \,:\, K \in \mathcal{C}_X,\ K \subset A \cap D\big\}. \tag{4}$$

Fix for the moment a compact subset $K \subset A \cap D$. Using Claim 1 we have the inequality

$$\omega^*(A) \geq \omega(K) + \omega^*(A \smallsetminus K).$$

Since we obviously have the inclusion $A \smallsetminus K \supset A \smallsetminus D$, the above inequality gives $\omega^*(A) \geq \omega(K) + \omega^*(A \smallsetminus D)$, which can be rw-written as

$$\omega^*(A) - \omega^*(A \smallsetminus D) \geq \omega(K), \text{ for all } K \in \mathcal{C}_X \text{ with } K \subset A \cap D.$$

Taking supremum in the right hand side, and using (4), we immediately get the desired inequality (3).

We now proceed with the proof of (3) for arbitrary A's. Fix A, and consider an arbitrary open set $E \supset A$. By Claim 2, we have

$$\omega^*(E) \geq \omega^*(E \cap D) + \omega^*(E \smallsetminus D).$$

Using the obvious inclusions $E \cap D \supset A \cap D$ and $E \smallsetminus D \supset A \smallsetminus D$, we then get

$$\omega^*(E) \geq \omega^*(A \cap D) + \omega^*(A \smallsetminus D).$$

The desired inequality (3) follows now by taking infimum in the left hand side, and using Remark 7.1.B. □

The most important consequence of Theorem 7.2 is the following.

COROLLARY 7.2. *Let X be a locally compact space, and let ω be a regular content on X. Then ω can be extended uniquely to a measure μ_ω on $Bor(X)$, with the following properties.*

(i) *μ is regular from above, with respect to the collection $\mathcal{T}_X$ of all open sets, that is*

$$\mu_\omega(B) = \inf\big\{\mu_\omega(D) \,:\, D \in \mathcal{T}_X,\ D \supset B\big\},\ \ \forall\, B \in Bor(X);$$

(ii) *for every open set $D \subset X$, one has the equality*

$$\mu_\omega(D) = \sup\big\{\mu_\omega(K) \,:\, K \in \mathcal{C}_X,\ K \subset D\big\}.$$

Conversely, if μ is a measure on $Bor(X)$ with properties (i) *and* (ii)*, and such that $\mu(K) < \infty$, $\forall\, K \in \mathcal{C}_X$, then $\mu\big|_{\mathcal{C}_X}$ is regular content.*

PROOF. If we denote by $\mathcal{M}_{\omega^*}(X)$ the σ-algebra of ω^*-measurable sets, then Theorem 7.2 gives the inclusion $Bor(X) \subset \mathcal{M}_{\omega^*}(X)$, so the existence follows by taking $\mu_\omega = \omega^*\big|_{Bor(X)}$. The fact that μ_ω has properties (i) and (ii) is trivial, by construction and by Remarks 7.1 and Proposition 7.1.

The uniqueness is trivial, since property (ii) uniquely defines μ_ω on open sets, and (i) then uniquely defines μ_ω on all Borel sets.

To prove the second assertion, assume μ is a measure on $Bor(X)$ with properties (i) and (ii), and let us show that $\omega = \mu\big|_{\mathcal{C}_X}$ is a regular content. The fact that ω is a content is trivial, so the only thing we must show is regularity. Fix some compact set $K \subset X$. It is clear that

$$\omega(K) \leq \inf\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ K \subset \text{Int}(L)\big\}.$$

To prove the converse, we use property (i), to find, for each $\varepsilon > 0$, an open set $D_\varepsilon \supset K$, such that $\mu(D_\varepsilon) \leq \mu(K) + \varepsilon$. If we choose, for each ε.), a compact set L_ε, such that

$$K \subset \text{Int}(L_\varepsilon) \subset L_\varepsilon \subset D_\varepsilon,$$

then we obviously have

$$\mu(K) \leq \mu(L_\varepsilon) \leq \mu(D_\varepsilon) \leq \mu(K) + \varepsilon,$$

so we get the inequality

$$\inf\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ K \subset \text{Int}(L)\big\} \leq \omega(K) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we get in fact the inequality

$$\inf\big\{\omega(L) \,:\, L \in \mathcal{C}_X,\ K \subset \text{Int}(L)\big\} \leq \omega(K),$$

and we are done. □

Definition. Let X be a locally compact space. A *Radon measure on X* is a measure μ on $Bor(X)$ with the following properties:

(i) $\mu(K) < \infty$, for all compact sets $K \subset X$;

(ii) for every open set D one has

$$\mu(D) = \sup\big\{\mu(K) \,:\, K \subset D,\ \ K \text{ compact}\big\};$$

(iii) for every Borel set B one has

$$\mu(B) = \inf\big\{\mu(B) \,:\, D \supset B,\ \ D \text{ open}\big\}.$$

By Corollary 7.2, the map $\omega \longmapsto \mu_\omega$ establishes a bijective correspondence between the set of all regular contents on X, and the set of all Radon measures on X. For a regullar content ω, the measure μ_ω is called the *Radon measure extension of* ω.

Proposition 7.2. *Let X be a locally compact space.*

(*a*) *If μ is a Radon measure on X, and $t \in [0,\infty)$, then $t\mu$ is also a Radon measure on X.*

(*b*) *If μ_1 and μ_2 are Radon measures on X, then $\mu_1 + \mu_2$ is also a Radon measure on X.*

Proof. Property (*a*) is trivial.

To prove property (*b*) let us denote $\mu_1 + \mu_2$ simply by μ. We first obvserve that μ is indeed a measure on $Bor(X)$, and we clearly have

$$\mu(K) = \mu_1(K) + \mu_2(K) < \infty,\ \ \forall\, K \in \mathcal{C}_X.$$

Let us show that μ satisfies condition (ii). Fix some open set $D \subset X$, and let us prove that

$$\mu(D) = \sup\big\{\mu(K) \,:\, K \in \mathcal{C}_X,\ K \subset D\big\}. \tag{5}$$

If $\mu(D) = \infty$, then either $\mu_1(D) = \infty$ or $\mu_2(D) = \infty$, so we get

$$\sup\big\{\max\big(\mu_1(K), \mu_2(K)\big) \,:\, K \in \mathcal{C}_X,\ K \subset D\big\} = \infty,$$

and since $\mu(K) \geq \max\big(\mu_1(K), \mu_2(K)\big)$, $\forall\, K \in \mathcal{C}_X$, the equality (5) immediately follows. Suppose now $\mu(D) < \infty$, which is equivalent to the fact that $\mu_1(D), \mu_2(D) < \infty$. Denote the right hand side of (5) by $\nu(D)$. For every $\varepsilon > 0$, using the fact that μ_1 and μ_2 are Radon measures, we can find two compact sets $K_1^\varepsilon, K_2^\varepsilon \subset D$, such that $\mu_1(K_1^\varepsilon) \geq \mu_1(D) - \frac{\varepsilon}{2}$ and $\mu_2(K_2^\varepsilon) \geq \mu_2(D) - \frac{\varepsilon}{2}$. Of course, the compact set $K_\varepsilon = K_1^\varepsilon \cup K_2^\varepsilon$ is still a subset of D, and satisfies

$$\mu_1(K_\varepsilon) \geq \mu_1(K_1^\varepsilon) \geq \mu_1(D) - \frac{\varepsilon}{2},$$
$$\mu_2(K_\varepsilon) \geq \mu_2(K_2^\varepsilon) \geq \mu_2(D) - \frac{\varepsilon}{2},$$

so we get $\mu(K_\varepsilon) = \mu_1(K_\varepsilon) + \mu_2(K_\varepsilon) \geq \mu_1(D) + \mu_2(D) - \varepsilon = \mu(D) - \varepsilon$. This proves that $\nu(D) \geq \mu(D) - \varepsilon$, and since this inequality is true for all $\varepsilon > 0$, we get $\nu(D) \geq \mu(D)$. The inequality $\nu(D) \leq \mu(D)$ is trivial.

We now show that μ satisfies condition (iii). Fix some set $A \in Bor(X)$, and let us prove that

$$\mu(A) = \inf\big\{\mu(D) \,:\, D \in \mathcal{T}_X,\ D \supset A\big\}. \tag{6}$$

If $\mu(A) = \infty$, there is nothing to prove. Suppose now $\mu(A) < \infty$, which is equivalent to the fact that $\mu_1(A), \mu_2(A) < \infty$. Denote the right hand side of (6) by $\lambda(A)$. For every $\varepsilon > 0$, using the fact that μ_1 and μ_2 are Radon measures, we can find two open sets $D_1^\varepsilon, D_2^\varepsilon \supset A$, such that $\mu_1(D_1^\varepsilon) \leq \mu_1(A) + \frac{\varepsilon}{2}$ and $\mu_2(D_2^\varepsilon) \leq \mu_2(A) + \frac{\varepsilon}{2}$. Then open set $D_\varepsilon = D_1^\varepsilon \cap D_2^\varepsilon$ still contains A, and satisfies

$$\mu_1(D_\varepsilon) \leq \mu_1(D_1^\varepsilon) \leq \mu_1(A) + \frac{\varepsilon}{2},$$
$$\mu_2(D_\varepsilon) \leq \mu_2(D_2^\varepsilon) \geq \mu_2(A) + \frac{\varepsilon}{2},$$

so we get $\mu(D_\varepsilon) = \mu_1(D_\varepsilon) + \mu_2(D_\varepsilon) \leq \mu_1(A) + \mu_2(A) + \varepsilon = \mu(A) + \varepsilon$. This proves that $\lambda(A) \leq \mu(A) + \varepsilon$, and since this inequality is true for all $\varepsilon > 0$, we get $\lambda(A) \leq \mu(A)$. The inequality $\lambda(A) \geq \mu(A)$ is trivial. □

Radon measures are also functorial with respect to proper maps, in the following sense.

Proposition 7.3. *Let X and Y be locally compact spaces, let $\Phi : X \to Y$ be a proper continuous map, and let μ be a Radon measure on X. Then the map $\nu : Bor(Y) \to [0, \infty]$, defined by*

$$\nu(B) = \mu\big(\Phi^{-1}(B)\big), \ \forall B \in Bor(Y),$$

is a Radon measure on Y.

Proof. First of all, remark that since Φ is continuous, it is Borel measurable, which means that

$$\Phi^{-1}(B) \in Bor(X), \ \forall B \in Bor(Y).$$

Secondly, by the well known properties of measures, the map ν is a measure.

We now check that ν is a Radon measure. First of all, if $K \subset Y$ is compact, then using the fact that Φ is proper, it means that $\Phi^{-1}(K)$ is compact in X, so we clearly get

$$\nu(K) = \mu\big(\Phi^{-1}(K)\big) < \infty.$$

To prove that ν satisfies condition (ii), start with some open set $D \subset Y$, and let us find a sequence $(L_n)_{n=1}^\infty$ of compact subsets of D, such that $\nu(D) = \lim_{n\to\infty} \nu(L_n)$. The set $\Phi^{-1}(D)$ is open, so there exists a sequence $(K_n)_{n=1}^\infty$ of compact subsets of $\Phi^{-1}(D)$, with

$$\nu(D) = \mu\big(\Phi^{-1}(D)\big) = \lim_{n\to\infty} \mu(K_n). \tag{7}$$

It we define the subsets $L_n = \Phi(K_n)$, then $(L_n)_{n\geq 1}$ is a sequence of compact subsets of D, and the inclusion $K_n \subset \Phi^{-1}(L_n)$ immediately gives $\nu(D) \geq \nu(L_n) \geq \mu(K_n)$, so by (7) we also get $\nu(D) = \lim_{n\to\infty} \nu(L_n)$.

To prove condition (iii) start with some arbitrary subset $B \in Bor(Y)$, and let us a sequence $(E_n)_{n=1}^\infty$ of open subset of Y, such that $\nu(B) = \lim_{n\to\infty} \nu(E_n)$, and $E_n \supset B$, $\forall n \geq 1$. Use the fact that μ is a Radon measure, to find a sequence $(D_n)_{n=1}^\infty$ of open subset of X, such that

$$\nu(B) = \mu\big(\Phi^{-1}(B)\big) = \lim_{n\to\infty} \mu(D_n), \tag{8}$$

and $D_n \supset \Phi^{-1}(B)$, $\forall n \geq 1$. Put $T_n = X \smallsetminus D_n$, so that T_n is closed, for each $n \geq 1$. By Proposition I.5.2, the sets $\Phi(T_n)$ are closed in Y, hence their complements $E_n = Y \smallsetminus \Phi(^{}T_n)$, $n \geq 1$ are open. Remark that we have the inclusions $B \subset E_n$, $\forall n \geq 1$. Otherwise, we would have $B \cap \Phi(T_n) \neq \varnothing$, forcing $T_n \cap \Phi^{-1}(B) \neq \varnothing$, which is impossible, since $\Phi^{-1}(B) \subset D_n = X \smallsetminus T_n$. Moreover, we also have the inclusions

$$\Phi^{-1}(B) \subset \Phi^{-1}(E_n) \subset D_n, \ \forall n \geq 1,$$

which then force

$$\nu(B) \leq \nu(E_n) \leq \mu(D_n), \ \ \forall\, n \geq 1.$$

Using by (8) this gives the equality $\lim_{n\to\infty} \nu(E_n) = \nu(B)$. □

Of course, if X is a compact Hausdorff space, then every Radon measure μ on X is finite. The following gives an interesting converse of this property, which also shows that sometimes functoriality can be present beyond the proper case described above.

PROPOSITION 7.4. *Let X be a locally compact space, let μ be a Radon measure on X, and let (θ, T) be a compactification of X. The following are equivalent:*

(i) $\mu(X) < \infty$*;*

(ii) *the map $\nu : Bor(T) \to [0, \infty)$, defined by*

$$\nu(B) = \mu\big(\theta^{-1}(B)\big), \ \ \forall\, B \in Bor(T),$$

is a Radon measure on T.

PROOF. Recall that the fact that (θ, T) is a compactification of X means that

- T is a compact Hausdorff space;
- $\theta : X \to T$ is continuous;
- $\theta(X)$ is open and dense in T;
- $\theta : X \to \theta(X)$ is a homeomorphism.

Without any loss of generality, we can assume that X is a dense open subset of T, and θ is the inclusion map. With this convention, the map ν is defined by

$$\nu(B) = \mu(B \cap X), \ \ \forall\, B \in Bor(T). \tag{9}$$

(i) ⇒ *(ii)*. Assume $\mu(X) < \infty$. It is clear that ν is a finite measure on $Bor(T)$, and in fact we have $\nu(T \smallsetminus X) = 0$.

The fact that $\nu(K) < \infty$, for every compact subset $K \subset T$ is of course trivial.

We now check the second condition in the definition. Fix some open subset $D \subset T$, and let us show that

$$\nu(D) = \sup\big\{\nu(K) \,:\, K \text{ compact}, \, K \subset D\big\}.$$

All we need is a sequence $(K_n)_{n=1}^\infty$ of compact subsets of D, with $\lim_{n\to\infty} \nu(K_n) = \nu(D)$. To get this sequence we simply use the fact that $D \cap X$ is open (in X), so we can find a sequence $(K_n)_{n=1}^\infty$ of compact subsets of $D \cap X$, with $\lim_{n\to\infty} \mu(K_n) = \mu(D \cap X) = \nu(D)$. Now we are done, because the fact that $K_n \subset X$, gives $\mu(K_n) = \nu(K_n)$, $\forall\, n \geq 1$.

We now check the third condition in the definition. Fix some set $B \in Bor(T)$, and let us show that

$$\nu(B) = \inf\big\{\nu(D) \,:\, D \subset T \text{ open}, \, D \supset B\big\}.$$

All we need is a sequence $(D_n)_{n=1}^\infty$ of open subsets of T, with $D_n \supset B$, $\forall\, n \geq 1$, and $\lim_{n\to\infty} \nu(D_n) = \nu(B)$. Start off by choosing a sequnce $(K_n)_{n=1}^\infty$ of compact subsets of X, such that $\lim_{n\to\infty} \mu(K_n) = \mu(X)$, we will get $\lim_{n\to\infty} \mu(X \smallsetminus K_n) = 0$ (the condition that $\mu(X) < \infty$ is essential here). If we define then the open sets $A_n = T \smallsetminus K_n$, then we will have $\nu(A_n) = \mu(A_n \cap X) = \mu(X \smallsetminus K_n)$, $\forall\, n \geq 1$, so we have

$$\lim_{n\to\infty} \nu(A_n) = 0. \tag{10}$$

Notice also that

$$A_n \supset T \smallsetminus X, \ \forall n \geq 1. \tag{11}$$

Use now the fact that μ is a Radon measure on X, and the fact that $B \cap X \in Bor(X)$, to find a sequence $(E_n)_{n=1}^{\infty}$ of open subsets of X, with $E_n \supset B \cap X$, $\forall n \geq 1$, and

$$\lim_{n\to\infty} \nu(E_n) = \mu(B \cap X). \tag{12}$$

Since X is open in T, it follows that all the E_n's are open in T. If we define $D_n = E_n \cup A_n$, then using (11) we have the inclusions

$$D_n = E_n \cup A_n \supset (B \cap X) \cup (T \smallsetminus X) \supset B, \ \forall n \geq 1,$$

as well as the inequalities

$$\begin{aligned}\mu(B \cap X) = \nu(B) \leq \nu(D_n) \leq \nu(E_n) + \nu(A_n) = \\ = \mu(E_n \cap X) + \nu(A_n) = \mu(E_n) + \nu(A_n), \ \forall n \geq 1,\end{aligned}$$

which, combined with (10) and (12), clearly give $\lim_{n\to\infty} \nu(D_n) = \mu(B\cap X) = \nu(B)$.

$(ii) \Rightarrow (i)$. This implication is trivial, because the fact that ν is a Radon measure forces $\mu(X) = \nu(X) \leq \nu(T) < \infty$. □

COMMENT. Assume μ is a Radon measure on a locally compact space X. Although the measure μ is regular from above with respect to open sets by (iii), in general, one cannot conclude that it is regular from below with respect to compact sets. The following example illustrates such an anomaly.

*Exercise 3**. Equipp the space $X = \mathbb{R}^2$ with the *disjoint union topology* defined by the decomposition $X = \bigcup_{y\in\mathbb{R}} \big(\mathbb{R} \times \{y\}\big)$. More explicitly, if we define, for each $A \subset X$, and each $y \in \mathbb{R}$, the set

$$A_y = \{x \in \mathbb{R} : (x, y) \in A\big\},$$

then a set $D \subset X$ is declared to be open, if and only if all subsets $D_y \subset \mathbb{R}$, $y \in \mathbb{R}$ are open (in the usual topology on $\mathbb{R}$). For each subset $A \subset X$, define its support

$$S_A = \{y \in \mathbb{R} : A_y \neq \varnothing\}.$$

Prove the following.

(i) A set $K \subset X$ is compact, if and only if its support S_K is finite and, for each $y \in S_K$, the set $K_y \subset \mathbb{R}$ is compact (in the usual topology on $\mathbb{R}$).

(ii) X is a locally compact space.

(iii) If we define, for every compact subset $K \subset X$, the number

$$\omega(K) = \sum_{y\in S_K} \lambda(K_y),$$

where λ is the Lebesgue measure on $\mathbb{R}$, then ω is a regular content on X.

(iv) Let μ denote the Radon measure extension of ω. Then for every open set $D \subset X$, one has the equality

$$\mu(D) = \sum_{y\in\mathbb{R}} \lambda(D_y),$$

where one uses the summation conventions discussed in II.2. (The sum in the right hand side is defined as the supremum of all finite sums.)

(v) If $B \in Bor(X)$ has uncountable support S_B, then $\mu(B) = \infty$.

(vi) Consider the y-axis $Y = \{0\} \times \mathbb{R} \subset X$ Show that F is closed in X (hence Borel), it has infinite measure $\mu(F) = \infty$, but $\mu(K) = 0$, for all compact subsets $K \subset F$.

HINTS: Using regularity from above, it suffices to prove (v) only when B is open. In this case use the fact that if a map $\alpha : \mathbb{R} \to \mathbb{R}$ is summable, then the set $\{t \in \mathbb{R} : \alpha(t) \neq 0\}$ is countable. For (vi), the equality $\mu(F) = \infty$ is a consequence of (v). To get the fact that all compact subsets of F have measure zero, use part (i).

REMARK 7.2. Let X be a locally compact space, and let μ be a Radon measure on X. We define the maximal outer extension of μ (see Section 5) by

$$\mu^*(A) = \inf\big\{\mu(B) : B \in Bor(X),\ B \supset A\big\},\ \ \forall\, A \subset X.$$

By the regularity from above, one has the equality

$$\mu^*(A) = \inf\big\{\mu(D) : D \in \mathcal{T}_X,\ D \supset A\big\},\ \ \forall\, A \subset X. \tag{13}$$

If one considers the regular content $\omega = \mu\big|_{\mathcal{C}_X}$, then $\mu^* = \omega^*$, the outer measure induced by ω. We also know that if we consider the σ-algebra $\mathcal{M}_{\mu^*}(X)$ of all μ^*-measurable subsets of X, we have the inclusion $Bor(X) \subset \mathcal{M}_{\mu^*}(X)$, and $\mu^*\big|_{Bor(X)} = \mu$.

Exercise 4. Consider the collection $\mathcal{D}$ of all subsets of $\mathbb{R}^n$, of the form

$$D = (a_1, b_1) \times \cdots \times (a_n, b_n),\ \ a_1 < b_1, \dots, a_n < b_n.$$

For every such D we define

$$\mathrm{vol}_n = \prod_{j=1}^{n} (b_j - a_j).$$

We define, for every bounded subset $B \subset \mathbb{R}^n$, the number

$$\mathbf{v}(B) = \inf\Big\{\sum_{p=1}^{N} \mathrm{vol}_n(D_p) : (D_p)_{p=1}^{N} \subset \mathcal{D},\ B \subset \bigcup_{p=1}^{N} D_p\Big\}. \tag{14}$$

(i) If we define $\mathcal{B} = \big\{B \subset \mathbb{R}^n : B \text{ bounded}\big\}$, then $\mathcal{B}$ is a ring, the map $\mathbf{v} : \mathcal{B} \to [0, \infty)$ is sub-additive, but not σ-sub-additive. In particular, $\mathbf{v}$ does not extend to an outer measure on $\mathbb{R}^n$.

(ii) If we consider the unit square $S = [0, 1]^n$, then the collection $\mathcal{N}_{\mathbf{v}}(S) = \big\{N \subset S : \mathbf{v}(N) = 0\big\}$ is a ring, but not a σ-ring.

(iii) When restricted to the collection $\mathcal{C}_{\mathbb{R}^n}$, of all compact subsets of $\mathbb{R}^n$, the map $\omega = \mathbf{v}\big|_{\mathcal{C}_{\mathbb{R}^n}}$ defines a regular content on $\mathbb{R}^n$.

(iv) The outer measure ω^*, defined by ω, is precisely the outer Lebesgue measure λ_n^*.

The above construction somehow belongs to the "prehistory" of measure theory. The map $\mathbf{v} : \mathcal{B} \to [0, \infty)$ is called the *Jordan content*. Bounded sets $N \subset \mathbb{R}^n$, with $\mathbf{v}(N) = 0$ are called *Jordan neglijeable*. The theory of Riemann integration (especially for functions of several variables) relies heavily on the use of Jordan neglijeable sets. Part (ii) shows that, when restricted to $Bor(S)$, the map $\mathbf{v}$ fails to be a measure. Parts (iii) and (iv) explain how the construction can be "fixed." The regular content $\omega = \mathbf{v}\big|_{\mathcal{C}_{\mathbb{R}^n}}$ is called the *Lebesgue content*. The correspondence $\omega \longmapsto \omega^*$ gives an alternative construction of the outer Lebesgue measure, which starts with its definition on compact sets as the Jordan content.

For Radon measures, the lack of regularity from below, with respect to compact sets, in somehow compensated by the following result (compare with Exercise 2 from Section 6).

LEMMA 7.1. *Let X be a locally compact space, let μ be a Radon measure on X, and let μ^* be the maximal outer extension of μ. For a subset $A \subset X$, with $\mu^*(A) < \infty$, the following are equivalent*

(i) *A is μ^*-measurable;*
(ii) *$\mu^*(A) = \sup\{\mu(K) : K \in \mathcal{C}_X,\ K \subset A\}$;*
(iii) *there exists a sequence $(K_n)_{n=1}^\infty$ of compact subsets of A, such that*

$$\mu^*\big(A \smallsetminus \big[\bigcup_{n=1}^\infty K_n\big]\big) = 0.$$

PROOF. $(i) \Rightarrow (ii)$. Suppose A is μ^*-measurable, and let us prove the equality (ii). Denote the right hand side of (ii) simply by $\nu(A)$. It is obvious, by the monotonicity of μ^*, and the fact that $\mu^*\big|_{Bor(X)} = \mu$, that we have the inequality $\mu^*(A) \geq \nu(A)$. To prove the other inequality we fix for the moment some $\varepsilon > 0$. Using (13), there exists an open set $D \supset A$, such that $\mu(D) \leq \mu^*(A) + \varepsilon$. Use property (ii) in the definition of Radon measures, to find some compact set $L \subset D$ such that

$$\mu(D) \leq \mu(L) + \varepsilon.$$

Since $\mu(D) = \mu(D \smallsetminus L) + \mu(L)$, and $\mu(L) \leq \mu(D) < \infty$, this inequality gives

$$\mu(D \smallsetminus L) \leq \varepsilon,$$

which, combined with the obvious inclusion $A \smallsetminus L \subset D \smallsetminus L$, yields

$$\mu^*(A \smallsetminus L) \leq \mu^*(D \smallsetminus L) = \mu(D \smallsetminus L) \leq \varepsilon. \tag{15}$$

Using (13) we can also find an open set $E \supset L \smallsetminus A$, such that

$$\mu(E) \leq \mu^*(L \smallsetminus A) + \varepsilon. \tag{16}$$

Since $L \smallsetminus A$ is μ^*-measurable, we have $\mu(E) = \mu^*(E) = \mu^*\big(E \smallsetminus (L \smallsetminus A)\big) + \mu^*(L \smallsetminus A)$. Since $\mu^*(L \smallsetminus A) \leq \mu^*(E) = \mu(E) < \infty$, the inequality (16) gives

$$\mu^*\big(E \smallsetminus (L \smallsetminus A)\big) \leq \varepsilon. \tag{17}$$

Consider the set $K = L \smallsetminus E$. It is obvious that K is compact, and we have the inclusion

$$K \subset L \smallsetminus (L \smallsetminus A) = L \cap A \subset A.$$

Moreover, we have

$$(L \cap A) \smallsetminus K \subset E \smallsetminus (L \smallsetminus A).$$

Using the inequality (17), we then get

$$\mu^*\big((L \cap A) \smallsetminus K\big) \leq \varepsilon.$$

Finally, the above inequality, combined with (15), gives

$$\mu^*(A \smallsetminus K) \leq \mu^*\big((L \cap A) \smallsetminus K\big) + \mu^*\big((A \smallsetminus L) \smallsetminus K\big) \leq \varepsilon + \mu^*(A \smallsetminus L) \leq 2\varepsilon.$$

Since $K \subset A$, we get

$$\mu^*(A) \leq \mu^*(A \smallsetminus K) + \mu^*(K) \leq 2\varepsilon + \mu(K) \leq 2\varepsilon + \nu(A).$$

Since the inequality $\mu^*(A) \leq 2\varepsilon + \nu(A)$ holds for all $\varepsilon > 0$, we get $\mu^*(A) \leq \nu(A)$, so (ii) follows.

(ii) ⇒ *(iii)*. Assume A satisfies (ii), and let us show that A has property (iii). For every integer $n \geq 1$, we use *(ii)* to find a compact set $K_n \subset A$, such that

$$\mu^*(A) \leq \mu(K_n) + \frac{1}{n}. \tag{18}$$

On the one hand, we have the inclusions $A \smallsetminus \big[\bigcup_{n=1}^{\infty} K_n\big] \subset A \smallsetminus K_p$, which give

$$\mu^*\big(A \smallsetminus \big[\bigcup_{n=1}^{\infty} K_n\big]\big) \leq \mu^*(A \smallsetminus K_p), \ \ \forall\, p \geq 1. \tag{19}$$

On the other hand, since K_p is measurable, we have the equality

$$\mu^*(A) = \mu^*(A \smallsetminus K_p) + \mu^*(K_p) = \mu^*(A \smallsetminus K_p) + \mu(K_p),$$

and then the fact that $\mu^*(A) < \infty$, combined with (18), will force

$$\mu^*(A \smallsetminus K_p) \leq \frac{1}{p}, \ \ \forall\, n \geq 1.$$

Using (19), this forces $\mu^*\big(A \smallsetminus \big[\bigcup_{n=1}^{\infty} K_n\big]\big) = 0$.

(iii) ⇒ *(i)*. This is pretty obvious. We define the sets $B = \bigcup_{n=1}^{\infty} K_n \subset A$, and $N = A \smallsetminus B$. Then $\mu^*(N) = 0$, so in particular, N is μ^*-measurable. Since B is Borel, it is also μ^*-measurable, so $A = B \cup N$ is indeed μ^*-measurable. □

The following result generalizes Lemma 7.1 to the σ-finite case.

Theorem 7.3. *Let X be a locally compact space, let μ be a Radon measure on X, and let μ^* be the maximal outer extension of μ. For a set $A \subset X$, the following are equivalent*

(i) *A is μ^*-measurable, and μ^*-σ-finite.*
(ii) *There exists sequences $(K_n)_{n=1}^{\infty} \subset \mathcal{C}_X$ and $(D_n)_{n=1}^{\infty} \subset \mathcal{T}_X$, such that*

$$\bigcup_{n=1}^{\infty} K_n \subset A \subset \bigcap_{n=1}^{\infty} D_n \quad \text{and} \quad \mu\big(\big[\bigcap_{n=1}^{\infty} D_n\big] \smallsetminus \big[\bigcup_{n=1}^{\infty} K_n\big]\big) = 0.$$

(The condition that A is μ^*-σ-finite means that there exists a sequence $(A_n)_{n=1}^{\infty}$ of subsets of X, with $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) < \infty$, for all $n \geq 1$.)

Proof. *(i)* ⇒ *(ii)*. Assume A is μ^*-measurable and μ^*-σ-finite.

Claim 1**:** *There exists a sequence $(A_n)_{n=1}^{\infty}$ of μ^*-measurable sets, such that $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) < \infty$, $\forall\, n \geq 1$.*

A priori, we only know that there exists a sequence $(A_n^0)_{n=1}^{\infty}$ of subsets of X (not assumed to be μ^*-measurable), with $A = \bigcup_{n=1}^{\infty} A_n^0$, and $\mu^*(A_n^0) < \infty$, $\forall\, n \geq 1$. Using (13), we can choose however, for each $n \geq 1$, an open set E_n, with $A_n^0 \subset E_n$, and $\mu(E_n) < \infty$. In particular, E_n is μ^*-measurable, and so will be $A_n = A \cap E_n$. We clearly have $A = \bigcup_{n=1}^{\infty} A_n$, and $\mu^*(A_n) \leq \mu^*(E_n) < \infty$, $\forall\, n \geq 1$.

Using Claim 1, we start off by writing $A = \bigcup_{n=1}^{\infty} A_n$, with the A_n's μ^*-measurable, and $\mu^*(A_n) < \infty$. For each $n \geq 1$, we use Lemma 7.1 to find a sequence $(L_n^p)_{p=1}^{\infty}$ of compact subsets of A_n, such that

$$\mu^*\big(A_n \smallsetminus \big[\bigcup_{p=1}^{\infty} L_n^p\big]\big) = 0.$$

Let us list the countable collection $\{L_n^p : p,n \geq 1\}$ as a sequence $(K_n)_{n=1}^\infty$, so that we have

$$\bigcup_{n=1}^\infty K_n = \bigcup_{n=1}^\infty \bigcup_{p=1}^\infty L_n^p \subset \bigcup_{n=1}^\infty A_n = A.$$

Claim 2: *The set $M = A \smallsetminus \big(\bigcup_{n=1}^\infty K_n\big)$ is μ^*-neglijeable, i.e. $\mu^*(M) = 0$.*

Indeed, if we define, for each $k \geq 1$ the set $M_k = A_k \smallsetminus \big[\bigcup_{n=1}^\infty K_n\big]$, then we have the obvious equality $M = \bigcup_{k=1}^\infty M_k$, and the inclusions

$$M_k = A_k \smallsetminus \Big[\bigcup_{p=1}^\infty \bigcup_{n=1}^\infty L_n^p\Big] \subset A_k \smallsetminus \Big[\bigcup_{p=1}^\infty L_k^p\Big],\ \ \forall\, k \geq 1,$$

which, by the choice of the L's, prove that $\mu^*(M_k) = 0$, $\forall\, k \geq 1$.

We proceed now with the construction of the D's. For each pair of integers (p,n), we use (13) to find an open set $E_n^p \supset A_n$, such that $\mu(E_n^p) \leq \mu^*(A_n) + \frac{1}{2^{p+n}}$. Since the A_n's are μ^*-measurable, we have

$$\mu(E_n^p) = \mu^*(E_n^p) = \mu^*(A_n) + \mu^*(E_n^p \smallsetminus A_n).$$

Since $\mu^*(A_n) < \infty$, by the choice of the E's, we will get

$$\mu^*(E_n^p \smallsetminus A_n) \leq \frac{1}{2^{p+n}},\ \ \forall\, p,n \geq 1. \tag{20}$$

We then define, for each $p \geq 1$, the open set $D_p = \bigcup_{n=1}^\infty E_n^p$. Notice that, for each $p \geq 1$, we have the inclusion $A = \bigcup_{n=1}^\infty A_n \subset \bigcup_{n=1}^\infty E_n^p = D_p$, and

$$D_p \smallsetminus A = \bigcup_{n=1}^\infty [E_n^p \smallsetminus A] \subset \bigcup_{n=1}^\infty [E_n^p \smallsetminus A_n].$$

Using (20), we then get

$$\mu^*(D_p \smallsetminus A) \leq \sum_{n=1}^\infty \mu^*(E_n^p \smallsetminus A_n) \leq \sum_{n=1}^\infty \frac{1}{2^{p+n}} = \frac{1}{2^p},\ \ \forall\, p \geq 1. \tag{21}$$

Since $A \subset D_p$, $\forall\, p \geq 1$, we get $A \subset \bigcap_{p=1}^\infty D_p$. Moreover, if we define the set $N = \big[\bigcap_{p=1}^\infty D_p\big] \smallsetminus A$, we obviously have the inclusions $N \subset D_p \smallsetminus A$, $\forall\, p \geq 1$, and then (21) clearly forces $\mu^*(N) = 0$.

Now we have $\bigcup_{n=1}^\infty K_n \subset A \subset \bigcap_{p=1}^\infty D_p$, and $\big[\bigcap_{p=1}^\infty D_p\big] \smallsetminus \big[\bigcup_{n=1}^\infty K_n\big] = N \cup M$, with $\mu^*(M) = \mu^*(N) = 0$, so we indeed have (ii).

The implication $(ii) \Rightarrow (i)$ is pretty obvious. If there exist sequences $(K_n)_{n=1}^\infty$ and $(D_n)_{n=1}^\infty$ as in (ii), then the sets $B = \bigcup_{n=1}^\infty K_n$ and $G = \bigcap_{n=1}^\infty D_n$ are Borel. Moreover, the inclusions $B \subset A \subset G$, give $A \smallsetminus B \subset G \smallsetminus B$, so we have $\mu^*(A \smallsetminus B) \leq \mu^*(G \smallsetminus B)$. By the second feature in (ii) we know that $\mu^*(G \smallsetminus B) = 0$, therefore the set $P = A \smallsetminus B$ is μ^*-neglijeable, hence μ^*-measurable. Since $A = B \cup P$, it follows that A is indeed μ^*-measurable. $\square$

Comment. The implication $(ii) \Rightarrow (i)$ in Theorem 7.3 holds without the μ^*-σ-finiteness assumption on A. In fact, condition (ii) actually forces A to be μ^*-σ-finite.

Corollary 7.3. *If μ is a Radon measure on X, and the set A is μ^*-measurable, and μ^*-σ-finite, then one has the equality*

$$\mu^*(A) = \sup\big\{\mu(K) : K \in \mathcal{C}_X,\ K \subset A\big\}.$$

Proof. Follow the first part of the proof of $(i) \Rightarrow (ii)$ to find a sequence $(K_n)_{n=1}^\infty$ of compact subsets of A, such that

$$\mu^*\big(A \smallsetminus \bigcup_{n=1}^\infty K_n]\big) = 0.$$

Since $\bigcup_{n=1}^\infty K_n$ is μ^*-measurable, this forces the equality

$$\mu^*(A) = \mu^*\big(\bigcup_{n=1}^\infty K_n\big) = \lim_{n\to\infty} \mu^*(K_1 \cup \cdots \cup K_n). \quad \square$$

Exercise 5.* Let X be a locally compact space, and let μ be a Radon measure on X. Suppose $\nu : Bor(X) \to [0, \infty]$ is a measure satisfying the following conditions:

(a) $\nu(B) \le \mu(B)$, $\forall\, B \in Bor(X)$;

(b) for every $B \in Bor(X)$, one has the implication $\nu(B) < \infty \Rightarrow \mu(B) < \infty$.

Prove that ν is a Radon measure on X. (Notice that, in the case when μ is finite, the condition (b) is superfluous.)

Hints: To prove condition (ii) in the definition of Radon measures, start with some open set $D \subset X$, and choose a sequence $K_1 \subset K_2 \subset \cdots \subset D$ of compact subsets, such that

$$\lim_{n\to\infty} \mu(K_n) = \mu(D),$$

and define the Borel set $B = \bigcup_{n=1}^\infty K_n \subset D$. Notice that we have the equalities $\mu(B) = \lim_{n\to\infty} \mu(K_n)$ and $\nu(B) = \lim_{n\to\infty} \nu(K_n)$. Argue that, when $\nu(D) = \infty$, we must have $\nu(B) = \infty$. When $\nu(D) < \infty$, show that $\mu(D \smallsetminus B) = 0$. In either case we get $\nu(B) = \nu(D)$.

The next result explains somehow the anomaly illustrated by Exercise 3.

Proposition 7.5. *If μ is a Radon measure on X, and let μ^* denote its maximal outer extension. For a subset $N \subset X$, the following are equivalent*

(i) *N is μ^*-measurable, and for every compact subset $K \subset N$, one has the equality $\mu(K) = 0$;*

(ii) *$\mu^*(D \cap N) = 0$, for all open subsets $D \subset X$ with $\mu(D) < \infty$;*

(iii) *N is locally μ^*-neglijeable, i.e.*

$$\mu^*(A \cap N) = 0, \text{ for all subsets } A \subset X \text{ with } \mu^*(A) < \infty.$$

Proof. $(i) \Rightarrow (ii)$. Assume N satisfies condition (i). Fix some open set $D \subset X$, with $\mu(D) < \infty$. Then the set $D \cap N$ is measurable, and $\mu^*(D \cap N) \le \mu(D) < \infty$. The equality $\mu^*(D \cap N) = 0$ then follows from (i), combined with Corollary 7.3.

$(ii) \Rightarrow (iii)$. Assume N satisfies condition (ii). Fix some arbitrary subset $A \subset X$, with $\mu^*(A) < \infty$. Using (13), there exists some open set $D \supset A$ with $\mu(D) < \infty$. Then we have the inequality $\mu^*(A \cap N) \le \mu^*(D \cap N)$, so condition (ii) will force $\mu^*(A \cap N) = 0$.

$(iii) \Rightarrow (i)$. Let N be locally μ^*-neglijeable. We know that local μ^*-neglijeability implies μ^*-measurability (see Section 5). The fact that $\mu(K) = 0$, for all compact subsets $K \subset N$ is also trivial. $\square$

Notation. Let μ be a Radon measure on the locally compact space X, and let μ^* be the maximal outer extension of μ. We denote the σ-algebra $\mathcal{M}_{\mu^*}(X)$, of all μ^*-measurable subsets of X, simply by $\mathfrak{M}_\mu(X)$, and we define the measure $\tilde{\mu} = \mu^*\big|_{\mathcal{M}_{\mu^*}(X)}$. Using the terminology introduced in Section 5, the pair $(\mathfrak{M}_\mu(X), \tilde{\mu})$ is the quasi-completion of $Bor(X)$ with respect to μ.

Our next goal is to examine the inclusion $Bor(X) \subset \mathfrak{M}_\mu(X)$ along the same lines used in the final part of Section 5. In preparation for the results that follow, it is helpful to introduce the following terminology.

Definition. Let μ be a Radon measure on the locally compact space X. A non-empty compact subset $K \subset X$, is said to be *μ-tight*, if it has the property

- *there is no compact non-empty proper subset $L \subsetneq K$, with $\mu(K) = \mu(L)$.*

Remark 7.3. Singleton sets are always μ-tight. If K is μ-tight, and $\mu(K) = 0$ then K must be a singleton.

For a non-empty compact set K with $\mu(K) > 0$, the μ-tightness is equivalent to the following condition[1]:

$$\left.\begin{array}{c} D \subset X \text{ open} \\ D \cap K \neq \varnothing \end{array}\right\} \Longrightarrow \mu(D \cap K) > 0. \tag{22}$$

Indeed, if K is μ-tight, and $D \subset X$ is an open set, such that $D \cap K \neq \varnothing$, then the compact set $L = K \smallsetminus D$ is either empty, or a proper subset of K. In either case, we get $\mu(L) < \mu(K)$, and then the equality $D \cap K = K \smallsetminus L$ gives $\mu(D \cap K) = \mu(K) - \mu(L) > 0$. Conversely, if K satisfies (22) and if L is a non-empty proper compact subset of K, then the set $D = X \smallsetminus L$ is open, and satisfies $D \cap K \neq \varnothing$. By (22) this forces $\mu(D \cap K) > 0$, and since we have $L = K \smallsetminus (D \cap K)$, we get $\mu(L) = \mu(K) - \mu(D \cap K) < \mu(K)$.

A μ-tight compact set K, with $\mu(K) > 0$, will be called *non-degenerate*.

Lemma 7.2. *Let X be a locally compact space, let μ be a Radon measure on X. Every non-empty compact set $K \subset X$ has a μ-tight compact subset $K_0 \subset K$, with $\mu(K_0) = \mu(K)$.*

Proof. If K is already tight, there is nothing to prove. Also, if $\mu(K) = 0$, then we can pick K_0 to be of the form $\{x\}$, with x any point in K.

For the remainder of the proof, we are going to assume that K is not μ-tight, and $\mu(K) > 0$. Consider the collection

$$\mathfrak{L} = \big\{L \in \mathfrak{C}_X \,:\, \varnothing \neq L \subsetneq K \text{ and } \mu(L) = \mu(K)\big\}.$$

Since K is not μ-tight, the collection $\mathfrak{L}$ is non-empty. One key property of the collection $\mathfrak{L}$ is the following.

Claim 1: *If $L_1, \dots, L_n \in \mathfrak{L}$, then $L_1 \cap \dots \cap L_n \in \mathfrak{L}$.*

Indeed, if we define the sets $A_j = K \smallsetminus L_j$, $j = 1, \dots, n$, then $\mu(A_1) = \dots = \mu(A_n) = 0$, and then the equality

$$K \smallsetminus [L_1 \cap \dots \cap L_n] = A_1 \cup \dots \cup A_n$$

will force $\mu\big(K \smallsetminus [L_1 \cap \dots \cap L_n]\big) = 0$, thus giving $\mu(L_1 \cap \dots \cap L_n) = \mu(K) > 0$. (The last inequality forces of course $L_1 \cap \dots \cap L_n \neq \varnothing$.)

Using the *finite intersection property*, it follows that the intersection $K_0 = \bigcap_{L \in \mathfrak{L}} L$ is non-empty.

Claim 2: $K_0 \in \mathfrak{L}$.

Obviously K_0 is compact non-empty proper subset of K, so the only thing we need to prove is the equality $\mu(K_0) = \mu(K)$. Consider the Borel subset

$$B = K \smallsetminus K_0 \subset K.$$

[1] Notice that using $D = X$, condition (22) actually forces $\mu(K) > 0$.

Since $B \subset K$, it follows that $\mu(B) < \infty$. By Corollary 7.3 we have

$$\mu(B) = \sup \{\mu(P) : P \text{ compact}, P \subset B\}. \tag{23}$$

Notice however that if $P \subset B$ is compact, then we have, by the definition of B, the equality

$$\bigcap_{L \in \mathcal{L}} (L \cap P) = \Big(\bigcap_{L \in \mathcal{L}} L\Big) \cap P = (K \smallsetminus B) \cap P = \varnothing,$$

so again by the *finite intersection property*, combined with Claim 1, it follows that there exists $L \in \mathcal{L}$, such that $P \cap L = \varnothing$. Then we have $\mu(K \smallsetminus L) = 0$, so the inclusion $P \subset K \smallsetminus L$ will force $\mu(P) = 0$. Using (23), this forces $\mu(B) = 0$.

We now show that K_0 is μ-tight. Indeed, if K_0 were not tight, we could find some non-empty compact proper subset $L \subsetneq K_0$, with $\mu(L) = \mu(K_0) = \mu(K)$. This will of course force L to belong to $\mathcal{L}$, and therefore it will force the inclusion $K_0 \subset L$, which is impossible. □

LEMMA 7.3. *Let X be a locally compact space, let ν be a Radon measure on X, and let $\mathcal{G}$ be a pair-wise disjoint collection of non-degenerate μ-tight compact sets. For any set $A \subset X$, with $\mu^*(A) < \infty$, the collection*

$$S_{\mathcal{G}}(A) = \{G \in \mathcal{G} : G \cap A \neq \varnothing\}$$

is at most countable.

PROOF. Since $\mu^*(A) < \infty$, by (13), there exists some open set $D \supset A$ with $\mu(D) < \infty$. It is obvious that $S_{\mathcal{G}}(A) \subset S_{\mathcal{G}}(D)$, so it suffices to prove that $S_{\mathcal{G}}(D)$ is at most countable.

On the one hand, we notice that, for every finite subset $\mathcal{F} \subset S_{\mathcal{G}}(D)$, one has

$$\sum_{G \in \mathcal{F}} \mu(G \cap D) = \mu\Big(\bigcup_{G \in \mathcal{F}} [G \cap D]\Big) \leq \mu(D) < \infty.$$

This means that the family $\big(\mu(G \cap D)\big)_{G \in S_{\mathcal{G}}(D)}$ is *summable*, and we have

$$\sum_{G \in S_{\mathcal{G}}(D)} \mu(G \cap D) \leq \mu(D) < \infty.$$

On the other hand, by Remark 7.3, we know that all the terms $\mu(G \cap D)$, $G \in S_{\mathcal{G}}(D)$ are are strictly positive. Using Proposition II.2.2, this forces $S_{\mathcal{G}}(D)$ to be countable. □

The main application of the above result is the following.

THEOREM 7.4. *Let X be a locally compact space, and let μ be a Radon measure on X. Then there exists a partition $\mathcal{F}$ of X into μ-tight compact sets, with the property that the set*

$$N_{\mathcal{F}} = \bigcup_{\substack{F \in \mathcal{F} \\ \mu(F) = 0}} F$$

is locally μ^*-neglijeable.

PROOF. Define the set

$$\mathbf{\Omega} = \{\mathcal{F} : \mathcal{F} \text{ pairwise disjoint collection of non-degenerate } \mu\text{-tight compact sets}\}.$$

We agree to consider the empty collection as an element of $\mathbf{\Omega}$, so that $\mathbf{\Omega}$ is non-empty. Equip the set $\mathbf{\Omega}$ with the order relation $\subset$ given by inclusion.

Claim 1: *The ordered set* $(\mathbf{\Omega}, \subset)$ *contains a maximal element.*

This is a straightforward application of Zorn's Lemma. Start with some subset $\mathbf{\Lambda}$ of $\mathbf{\Omega}$, which is totally ordered with respect to $\subset$, and let us show that there is an upper bound for $\mathbf{\Lambda}$ (in $\mathbf{\Omega}$). If we write $\mathbf{\Lambda} = \{\mathcal{G}_i : i \in I\}$, we define the collection $\mathcal{G} = \bigcup_{i\in I} \mathcal{G}_i$. It is clear that every element in $\mathcal{G}$ is a non-degenerate μ-tight compact set. If $K, L \in \mathcal{G}$ are different elements, then there exist $i, j \in I$ with $K \in \mathcal{G}_i$ and $L \in \mathcal{G}_j$. Since $\mathbf{\Lambda}$ is totally ordered, we either have $\mathcal{G}_i \subset \mathcal{G}_j$, or $\mathcal{G}_j \subset \mathcal{G}_i$. In either case, we conclude that there exists some $k \in I$, such that $K, L \in \mathcal{G}_k$, and then $K \cap L = \varnothing$. This shows that $\mathcal{G}$ is pairwise disjoint, hence $\mathcal{G}$ belongs to $\mathbf{\Omega}$. It is obvious that $\mathcal{G}$ is an upper bound for $\mathbf{\Lambda}$.

Having proven Claim 1, we fix a maximal collection $\mathcal{G} \in \mathbf{\Omega}$, and we define the set $T = \bigcup_{G\in\mathcal{G}} G$. It is quite possible that $\mathcal{G} = \varnothing$. In that case we define $T = \varnothing$.

Claim 2: *For every compact subset* $K \subset X \smallsetminus T$, *one has* $\mu(K) = 0$.

We prove this by contradiction. Assume $\mu(K) > 0$. By Lemma 7.3 there exists a μ-tight compact subset $K_0 \subset K$, with $\mu(K_0) = \mu(K) > 0$ (in particular K_0 is non-degenerate). But then the collection $\mathcal{G} \cup \{K_0\}$ would obviously contradict the maximality of $\mathcal{G}$.

Claim 3: *Whenever* $D \subset X$ *is an open set with* $\mu(D) < \infty$, *it follows that the set* $D \smallsetminus T$ *is Borel, and* $\mu(D \smallsetminus T) = 0$.

By Lemma 7.3, the collection

$$S_{\mathcal{G}}(D) = \big\{G \in \mathcal{G} : G \cap D \neq \varnothing\big\}$$

is at most countable. Now we have

$$D \cap T = \bigcup_{G\in S_{\mathcal{G}}(D)} (D \cap G),$$

so $D \cap T$ is a countable union of Borel sets, hence $D \cap T$ itself is Borel, and so will be $D \smallsetminus T = D \smallsetminus (D \cap T)$. Since $\mu(D \smallsetminus T) \le \mu(D) < \infty$, by Corollary 7.3, we have

$$\mu(D \smallsetminus T) = \sup\big\{\mu(K) : K \text{ compact, } K \subset D \smallsetminus T\big\}.$$

By Claim 2 this, forces $\mu(D \smallsetminus T) = 0$.

Going back to the proof of the theorem, we notice that, by Claim 2, none of the singletons $\{x\}$, $x \in X \smallsetminus T$, has positive measure. We can then define collection

$$\mathcal{F} = \mathcal{G} \cup \big\{\{x\} : x \in X \smallsetminus T\big\},$$

which is obviously a partition of X into μ-tight compact sets. For this partition, we obviously have the equality $N_{\mathcal{F}} = X \smallsetminus T$. By Claim 3, we have

$$\mu^*(N_{\mathcal{F}} \cap D) = 0, \text{ for all open sets } D \subset X \text{ with } \mu(D) < \infty.$$

By Proposition 7.2, it follows that $N_{\mathcal{F}}$ is indeed locally μ^*-neglijeable. □

Definition. Let X be a locally compact space, and let μ be a Radon measure on X. A partition $\mathcal{F}$ of X into μ-tight compact sets, with the property stated in Theorem 7.3, will be called *non-degenerate*.

The existence of such partitions is significant, as indicated below.

Theorem 7.5. *Let* X *be a locally compact space, let* μ *be a Radon measure on* X, *and let* $\mathcal{F}$ *be a non-degenerate partition of* X *into* μ*-tight compact sets. Then*[2] $\mathcal{F}$ *is a sufficient* μ*-finite* $Bor(X)$*-partition of* X.

[2] See Section 5 for the terminology.

Proof. What we to prove are the following properties:

(i) $\mathcal{F}$ is pairwise disjoint, and $\bigcup_{F\in\mathcal{F}} F = X$;
(ii) $\mathcal{F} \subset \mathcal{B}$, and $\mu(F) < \infty$, for all $F \in \mathcal{F}$;
(iii) for every set $B \in Bor(X)$, with $\mu(B) < \infty$, one has the equality[3]

$$\mu(B) = \sum_{F\in\mathcal{F}} \mu(B \cap F). \tag{24}$$

Conditions (i) and (ii) are obvious.

To prove condition (iii) we define the sub-collection $\mathcal{G} = \{F \in \mathcal{F} : \mu(F) > 0\}$, so that $\mathcal{G}$ consists on non-degenerate μ-tight compact sets, and the set

$$N_{\mathcal{F}} = X \smallsetminus \big(\bigcup_{F\in\mathcal{G}} F\big)$$

is locally μ^*-neglijeable. Assume now $B \in Bor(X)$ has $\mu(B) < \infty$. By Lemma 7.3, the collection

$$S_{\mathcal{G}}(B) = \big\{F \in \mathcal{G} : B \cap F \neq \varnothing\big\}$$

is at most countable. In particular, the set

$$B_0 = \bigcup_{F\in S_{\mathcal{G}}(B)} (B \cap F) = B \smallsetminus N_{\mathcal{F}}$$

is Borel, and so will be $B \smallsetminus B_0 = B \cap N_{\mathcal{F}}$. On the one hand, since $B \smallsetminus B_0$ is a subset of $N_{\mathcal{F}}$, it follows that $B \smallsetminus B_0$ is locally μ^*-neglijeable. On the other hand, since $B \smallsetminus B_0$ is a subset of B, it follows that $\mu(B \smallsetminus B_0) < \infty$. This clearly forces $\mu(B \smallsetminus B_0) = 0$, so we have the equality

$$\mu(B) = \mu(B_0) = \sum_{F\in S_{\mathcal{G}}(B)} \mu(B \cap F). \tag{25}$$

Notice that, if $F \in \mathcal{F} \smallsetminus S_{\mathcal{G}}(B)$, then either $F \notin \mathcal{G}$, in which case we have $\mu(F) = 0$, or $F \in \mathcal{G} \smallsetminus S_{\mathcal{G}}(B)$, in which case we have $\mu(B \cap F) = 0$. This shows that

$$\mu(B \cap F) = 0,\ \ \forall\, F \in \mathcal{F} \smallsetminus S_{\mathcal{G}}(B),$$

so the equality (25) immediately gives (24). □

Corollary 7.4. *Under the hypothesis above, the collection $\mathcal{F}$ is a $\tilde{\mu}$-finite decomposition for $\mathfrak{M}_\mu(X)$.*

Proof. Immediate from Corollary 5.3. □

In the remainder of this section we discuss two basic examples of methods for constructing (regular) contents.

To introduce the first construction, let us recall some notations and terminology introduced in II.5 For a locally compact space X, and $\mathbb{K}$ one of the fields $\mathbb{R}$ or $\mathbb{C}$, we denote by $C_c^{\mathbb{K}}(X)$ the space of all continuous functions $f : X \to \mathbb{K}$, with compact support. A $\mathbb{R}$-linear map $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ is said to be *positive*, if it has the property:

$$f \in C_c^{\mathbb{R}}(X),\ f \geq 0 \Rightarrow \phi(f) \geq 0.$$

With these notations, we have the following result.

[3] Here we use the summation convention from II.2

Proposition 7.6. *Let X be a locally compact space, and let $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ be a positive $\mathbb{R}$-linear map. For every compact subset $K \subset X$, define the number*

$$\omega_\phi(K) = \inf\big\{\phi(f) \,:\, f \in C_c^{\mathbb{R}}(X),\ f \geq \varkappa_K\big\}.$$

Then the map $\mathcal{C}_X \ni K \longmapsto \omega_\phi(K) \in [0,\infty)$ is a regular content on X.

Proof. The inequality $f \geq \varkappa_K$ forces $f \geq 0$, so we indeed have $\omega_\phi(K) \geq 0$, $\forall\, K \in \mathcal{C}_X$. We now check conditions (i)-(iv) in the definition of a content.

The constant function 0 satisfies $0 \geq \varkappa_\varnothing$, which immediately gives the equality $\omega_\phi(\varnothing) = 0$, so condition (i) is satisfied.

By the definition of ω_ϕ, it is clear that one has the implication

$$K, L \in \mathcal{C}_X,\ K \subset L \Longrightarrow \omega_\phi(K) \leq \omega_\phi(L),$$

thus giving condition (ii).

To check condition (iii), suppose $K, L \in \mathcal{C}_X$, and let us prove the inequality

$$\omega_\phi(K \cup L) \leq \omega_\phi(K) + \omega_\phi(L). \tag{26}$$

Start with some $\varepsilon > 0$, and choose functions $f, g \in C_c^{\mathbb{R}}(X)$, such that $f \geq \varkappa_K$, $g \geq \varkappa_L$, $\phi(f) \leq \omega_\phi(K) + \varepsilon$, and $\phi(g) \leq \omega_\phi(L)$. If we consider the function $h = f + g \in C_c^{\mathbb{R}}(X)$, then we clearly have $h \geq \varkappa_{K\cup L}$, so we will have

$$\omega_\phi(K \cup L) \leq \phi(h) = \phi(f+g) = \phi(f) + \phi(g) \leq \omega_\phi(K) + \omega_\phi(L) + 2\varepsilon.$$

Since the inequality $\omega_\phi(K \cup L) \leq \omega_\phi(K) + \omega_\phi(L) + 2\varepsilon$ holds for arbitrary $\varepsilon > 0$, it will clearly force (26)

Finally, to check condition (iv) we need start with two disjoint sets $K, L \in \mathcal{C}_X$, and we prove the equality

$$\omega_\phi(K \cup L) = \omega_\phi(K) + \omega_\phi(L). \tag{27}$$

By (26) it only suffices to show the inequality

$$\omega_\phi(K \cup L) \geq \omega_\phi(K) + \omega_\phi(L). \tag{28}$$

Start with some arbitrary $\varepsilon > 0$, and choose a function $f \in C_c^{\mathbb{R}}(X)$, with $f \geq \varkappa_{K\cup L}$ and $\phi(f) \leq \omega_\phi(K \cup L) + \varepsilon$. Use Uryshon Lemma for locally compact spaces (Theorem I.5.1) to find a continuous map $\theta : X \to [0,1]$, such that $\theta\big|_K = 1$ and $\theta\big|_L = 0$. The functions $g = f\theta$ and $h = f(1-\theta)$ are obviously continuous, and have compact supports. Moreover, one has the inequalities $g \geq \varkappa_K$ and $h \geq \varkappa_L$. Since $g + h = f$, we get

$$\omega_\phi(K \cup L) + \varepsilon \geq \phi(f) = \phi(g+h) = \phi(g) + \phi(h) \geq \omega_\phi(K) + \omega_\phi(L).$$

Since the inequality $\omega_\phi(K \cup L) + \varepsilon \geq \omega_\phi(K) + \omega_\phi(L)$ holds for all $\varepsilon > 0$, it will clearly force the inequality (28)

So far, we have shown that ω_ϕ is a content. We now prove that ω_ϕ is regular, which means that, for every $K \in \mathcal{C}_X$, one has the equality

$$\omega_\phi(K) = \inf\big\{\omega_\phi(L) \,:\, L \in \mathcal{C}_X,\ K \subset \operatorname{Int}(L).$$

By property (ii) we always have the inequality

$$\omega_\phi(K) \leq \inf\big\{\omega_\phi(L) \,:\, L \in \mathcal{C}_X,\ K \subset \operatorname{Int}(L),$$

so all we need to prove is the inequality

$$\omega_\phi(K) \geq \inf\big\{\omega_\phi(L) \,:\, L \in \mathcal{C}_X,\ K \subset \operatorname{Int}(L). \tag{29}$$

Start with some arbitrary $\varepsilon > 0$, and choose a function $f \in C_c^{\mathbb{R}}(X)$ with $f \geq \varkappa_K$, and $\phi(f) \leq \omega_\phi(K) + \varepsilon$. Consider the function $g = (1+\varepsilon)f$, and the set

$$D = \{x \in X \,:\, g(x) > 1\}.$$

Obviously D is an open set, and since $f(x) \geq 1$, $\forall\, x \in K$, we get $g(x) \geq 1 + \varepsilon > 1$, $\forall\, x \in K$. In particular, this gives the inclusion $K \subset D$. Apply then Lemma I.5.1 to find some compact set $L \subset D$, with $K \subset \text{Int}(L)$. Since $g(x) > 1$, $\forall\, x \in L$, we clearly have

$$\omega_\phi(L) \leq \phi(g) = (1+\varepsilon)\phi(f) \leq (1+\varepsilon)(\omega_\phi(K) + \varepsilon).$$

This argument shows that, if we denote the right hand side of (29) by $\nu(K)$, then we have the inequality

$$\nu(K) \leq (1+\varepsilon)(\omega_\phi(K) + \varepsilon).$$

Since this inequality holds for all $\varepsilon > 0$, it will force the inequality $\nu(K) \leq \omega_\phi(K)$, thus proving (29). □

Definition. Let X be a locally compact space, and let $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ be a positive $\mathbb{R}$-linear map. We apply Corollary 7.2 to the regular content ω_ϕ, and we will denote the Radon measure extension of ω_ϕ simply by μ_ϕ. The measure μ_ϕ on $Bor(X)$ is called the *Riesz measure associated with* ϕ.

An interesting property, which will later be generalized, is the following.

Lemma 7.4 (Mean Value Property). *Let X be a locally compact space, let $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ be a positive $\mathbb{R}$-linear map, and let μ_ϕ be the Riesz measure associated with ϕ. For any function $f \in C_c^{\mathbb{R}}(X)$, and any compact subset $K \subset X$, with $K \supset \operatorname{supp} f$, one has the inequality*

$$\Big[\min_{x\in K} f(x)\Big] \cdot \mu_\phi(K) \leq \phi(f) \leq \Big[\max_{x\in K} f(x)\Big] \cdot \mu_\phi(K). \tag{30}$$

Proof. Since $\min_{x\in K} f(x) = -\max_{x\in K}(-f)(x)$, it suffices to prove only the inequality

$$\phi(f) \leq \Big[\max_{x\in K} f(x)\Big] \cdot \mu_\phi(K). \tag{31}$$

Fix $f \in C_c^{\mathbb{R}}(X)$, as well as the compact set $K \supset \operatorname{supp} f$. Denote the number $\max_{x\in K} f(x)$ simply by M.

If $M < 0$ the inequality is pretty clear, because the function $g = \frac{f}{M}$ satisfies $g \geq \varkappa_K$, which gives $\phi(g) \geq \omega_\phi(K) = \mu_\phi(K)$, and then multiplying by M immediately gives (31).

The case $M = 0$ is also trivial, since this forces $f \leq 0$, so we get $\phi(f) \leq 0$.

Assume $M > 0$. Fix for the moment some $\varepsilon > 0$, and choose some function $h \in C_c^{\mathbb{R}}(X)$, with $h \geq \varkappa_K$, and $\phi(h) \leq \mu_\phi(K) + \varepsilon$.

Let us observe that $Mh - f \geq 0$. Indeed, if we start with some arbitrary point $x \in X$, then either $x \in K$, in which case we have $Mh(x) \geq M \geq f(x)$, or we have $x \in X \smallsetminus K$, in which case $Mh(x) \geq 0 = f(x)$.

Using the positivity of ϕ we then get $\phi(Mh - f) \geq 0$, which by the choice of h gives

$$\phi(f) \leq \phi(Mh) = M\phi(h) \leq M\big(\mu_\phi(K) + \varepsilon\big).$$

Since the inequality $\phi(f) \leq M\big(\mu_\phi(K) + \varepsilon\big)$ holds for arbitrary $\varepsilon > 0$, it will clearly force $\phi(f) \leq M\mu_\phi(K)$. □

The Riesz measure can be implicitly characterized by the following result.

PROPOSITION 7.7. *With the notations above, the Riesz measure μ_ϕ is the unique Radon measure which has the interpolation property:*

(I_ϕ) *whenver $F \subset X$ is compact, $D \subset X$ is open, and $f \in C_c^{\mathbb{R}}(X)$ satisfies $\varkappa_F \leq f \leq \varkappa_D$, it follows that one has the inequality*

$$\mu_\phi(F) \leq \phi(f) \leq \mu_\phi(D).$$

PROOF. Let us first show that μ_ϕ has property (I_ϕ). Start with F, D and f as in (I_ϕ). Since $\mu_\phi(F) = \omega_\phi(F)$, by the definition of ω_ϕ, we immediately get the inequality $\mu_\phi(F) \leq \phi(f)$.

To prove the inequality $\phi(f) \leq \mu_\phi(D)$, we need some preparations. For every integer $n \geq 1$ we define the sets

$$A_n = \big\{x \in X \,:\, f(x) > \frac{1}{n}\big\} \text{ and } B_n = \big\{x \in X \,:\, f(x) \geq \frac{1}{n}\big\}.$$

Define also the set $E = \{x \in X \,:\, f(x) > 0\}$, so that $\overline{E} = \operatorname{supp} f$. (Here we use the obvious fact that $f \geq 0$.) The sets A_n, $n \geq 1$ are open. The sets B_n, $n \geq 1$ are closed subsets of $E \subset \overline{E}$, hence they are compact. Notice also that we have the inclusions

$$A_1 \subset B_1 \subset A_2 \subset B_2 \subset \cdots \subset E \subset D.$$

For every $n \geq 1$, we use Urysohn Lemma to find a continuous function $h_n : X \to [0,1]$, with $h_n\big|_{B_n} = 1$ and $h_n\big|_{X \smallsetminus A_{n+1}} = 0$. On the one hand, we notice that the function $f(1-h_n)$ has the support contained in the compact set $\overline{E} \smallsetminus A_n \subset X \smallsetminus A_n$. Moreover, since we clearly have $f(x) \leq \frac{1}{n}$, $\forall\, x \in X \smallsetminus A_n$, by Lemma 7.4 we get the inequality

$$\phi(f) = \phi(fh_n) + \phi\big(f(1-h_n)\big) \leq \phi(fh_n) + \frac{\mu_\phi(\overline{E} \smallsetminus A_n)}{n} \leq \phi(fh_n) + \frac{\mu_\phi(\overline{E})}{n}, \ \ \forall\, n \geq 1,$$

which shows that

$$\phi(f) \leq \limsup_{n\to\infty} \phi(fh_n). \tag{32}$$

On the other hand, for each $n \geq 1$, the function fh_n has support contained in B_{n+1}, and $(fh_n)(x) \leq 1$, $\forall\, x \in B_{n+1}$, so again by Lemma 7.4 combined with the inclusion $B_{n+1} \subset D$, we get

$$\phi(fh_n) \leq \mu_\phi(B_{n+1}) \leq \mu_\phi(D).$$

Using (32) we immediately get $\phi(f) \leq \mu_\phi(D)$.

We now prove the uniqueness. Let μ be a Radon measure with property (I_ϕ).

Claim 1: *For any compact set $K \subset X$ and any open set $D \subset X$, with $K \subset D$, one has the inequality*

$$\mu_\phi(K) \leq \mu(D).$$

Choose a compact set $L \subset X$, with $K \subset \operatorname{Int}(L) \subset L \subset D$, and use Urysohn Lemma to find a continuous function $f : X \to [0,1]$ such that $f\big|_K = 1$ and $f\big|_{X \smallsetminus \operatorname{Int}(L)} = 0$. In particular, f has compact support, and satisfies $\varkappa_K \leq f \leq \varkappa_D$. Using ($\mathrm{I}_\phi$) for μ_ϕ and for μ, we then get $\mu_\phi(K) \leq \phi(f) \leq \mu(D)$, and we are done.

Claim 2: *for every compact set $K \subset X$, one has the equality $\mu_\phi(K) = \mu(K)$.*

On the one hand, by the definition of the Radon measure, we have

$$\mu(K) = \inf \big\{ \mu(D) \,:\, D \subset X \text{ open, with } D \supset K \big\}.$$

By Claim 1, this immediately gives the inequality $\mu_\phi(K) \le \mu(K)$. On the other hand, if we choose, for every $\varepsilon > 0$, a function $f_\varepsilon \in C_c^{\mathbb{R}}(X)$ with $f \ge \varkappa_K$ and $\phi(f) \le \mu_\phi(K) + \varepsilon$, then the function $g_\varepsilon = \min\{f_\varepsilon, 1\}$ will also satisfy $g_\varepsilon \ge \varkappa_K$, and $\phi(g_\varepsilon) \le \phi(f_\varepsilon) \le \mu_\phi(K) + \varepsilon$. Applying (I_ϕ) for μ, with $C = K$ and $X = D$ will then force $\mu(K) \le \phi(g_\varepsilon) \le \mu_\phi(K) + \varepsilon$. Since the inequality $\mu(K) \le \mu_\phi(K) + \varepsilon$ holds for all $\varepsilon > 0$, it will force $\mu(K) \le \mu_\phi(K)$.

Having proven Claim 2, we now see that, using condition (iii) in the definition of Radon measures, we get the equality $\mu(D) = \mu_\phi(D)$, for all open sets $D \subset X$. Using condition (ii) from the definition, it then follows that $\mu(B) = \mu_\phi(B)$, $\forall\, B \in Bor(X)$. □

Comment. The *Riesz correspondence*

$$\left\{ \begin{array}{c} \text{positive } \mathbb{R}\text{-linear maps} \\ C_c^{\mathbb{R}}(X) \to \mathbb{R} \end{array} \right\} \ni \phi \longmapsto \mu_\phi \in \left\{ \begin{array}{c} \text{Radon measures} \\ \text{on } X \end{array} \right\}.$$

will be studied in Chapter IV, where we will eventially prove the fact that it is bijective. At this point we simply regard it as a method of constructing Radon measures.

Proposition 7.8. *Let X be a locally compact space. Then the Riesz correspondence is "linear" in the following sense.*

- (i) *If $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ is a positive $\mathbb{R}$-linear map, and $t \in [0, \infty)$, then $t\phi$ is also a positive $\mathbb{R}$-linear map, and one has the equality $\mu_{t\phi} = t\mu_\phi$.*
- (ii) *If $\phi_1, \phi_2 : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ are positive $\mathbb{R}$-linear maps, then $\phi_1 + \phi_2$ is also a positive $\mathbb{R}$-linear map, and one has the equality $\mu_{\phi_1+\phi_2} = \mu_{\phi_1} + \mu_{\phi_2}$.*

Proof. (i). Assume ϕ is positive and $t \in [0, \infty)$. The fact that $t\phi$ is positive is trivial. We know, by Proposition 7.2, that $t\mu_\phi$ is a radon measure. Then the equality $\mu_{t\phi} = t\mu_\phi$ follows from Proposition 7.5, combined with the obvious fact that $\mu_{t\phi}$ has the interpolation property $(\mathrm{I}_{t\phi})$

(ii). If ϕ_1 and ϕ_2 are positive, then so is $\phi_1 + \phi_2$. Define $\psi = \phi_1 + \phi_2$, and $\nu = \mu_{\phi_1} + \mu_{\phi_2}$. By Proposition 7.2, we again know that ν is a Radon measure. The equality $\mu_\psi = \nu$ follows from Proposition 7.5, combined with the obvious fact that ν has the interpolation property (I_ψ) □

The Riesz correspondence is also functorial, with respect to proper maps, in the following sense.

Proposition 7.9. *Let X and Y be locally compact spaces, let $\Phi : X \to Y$ be a proper continuous map, and let $\phi : C_c^{\mathbb{R}}(X)$ be a positive linear map.*

- (i) *Whenever $f : Y \to \mathbb{R}$ is a continuous function with compact support, it follows that the composition $f \circ \Phi : X \to \mathbb{R}$ is also a continuous function with compact support.*
- (ii) *The map*

$$\psi : C_c^{\mathbb{R}}(Y) \ni f \longmapsto \phi(f \circ \Phi) \in \mathbb{R}$$

is $\mathbb{R}$-linear and positive.

(iii) *If μ_ϕ is the Riesz measure on X defined by ϕ, and if μ_ψ is the Riesz measure on Y defined by ψ, then one has the equality*

$$\mu_\psi(B) = \mu_\phi\big(\Phi^{-1}(B)\big), \ \forall B \in Bor(Y).$$

PROOF. (i). This statement is trivial, since Φ is proper.

(ii). The linearity of ψ is a consequence of the linearity of the map

$$T : C_c^{\mathbb{R}}(Y) \ni f \longmapsto f \circ \Phi \in C_c^{\mathbb{R}}(X),$$

and of the obvious equality $\psi = \phi \circ T$.

(iii). Use Proposition 7.3, which states that the map $\nu : Bor(Y) \to [0, \infty]$, defined by

$$\nu(B) = \mu_\phi\big(\Phi^{-1}(B)\big), \ \forall B \in Bor(Y),$$

is a Radon measure. In order to prove statement (iii), which reads $\mu_\psi = \nu$, we observe that, using Proposition 7.5, it suffices to prove that ν has the interpolation property (I_ψ). Fix then a compact set K and an open set $D \subset Y$, as well as a function $f \in C_c^{\mathbb{R}}(Y)$, such that $\varkappa_K \leq f \leq \varkappa_D$, and let us prove the inequalities

$$\nu(K) \leq \psi(f) \leq \nu(D). \tag{33}$$

Define the compact set $L = \Phi^{-1}(K)$ (here we use the fact that Φ is proper), and define the open set $E = \Phi^{-1}(D) \subset X$, so that $\nu(K) = \mu_\phi(L)$ and $\nu(D) = \mu_\phi(E)$. If we define, using (i), the function $g = f \circ \Phi \in C_c^{\mathbb{R}}(X)$, then we have $\psi(f) = \phi(g)$, and the inequalities (33) are the same as the inequalities

$$\mu_\phi(L) \leq \phi(g) \leq \mu_\phi(E).$$

But these inequalities follow immediately from the interpolation property of μ_ϕ, combined with the obvious inequalities $\varkappa_L \leq g \leq \varkappa_E$. □

REMARKS 7.4. Let X be a locally compact space, let $\phi : C_c^{\mathbb{R}}(X) \to \mathbb{R}$ be a positive $\mathbb{R}$-linear map, and let μ_ϕ be the Riesz measure defined by ϕ.

A. One has the equality

$$\mu_\phi(X) = \sup\big\{\phi(f) \,:\, f \in C_c^{\mathbb{R}}(X), \ 0 \leq f \leq 1\big\}. \tag{34}$$

Indeed, if we denote the right hand side of (34) by M, then the inequality $\mu_\phi(X) \geq M$ is immediate from the interpolation property. In fact, if for each compact set $K \subset X$, we choose (use Urysohn Lemma) some continuous function $f_K : X \to [0, 1]$, with compact support, such that $f_K\big|_K = 1$, then by the interpolation property we get $M \geq \phi(f_K) \geq \mu_\phi(K)$, so we have

$$M \geq \sup\big\{\mu_\phi(K) \,:\, K \in \mathcal{C}_X\big\} = \mu_\phi(K).$$

B. As a consequence of the equality (34), and of Remark II.5.4, we get the equivalence

$$\phi \text{ continuous } \Leftrightarrow \mu_\phi(X) < \infty.$$

Moreover, in this case one has the equality $\|\phi\| = \mu_\phi(X)$.

C. Assume X is non-compact, and ϕ is continuous. Then ϕ can be extended to a positive linear function ϕ' on the completion $C_0^{\mathbb{R}}(X)$ of $C_c^{\mathbb{R}}(X)$. In this case the Riesz correspondence has a nice connection with the Alexandrov compactification $X^\alpha = X \sqcup \{\infty\}$ (see I.5 and II.5). Recall that $C_0^{\mathbb{R}}(X)$ is identified with the space of all continuous functions $f : X^\alpha \to \mathbb{R}$ with $f(\infty) = 0$. Moreover, ϕ' has a unique extension to a positive linear map $\psi : C^{\mathbb{R}}(X^\alpha) \to \mathbb{R}$, with $\|\phi\| = \|\phi'\| = \|\phi\|$.

We can then consider two Riesz measures μ_ϕ on X, and μ_ψ on X^α. One has the equality

$$\mu_\psi(B) = \mu_\phi(B \cap X),\ \ \forall\, B \in Bor(X^\alpha). \tag{35}$$

First of all, remark that

$$\mu_\psi(K) = \mu_\phi(K),\ \ \forall\, K \in \mathcal{C}_X. \tag{36}$$

This is a consequence of the fact that for every $g \in C^{\mathbb{R}}(X^\alpha)$ with $g \geq \varkappa_K$, there exists some $f \in C_c^{\mathbb{R}}(X)$, with $g \geq f \geq \varkappa_K$ (Simply take $f = gh$, for some continuous function $h : X \to [0,1]$ with compact support, with $h\big|_K = 1$.) Using (36), we immediately get the equality

$$\mu_\psi(B \cap X) = \mu_\phi(B \cap X),\ \ \forall\, B \in Bor(X^\alpha). \tag{37}$$

Using this with $B = X$, we get

$$\mu_\psi(X) = \mu_\phi(X) = \|\phi\| = \|\psi\| = \mu_\psi(X^\alpha),$$

which forces $\mu_\psi(\{\infty\}) = 0$, and then (35) is immediate from (37)

Exercise 6. Consider the case when $X = \mathbb{R}^n$. For every continuous function $f : \mathbb{R}^n \to \mathbb{R}$, with compact support, we define

$$\phi(f) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n)\, dx_1\, dx_2 \cdots dx_n,$$

where the numbers $a_1 < b_1, \dots, a_n < b_n$ are chosen (arbitrarily) such that

$$\operatorname{supp} f \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].$$

(One can show that the multiple integral is independent of the choice of the a's and the b's.) It is obvious that this way we have constructed a positive $\mathbb{R}$-linear map $\phi : C_c^{\mathbb{R}}(\mathbb{R}^n) \to \mathbb{R}$. The Riesz measure μ_ϕ, defined by ϕ, is precisely the Lebesgue measure λ_n.

Hint: Compute the values of μ_ϕ on compact boxes.

We conclude this section with an important result from harmonic analysis. The main object of study is explained in the following.

Definition. A *topological group* is a group G, which comes also equipped with a topology, which is compatible with the group structure in the sense that the map

$$G \times G \ni (g, h) \longmapsto gh^{-1} \in G$$

is continuous. Remark that is equivalent to the fact that both maps $G \times G \ni (g, h) \longmapsto gh \in G$ and $G \ni g \longmapsto g^{-1} \in G$ are continuous. To avoid any complications, *all topological groups are assumed to be Hausdorff.*

Examples 7.2. A. Any group becomes a topological group, when equipped with the *discrete topology.* (This is the topology in which every subset is open.)

B. The group $(\mathbb{R}^n, +)$ is a topological group, when equipped with the norm topology.

C. The unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is a topological group, when equipped with the unsual multiplication, and the topology induced from $\mathbb{C}$. More generally, for an integer $n \geq 1$, the n-dimensional torus $\mathbb{T}^n$, equipped with coordinate-wise multiplication, and the product topology, is a topological group.

D. Given an integer $n \geq 1$, the group $GL_n(\mathbb{R})$, of all invertible $n \times n$ matrices (with matrix multiplication as the group operation), is a topological group, when equipped with the topology comming from the identification of $GL_n(\mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$.

NOTATIONS. Let G be a group. For a subset $A \subset G$ and an element $g \in G$, we define the left and right translations of A by g, as the sets

$$gA = \{gh : h \in A\} \text{ and } Ag = \{hg : h \in A\}.$$

For two subsets $A, B \subset G$, we define

$$A \cdot B = \{hk : h \in A,\ k \in B\}.$$

Finally, for a subset $A \subset G$, we define $A^{-1} = \{h^{-1} : h \in A\}$.

REMARK 7.5. There is some similarity between topological groups and metric spaces. The subsets that paly role of open balls are the open neighborhoods of the identity. More explicitly, if G is a topological group, with identity element e, then one has the equalities

$$\begin{aligned}\{N \subset G : N \text{ open neighborhood of } g\} &= \\ = \{gV \subset G : V \text{ open neighborhood of } e\} &= \\ = \{Wg \subset G : W \text{ open neighborhood of } e\}&.\end{aligned}$$

For example, given a metric space (X, d), a map $f : G \to X$ is continuous at some point $g \in G$, if and only if, for every $\varepsilon > 0$, there exists some neighborhood V_ε of e, such that

$$d\big(f(gh), f(g)\big) < \varepsilon,\ \ \forall\, h \in V_\varepsilon.$$

The following two results will be used several times.

LEMMA 7.5. *Suppose G is a topological group, with identity element e. For any open neighborhood U of e, there exists an open neighborhoods V of e, such that $V = V^{-1}$ and $V \cdot V \subset U$.*

PROOF. Fix the open neighborhood U. Use the continuity of the map $G \times H \ni (g, h) \to gh \in G$, at (e, e), to find an open neighborhood D of (e, e) in $G \times G$, such that

$$gh \in U,\ \ \forall\, (g, h) \in D.$$

Since D is open in the product topology, there exist open neighborhoods U_1 and U_2, of e, such that $U_1 \times U_2 \subset D$. Then we obviously have

$$U_1 \cdot U_2 \subset U.$$

Consider the open neighborhood $W = U_1 \cap U_2$. We still have $W \cdot W \subset U$. Finally, using the continuity of the map $G \ni g \longmapsto g^{-1} \in G$, it follows that W^{-1} is also a neighborhood of e. Then we are done, if we take $V = W \cap W^{-1}$. □

PROPOSITION 7.10. *Let G be a topological group, and let $K, L \subset G$ be two compact disjoint sets. Then there exists an open neighborhood V of the identity element e, such that $V = V^{-1}$ and $(K \cdot V) \cap (L \cdot V) = (V \cdot K) \cap (V \cdot L) = \varnothing$.*

PROOF. Consider the continuous map $\phi : G \times G \ni (g, h) \longmapsto gh^{-1} \in G$, and the compact set $C = (K \times L) \cup (L \times K) \subset G \times G$. Since ϕ is continuous, it follows that $\phi(C)$ is a compact subset of G. The condition $K \cap L = \varnothing$ obviously gives the fact that $e \notin \phi(C)$. Since $\phi(C)$ is closed, there exists some open neighborhood U of e, such that $\phi(C) \cap U = \varnothing$. Use Lemma 7.5 to find some open neigborhood V of e, such that $V = V^{-1}$ and $V \cdot V \subset U$.

We now show that $(K \cdot V) \cap (L \cdot V) = \varnothing$. Suppose the contrary, i.e. there exist $g \in K$, $h \in L$, and $v, w \in V$, such that $gv = hw$. Then we get $h^{-1}g = wv^{-1} \in V \cdot V^{-1} = V \cdot V \subset U$, which is impossible, since $h^{-1}g$ also belongs to $\phi(C)$.

Finally, let us show that we also have $(V \cdot K) \cap (W \cdot L) = \varnothing$. Suppose the contrary, i.e. there exist $g \in K$, $h \in L$, and $v, w \in V$, such that $vg = wh$. Then we get $hg^{-1} = w^{-1}v \in V^{-1} \cdot = V \cdot V \subset U$, which is impossible, since gh^{-1} also belongs to $\phi(C)$. □

In what follows we are going to restrict our attention to those topological groups which are locally compact in their respective topology. The topological groups listed in Examples 7.2.A-D are all locally compact.

Definition. Let G be a locally compact group. A Radon measure μ on G is called a *Haar measure on* G, if $\mu(G) > 0$, and μ has the *left invariance property:*

$$\mu(gA) = \mu(A), \ \forall\, g \in G, \ A \in Bor(G).$$

Remark that, for every $g \in G$ the map $\ell_g : G \ni h \longmapsto gh \in G$ is a homeomorphism, so for a subset $A \subset G$, one has the equivalence $A \in Bor(G) \Leftrightarrow gA = \ell_g(A) \in Bor(G)$. Likewise, the map $r_g : G \ni h \longmapsto hg \in G$ is a hoemorphism, so $A \in Bor(X) \Leftrightarrow Ag \in Bor(G)$.

Remark 7.6. Let G be a locally compact group. For any element $g \in G$, and any function $F \in C_c^{\mathbb{R}}(G)$, we define the continuous functions $L_gF, R_gF : G \to \mathbb{R}$ by $L_gF = F \circ \ell_{g^{-1}}$ and $R_gF = f \circ r_g$. In other words,

$$(L_gF)(h) = F(g^{-1}h) \text{ and } (R_gF)(h) = F(hg), \ \forall\, h \in G.$$

It is fairly obvious that L_gF and R_gF both have compact support. Moreover, for a fixed $g \in G$, the maps $L_g, R_g : C_c^{\mathbb{R}}(G) \to C_c^{\mathbb{R}}(G)$ are linear, and continuous in the norm defined in Exercise 5. One has the equalities

$$L_{gh} = L_g \circ L_h \text{ and } R_{gh} = R_g \circ R_h, \ \forall\, g, h \in G,$$

as well as $L_e = R_e = \mathrm{Id}$, where e denotes the identity element in G.

The following result gives a sufficient condition for a Riesz measure to be a Haar measures.

Proposition 7.11. *Let G be a locally compact group, and let $\phi : C_c^{\mathbb{R}}(G) \to \mathbb{R}$ be a positive $\mathbb{R}$-linear map, which is not identically zero, and has the left invariance property:*

$$\phi \circ L_g = \phi, \ \forall\, g \in G.$$

Then the Riesz measure μ_ϕ is a Haar measure on G.

Proof. The key property we need is contained in the following

Claim **1**: *For any $g \in G$, and any compact subset $K \subset G$, one has the equality $\mu_\phi(gK) = \mu_\phi(K)$.*

Fix for the moment $g \in G$, as well as the compact set $K \subset G$. The set gK is compact, so we have

$$\mu_\phi(gK) = \inf\big\{\phi(F) \,:\, F \in C_c^{\mathbb{R}}(G), \ F \geq \varkappa_{gK}\big\}. \tag{38}$$

Notice that if $F \in C_c^{\mathbb{R}}(G)$ satisfies $F \geq \varkappa_{gK}$, this means that $F(gh) \geq \varkappa_{gK}(gh)$, $\forall\, h \in G$. Notice that, for any $h \in G$, one has the equivalences

$$\varkappa_{gK}(gh) = 1 \Leftrightarrow gh \in gK \Leftrightarrow h \in K,$$

which means that

$$\varkappa_{gK}(gh) = \varkappa_K(h), \ \ \forall\, h \in K.$$

The inequality $F \geq \varkappa_{gK}$ then gives

$$F(gh) \geq \varkappa_K(h), \ \ \forall\, h \in G,$$

which reads

$$L_{g^{-1}} f \geq \varkappa_K.$$

Using the invariance property, we get

$$\mu_\phi(K) \leq \phi\big(L_{g^{-1}}(F)\big) = (\phi \circ L_{g^{-1}})(F) = \phi(F).$$

In other words, we have

$$\phi(F) \geq \mu_\phi(K), \text{ for all } F \in C_c^{\mathbb{R}}(G) \text{ with } F \geq \varkappa_{gK}.$$

Using (38) this immediately gives

$$\mu_\phi(K) \leq \mu_\phi(gK).$$

Applying the same inequality with g replaced by g^{-1} and K replaced by gK, yields

$$\mu_\phi(gK) \leq \mu_\phi\big(g^{-1}(gK)\big) = \mu_\phi(K),$$

so the Claim follows.

Claim 2: *For any $g \in G$, and any open subset $D \subset G$, one has the equality $\mu_\phi(gD) = \mu_\phi(D)$.*

For a compact subset $L \subset G$, one clearly has the equivalence $L \subset gD \Leftrightarrow g^{-1} \subset D$. So, using Claim 1, for every compact subset $L \subset gD$, one has

$$\mu_\phi(L) = \mu_\phi(g^{-1}L) \leq \mu_\phi(D),$$

and using property (iii) for Radon measures, we immediately get the inequality

$$\mu_\phi(gD) = \sup\big\{\mu_\phi(L) \,:\, L \text{ compact, } L \subset gD\big\} \leq \mu_\phi(D).$$

The inequality $\mu_\phi(D) \leq \mu_\phi(gD)$ is proven by replacing g with g^{-1} and D with gD, in the above inequality.

We now prove that μ_ϕ is a Haar measure. Start with some Borel set $A \subset G$. For every open set $D \supset gA$, one has the inclusion $g^{-1}D \supset A$, which using Claim 2, gives $\mu_\phi(D) = \mu_\phi(g^{-1}D) \geq \mu_\phi(A)$. Using property (ii) in the definition of Radon measures, we then have

$$\mu_\phi(gA) = \inf\big\{\mu_\phi(D) \,:\, D \text{ open, } D \supset gA\big\} \geq \mu_\phi(A).$$

The inequality $\mu_\phi(A) \geq \mu_\phi(gA)$ is proven by replacing g with g^{-1} and A with gA, in the above inequality. □

Comment. Later on, in Chapter IV, we are going to prove that the left invariance property of ϕ is also a necessary condition for μ_ϕ to be a Haar measure.

Examples 7.3. Let us examine the examples 7.2.A-D and let us construct Haar measures on these groups.

A. On a discrete group G, one has the *counting measure* $\mu(A) = \operatorname{Card} A$, $\forall\, A \subset G$, which is obviously a Haar measure.

B. On $(\mathbb{R}^n, +)$, the Lebesgue measure is a Haar measure.

C. On the n-dimensional torus $\mathbb{T}^n$, we consider the Riesz measure μ_Λ, associated with the positive $\mathbb{R}$-linear map $\Lambda : C^{\mathbb{R}}(\mathbb{T}^n) \to \mathbb{R}$, defined by

$$\Lambda(F) = \int_0^1 \cdots \int_0^1 F(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})\, d\theta_1 \dots d\theta_n.$$

It is not hard to see that $\Lambda \circ L_g = \Lambda$, $\forall\, g \in \mathbb{T}^n$. One easy way is to check directly the equality $(\Lambda \circ L_g)(P) = \Lambda(P)$, for functions of the form $P(z_1, \dots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$, with $m_1, \dots, m_n \in \mathbb{Z}$, and then use continuity and the Stone-Weierstrass Theorem which gives the fact that the linear span of all these P's is dense in $C^{\mathbb{R}}(\mathbb{T}^n)$. Using Proposition 7.6 it follows that μ_Λ is a Haar measure on $\mathbb{T}^n$.

D. The construction of a Haar measure on $GL_n(\mathbb{R})$ is outlined in the following.

*Exercise 7**. Identify $GL_n(\mathbb{R})$ as an open subset in $\mathbb{R}^{n^2}$. For every continuous function $F : GL_n(\mathbb{R}) \to \mathbb{R}$, with compact support, $\breve{F} : \mathbb{R}^{n^2} \to \mathbb{R}$ by

$$\breve{F}(\mathbf{x}) = \begin{cases} F(\mathbf{x}) \cdot |\det \mathbf{x}|^{-n} & \text{if } \mathbf{x} \in GL_n(\mathbb{R}) \\ 0 & \text{if } \mathbf{x} \in \mathbb{R}^{n^2} \smallsetminus GL_n(\mathbb{R}) \end{cases}$$

and we define

$$\psi(F) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_{n^2}}^{b_{n^2}} f(x_1, x_2, \dots, x_{n^2})\, dx_1\, dx_2 \cdots dx_{n^2},$$

where the numbers $a_1 < b_1, \dots, a_{n^2} < b_{n^2}$ are chosen (arbitrarily) such that

$$\operatorname{supp} \breve{F} \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_{n^2}, b_{n^2}].$$

(On has the equality $\operatorname{supp} \breve{F} = \operatorname{supp} F$, and the multiple integral is independent of the choice of the a's and the b's.) Prove that $\psi \circ L_{\mathbf{s}} = \psi$, $\forall\, \mathbf{s} \in GL_n(\mathbb{R})$. Conclude that the Riesz measure μ_ψ associated with ψ is a Haar measure on $GL_n(\mathbb{R})$.

Hints: Fix $\mathbf{s} \in GL_n(\mathbb{R})$. The map $\ell_{\mathbf{s}^{-1}} : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ has an obvious linear extension $\Phi_{\mathbf{s}} : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, defined by

$$\Phi_{\mathbf{s}}(\mathbf{x}) = \mathbf{s}^{-1}\mathbf{x},\ \ \forall\, \mathbf{x} \in \mathbb{R}^{n^2},$$

where the vector space $\mathbb{R}^{n^2}$ is identified with $Mat_{n\times n}(\mathbb{R})$. Fix now $F \in C_c^{\mathbb{R}}\big(GL_n(\mathbb{R})\big)$ and consider the function $H = F \circ \ell_{\mathbf{s}^{-1}}$, so that $(\psi \circ L_{\mathbf{s}})(F) = \psi(H)$. Prove the equality

$$\breve{H}(\mathbf{x}) = \breve{F}\big(\Phi_{\mathbf{s}}(x)\big) \cdot |\det \mathbf{s}|^{-n},\ \ \forall\, \mathbf{x} \in \mathbb{R}^n.$$

Prove that the Jacobian of $\Phi_{\mathbf{s}}$ is given as

$$\big|\det[(D\Phi_{\mathbf{s}})(\mathbf{x})]\big| = |\det \mathbf{s}|^{-n},\ \ \forall\, \mathbf{x} \in \mathbb{R}^n.$$

Use this equality, combined with the above formula for $\breve{H}$, to get the equality $\psi(H) = \psi(F)$, as a result of the change of variable theorem. (Use the fact that in the definition of ψ, instead of integrating over rectangles one can integrate over arbitrary compact sets $\Omega \subset GL_n(\mathbb{R})$, with Jordan neglijeable boundary, and $\operatorname{Int}(\Omega) \supset \operatorname{supp} F$.)

Comments. The Haar measures defined in Examples 7.3.A-D are peculiar in the sense that they also have the *right invariance property:*

$$\mu(Ag) = \mu(A),\ \ \forall\, g \in G,\ A \in Bor(G).$$

In general such a property does not hold. At this point, we can only speculate on this matter, by examining the following example.

*Exercise 8**. Consider the group G of all affine orientation preserving affine transformations of $\mathbb{R}$, i.e. the collection

$$G = \{T_{ab} : a, b \in \mathbb{R},\ a > 0\},$$

where $T_{ab} : \mathbb{R} \ni x \longmapsto ax + b \in \mathbb{R}$. (Some people call this the "$ax + b$" group.) It is not hard to see that compositions and inverses of such transformations are again of this form. In fact one can identify G as the subgroup of $GL_2(\mathbb{R})$ given by

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a, b \in \mathbb{R},\ a > 0 \right\}.$$

The topology on G is the one induced from this inclusion. Equivalently, G can be identified with the right half-plane $(0, \infty) \times \mathbb{R}$. We use this identification to define a positive $\mathbb{R}$-linear map $\Lambda : C_c^{\mathbb{R}}(G) \to \mathbb{R}$ as follows. For every $F \in C_c^{\mathbb{R}}(G)$, we choose $0 < c_1 < d_1$ and $c_2 < d_2$, such that $\operatorname{supp} F \subset [c_1, d_1] \times [c_2, d_2]$, and we define

$$\Lambda(F) = \int_{c_1}^{d_1} \int_{c_2}^{d_2} \frac{F(a,b)}{a^2}\, da\, db.$$

The integral does not depend on the particular choice of the rectangle. Prove that $\Lambda \circ L_g = \Lambda$, $\forall\, g \in G$, so that the Riesz measure μ_Λ is a Haar measure. In general the equality $\Lambda \circ R_g = \Lambda$ fails. As indicated in the comment that followed Proposition 7.6, the fact that $\Lambda \circ R_g \neq \Lambda$ would prevent the Riesz measure μ_Λ from having the right invariance property.

HINTS: Use similar arguments to the ones in Exercise 8. If $g = T_{ab} \in G$, then the map $\ell_{g^{-1}} : G \to G$ extends to a linear map $\Phi_g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$\Phi_g(x, y) = (ax + by, y),\ \ \forall\, (x, y) \in \mathbb{R}^2.$$

Argue as in Exercise 8, and use the change of variable theorem.

Exercise 9. As indicated above, in general, Haar measures need not have the right invariance property. Prove that when μ is a Haar measure on G, then the map $\nu : Bor(G) \to [0, \infty]$ defined by

$$\nu(B) = \mu(B^{-1}),\ \ \forall\, B \in Bor(G),$$

is a Radon measure, which has the right invariance property.

HINT: The map $G \ni g \longmapsto g^{-1} \in G$ is a homeomorphism.

The main result we are interested in is the *existence of a Haar measure*. The following result reduces the problem to the existence of a left invariant content.

LEMMA 7.6. *Let G be a locally compact group, and let ω be a content on G, with the left invariance property:*

$$\omega(gK) = \omega(K),\ \ \forall\, g \in G,\ K \in \mathcal{C}_G.$$

If ω is not identically zero, then the outer measure ω^*, induced by ω, also has the left invariance property:

$$\omega^*(gA) = \omega(A),\ \ \forall\, g \in G,\ A \subset G.$$

The measure $\mu = \omega^*\big|_{Bor(G)}$ is a Haar measure on G.

PROOF. We trace the construction outlined in Theorem 7.1. Denote by $\mathcal{T}_G$ the collection of all open subsets of G, and define the map $\hat{\omega} : \mathcal{T}_G \to [0, \infty]$ by

$$\hat{\omega}(D) = \sup \{\omega(K) : K \in \mathcal{C}_G,\ K \subset D\},\ \ \forall\, D \in \mathcal{T}_G.$$

The outer measure ω^* is then defined by

$$\omega^*(A) = \inf\big\{\hat{\omega}(D) \,:\, D \in \mathcal{T}_G,\ D \supset A\big\},\ \ \forall\, A \subset G.$$

Claim: *The map $\hat{\omega} : \mathcal{T}_G \to [0, \infty]$ has the left invariance property:*

$$\hat{\omega}(gD) = \hat{\omega}(D),\ \ \forall\, g \in G,\ D \in \mathcal{T}_G.$$

Start with some arbitrary compact subset $K \subset gD$. Then $g^{-1}K$ is a compact subset of D, so by the left invariance property of ω, we get

$$\omega(K) = \omega(g^{-1}K) \le \hat{\omega}(D).$$

This means that we have $\omega(K) \le \hat{\omega}(D)$, for all compact subsets $K \subset gD$, so by the definition of $\hat{\omega}$ we get

$$\hat{\omega}(gD) = \sup\big\{\omega(K) \,:\, K \in \mathcal{C}_G,\ K \subset gD\big\} \le \hat{\omega}(D).$$

The other inequality $\hat{\omega}(D) \le \hat{\omega}(gD)$, follows from the one above if we replace g with g^{-1} and D with gD.

We are now in position to prove that ω^* has the left invariance property. Fix for the moment $A \subset G$ and $g \in G$. For every open set $D \supset gA$, one has $g^{-1}D \supset A$, so by the Claim we get

$$\hat{\omega}(D) = \hat{\omega}(g^{-1}D) \ge \omega^*(A).$$

Since we have $\hat{\omega}(D) \ge \omega^*(A)$, for all open sets $D \supset gA$, by the definition of ω^*, we get

$$\omega^*(gA) = \inf\big\{\hat{\omega}(D) \,:\, D \in \mathcal{T}_G,\ D \supset gA\big\} \ge \omega^*(A).$$

The other inequality $\omega^*(A) \ge \omega^*(gA)$, follows from the one above if we replace g with g^{-1} and A with gA.

In order to prove that μ is a Haar measure, all we need to prove is the fact that $\mu(G) > 0$. Start with some compact subset $K \subset G$, with $\omega(K) > 0$. We have

$$\mu(G) \ge \mu(K) = \omega^*(K) = \breve{\omega}(K) \ge \omega(K) > 0,$$

and we are done. ☐

Before we prove the existence of Haar measures, we need more preparations.

Notations. Let G be a group. For two non-empty subsets $A, B \subset G$, we write $A \prec B$, if there exist elements $g_1, \dots, g_n \in G$, such that $A \subset g_1B \cup \dots \cup g_nB$. In this case we define the number

$$[A : B] = \min\{n \in \mathbb{N} \,:\, \text{there exist } g_1, \dots, g_n \in G \text{ with } K \subset g_1V \cup \dots \cup g_nV\big\}.$$

The following result will be useful.

Lemma 7.7. *Let G be a group.*

(i) *If $A, B \subset G$ are non-empty sets with $A \subset B$, then $A \prec B$, and $[A : B] = 1$.*

(ii) *The relation $\prec$ is transitive, i.e. whenever $A, B, C \subset G$ are non-empty subsets satisfying $A \prec B$ and $B \prec C$, it follows that $A \prec C$. Moreover, in this case one has the inequality*

$$[A : C] \le [A : B] \cdot [B : C].$$

(iii) *The relation $\prec$ is compatible with left translations. This means that for any two elements $g, h \in G$, and any two non-empty subsets $A, B \subset G$, one has the equivalence $A \prec B \Leftrightarrow gA \prec hB$. Moreover, in this case one has*

$$[gA : hB] = [A : B].$$

(iv) *If $A, B, C \subset G$ are non-empty subsets such that $A \prec C$ and $B \prec C$, then $A \cup B \prec C$. Moreover, in this case one has the inequality*

$$[A \cup B : C] \leq [A : C] + [B : C].$$

(v) *If $A, B, C \subset G$ are non-empty sets, such that $A \prec C$, $B \prec C$, and $(A \cdot C^{-1}) \cap (B \cdot C^{-1}) = \varnothing$, then one has the equality*

$$[A \cup B : C] = [A : C] + [B : C].$$

Proof. (i) This part is trivial.

(ii) Put $m = [A : B]$ and $n = [B : C]$. Choose $g_1, \dots, g_m, h_1, \dots, h_n \in G$, such that $A \subset g_1 B \cup \dots \cup g_m B$, and $B \subset h_1 C \cup \dots \cup h_n C$. We then obviously have the inclusion

$$A \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (g_i h_j) C,$$

which proves that $A \prec C$, but also shows that $[A : C] \leq mn$.

(iii) This follows immediately from (ii) plus the obvious relations $A \prec gA \prec A$, $B \prec hB \prec B$, and the equalities

$$[A : gA] = [gA : A] = [B : hB] = [hB : B] = 1.$$

(iv) Let $m = [A : C]$ and $n = [B : C]$. Choose $g_1, \dots, g_m, g_{m+1}, \dots, g_{m+n} \in G$ such that $A \subset g_1 C \cup \dots \cup g_m C$ and $B \subset g_{m+1} C \cup \dots \cup g_{m+n} C$. This clearly shows that $A \cup B \prec C$ and $[A \cup B : C] \leq m + n$.

(v) Let $p = [A \cup B : C]$, and choose $g_1, \dots, g_p \in G$, such that $A \cup B \subset g_1 C \cup \dots \cup g_p C$. Define the sets

$$M = \big\{ j \in \{1, \dots, p\} : A \cap g_j C \neq \varnothing \big\} \text{ and } N = \big\{ k \in \{1, \dots, p\} : B \cap g_k C \neq \varnothing \big\}.$$

Notice that $M \cap N = \varnothing$. Indeed, if there exists $j \in M \cap N$, this means that on the one hand, we have $A \cap g_j C \neq \varnothing$, which gives $g_j \in A \cdot C^{-1}$, and on the other hand, we have $B \cap g_j C \neq \varnothing$ which gives $g_j \in B \cdot C^{-1}$. But this clearly contradicts the assumption that $(A \cdot C^{-1}) \cap (B \cdot C^{-1}) = \varnothing$.

By the definition of M and N, we clearly have the inclusions

$$A \subset \bigcup_{j \in M} g_j C \text{ and } B \subset \bigcup_{k \in N} g_k C.$$

These immediately give the inequalities $[A : C] \leq \operatorname{card} M$ and $[B : C] \leq \operatorname{card} N$.

Since M and N are disjoint, and $M \cup N \subset \{1, \dots, p\}$, these inequalities give

$$[A : C] + [B : C] \leq \operatorname{card} M + \operatorname{card} N = \operatorname{card}(M \cup N) \leq p = [A \cup B : C].$$

Using part (iv), we see that in fact we have equality $[A : C] + [B : C] = [A \cup B : C]$. □

Remark 7.7. If G is a topological group with identity element e, and if V is a neighborhood of e, then $K \prec V$, for every compact subset of G. Indeed, if we choose some open set D with $e \in D \subset V$, then using the compactness of K, and the obvious inclusion $K \subset \bigcup_{g \in K} gD$, it follows that there exists $g_1, \dots, g_n \in K$, such that $K \subset g_1 D \cup \dots \cup g_n D \subset g_1 V \cup \dots \cup g_n V$.

With these preparations we are in position to prove the following fundamental result.

THEOREM 7.6. *Let G be a locally compact group, and let A be a compact neighborhood of the identity element. Then there exists a Haar measure μ on G, such that $\mu(A) = 1$.*

PROOF. Denote the identity element of G by e. Throughout the proof the compact neighborhood A of e will be fixed. For every non-empty compact set $K \subset G$, we define $m(K) = [K : A]$. We also put $m(\varnothing) = 0$.

Let us define $\mathcal{V}$ to be the collection of all neighborhoods of e. For every $V \in \mathcal{V}$, we denote by $\mathbf{\Omega}(V)$ the set of all maps $\omega : \mathcal{C}_G \to [0, \infty)$ with the following properties

(i) $0 \leq \omega(K) \leq m(K)$, $\forall\, K \in \mathcal{C}_G$;
(ii) $\omega(A) = 1$;
(iii) $K, L \in \mathcal{C}_G,\ K \subset L \Rightarrow \omega(K) \leq \omega(L)$;
(iv) $\omega(K \cup L) \leq \omega(K) + \omega(L)$, $\forall\, K, L \in \mathcal{C}_G$;
(v) $\omega(gK) = \omega(K)$, $\forall\, g \in G,\ K \in \mathcal{C}_G$.
(vi) $K, L \in \mathcal{C}_G,\ (K \cdot V) \cap (L \cdot V) = \varnothing \Rightarrow \omega(K \cup L) = \omega(K) + \omega(L)$.

Claim 1: *For every $V \in \mathcal{V}$, the set $\mathbf{\Omega}(V)$ is non-empty.*

Fix V. We shall prove this Claim by an explicit construction of an element $\omega \in \mathbf{\Omega}(V)$. Define $\omega(\varnothing) = 0$, and define

$$\omega(K) = \frac{[K : V^{-1}]}{[A : V^{-1}]},$$

for all non-empty compact subsets $K \subset G$. The fact that ω has properties (i)-(vi) is immediate from Lemma 7.7.

Let us regard the sets $\mathbf{\Omega}(V)$, $V \in \mathcal{V}$ as subsets of the product space

$$\mathbf{P} = \prod_{K \in \mathcal{C}_G} [0, m(K)].$$

Notice that, when we equip $\mathbf{P}$ with the product topology, it becomes a compact space, by Tihonov's Theorem.

Claim 2: *For every $V \in \mathcal{V}$, the set $\mathbf{\Omega}(V)$ is closed in $\mathbf{P}$.*

Define, for any $K \in \mathcal{C}_G$, the map

$$\pi_K : \mathbf{P} \ni \omega \longmapsto \omega(K) \in \mathbb{R}.$$

By the definition of the topology of $\mathbf{P}$, all maps $\pi_K : \mathbf{P} \to \mathbb{R}$ are continuous. For any two sets $K, L \in \mathcal{C}_G$, consider the functions $F_{KL}, T_{KL} : \mathbf{P} \to \mathbb{R}$, defined by

$$F_{KL}(\omega) = \omega(K) - \omega(L) \text{ and } T_{KL}(\omega) = \omega(K \cup L) - \omega(K) - \omega(L),\ \ \forall\, \omega \in \mathbf{P}.$$

Since we have $F_{KL} = \pi_K - \pi_L$ and $T_{KL} = \pi_{K \cup L} - \pi_K - \pi_L$, it follows that the maps $F_{KL}, T_{KL} : \mathbf{P} \to \mathbb{R}$, $K, L \in \mathcal{C}_G$, are all continuous. As a consequence of the continuity of these maps, it follows that, for any two sets $K, L \in \mathcal{C}_G$, the sets

$$\begin{aligned}
\mathbf{\Gamma}(K, L) &= \{\omega \in \mathbf{P} : \omega(K) \leq \omega(L)\} = F_{KL}^{-1}\big((-\infty, 0]\big),\\
\mathbf{\Theta}^-(K, L) &= \{\omega \in \mathbf{P} : \omega(K \cup L) \leq \omega(K) + \omega(L)\} = T_{KL}^{-1}\big((-\infty, 0]\big),\\
\mathbf{\Theta}^+(K, L) &= \{\omega \in \mathbf{P} : \omega(K \cup L) \geq \omega(K) + \omega(L)\} = T_{KL}^{-1}\big([0, \infty)\big)
\end{aligned}$$

are closed subsets of $\mathbf{P}$. It then follows that the sets

$$\begin{aligned}
\mathbf{\Omega}^1 &= \big\{\omega \in \mathbf{P} \,:\, \omega(A) = 1\big\} = \pi_A^{-1}\big(\{1\}\big),\\
\mathbf{\Omega}^2 &= \bigcap_{\substack{(K,L)\in\mathcal{C}_G\times\mathcal{C}_G\\ K\subset L}} \mathbf{\Gamma}(K,L),\\
\mathbf{\Omega}^3 &= \bigcap_{(K,L)\in\mathcal{C}_G\times\mathcal{C}_G} \mathbf{\Theta}^-(K,L),\\
\mathbf{\Omega}^4 &= \bigcap_{K\in\mathcal{C}_G}\bigcap_{g\in G}\big[\mathbf{\Gamma}(K,gK)\cap\mathbf{\Gamma}(gK,K)\big],
\end{aligned}$$

are all closed, so the intersection

$$\mathbf{\Omega}^5 = \mathbf{\Omega}^1 \cap \mathbf{\Omega}^3 \cap \mathbf{\Omega}^3 \cap \mathbf{\Omega}^4$$

is again closed. Notice that

$$\mathbf{\Omega}^5 = \big\{\omega \in \mathbf{P} \,:\, \omega \text{ has properties (i)-(v) }\big\}.$$

Finally, if we define, for every $V \in \mathcal{V}$, the set

$$\mathbf{\Omega}^6_V = \bigcap_{\substack{(K,L)\in\mathcal{C}_G\times\mathcal{C}_G\\ (K\cdot V)\cap(L\cdot V)=\varnothing}} \mathbf{\Theta}^+(K,L),$$

then $\mathbf{\Omega}^6_V$ is also closed, and so will then be the intersection $\mathbf{\Omega}^5 \cap \mathbf{\Omega}^6_V = \mathbf{\Omega}(V)$.

Claim 3: *The intersection $\bigcap_{V\in\mathcal{V}} \mathbf{\Omega}(V)$ is non-empty.*

Remark that, if $V_1, V_2 \in \mathcal{V}$ are such that $V_1 \subset V_2$, then we have the inclusion $\mathbf{\Omega}(V_1) \subset \mathbf{\Omega}(V_2)$. Indeed, if ω belongs to $\mathbf{\Omega}(V_1)$, then properties (i)-(v) are clear. To check property (vi) for V_2 we need to show that whenever $K, L \subset G$ are compact sets, with $(K \cdot V_2) \cap (L \cdot V_2) = \varnothing$, it follows that $\omega(K \cup L) = \omega(K) + \omega(L)$. This is however trivial, since the inclusion $V_1 \subset V_2$ forces $(K \cdot V_1) \cap (L \cdot V_1) = \varnothing$, and then the desired equality follows from the property (vi) for V_1. We now see that, for any finite number of sets $V_1, \dots, V_n \in \mathcal{V}$, we have the inclusion

$$\mathbf{\Omega}(V_1 \cap \dots \cap V_n) \subset \mathbf{\Omega}(V_1) \cap \dots \cap \mathbf{\Omega}(V_n),$$

which by Claim 1, proves that $\mathbf{\Omega}(V_1) \cap \dots \cap \mathbf{\Omega}(V_n) \neq \varnothing$. Using Claim 2, and the compactness of $\mathbf{P}$, the Claim immediately follows.

Pick now an element $\omega \in \bigcap_{V\in\mathcal{V}} \mathbf{\Omega}(V)$.

Claim 4: *The map $\omega : \mathcal{C}_G \to [0,\infty)$ is a content on G with the left invariance property*

$$\omega(gK) = \omega(K),\ \ \forall\, g \in G,\ K \in \mathcal{C}_G.$$

Moreover, one has the equality $\omega(A) = 1$.

The fact that $\omega(A) = 1$ is clear, from condition (ii) in the definition of $\mathbf{\Omega}(V)$. The left invariance property follows from condition (v). In order to prove that ω is a content, we need to prove

(*a*) $\omega(\varnothing) = 0$;
(*b*) $K, L \in \mathcal{C}_G,\ K \subset L \Rightarrow \omega(K) \le \omega(L)$;
(*c*) $\omega(K \cup L) \le \omega(K) + \omega(L),\ \forall\, K, L \in \mathcal{C}_G$;
(*d*) $K, L \in \mathcal{C}_G,\ K \cap L = \varnothing \Rightarrow \omega(K \cup L) = \omega(K) + \omega(L)$.

Properties (a), (b), and (c) are clear, because every element in $\mathbf{\Omega}(V)$, $V \in \mathcal{V}$ satisfies them. (Property (a) is a consequence of condition (i), property (b) is a consequence of (iii), and property (c) is a consequence of (iv).) To prove property (d), we start with two disjoint compact sets K and L, and we use Proposition 7.5 to find some $V \in \mathcal{V}$ such that $(K \cdot V) \cap (L \cap V) = \varnothing$. Then we use the fact that ω belongs to $\mathbf{\Omega}(V)$, and by condition (vi) we indeed get $\omega(K \cup L) = \omega(K) + \omega(L)$.

Having proven Claim 4, we now define the measure $\mu_0 = \omega^*\big|_{Bor(G)}$. By Lemma 7.7, μ_0 is a Haar measure on G. Notice that $\mu_0(A) = \breve{\omega}(A) \geq \omega(A) = 1$, so if we define $\mu : Bor(G) \to [0, \infty]$ by (use the convention $\infty/\mu_0(A) = \infty$)

$$\mu(B) = \frac{\mu_0(B)}{\mu_0(A)}, \ \ \forall\, B \in Bor(G),$$

then μ is a Haar measure on G, and satisfies $\mu(A) = 1$. □

COMMENT. Eventually (see Chapter IV) we are going to improve on the above result by proving the uniqueness of μ.

In concrete examples, it is possible to prove uniqueness.

*Exercise 10**. Let $S = [0, 1]^n$ be the unit square in $\mathbb{R}^n$, and let μ be a Haar measure on $(\mathbb{R}^n, +)$, with $\mu(S) = 1$. Prove that μ coincides with the n-dimensional Lebesgue measure λ_n.

HINT: Consider first the half open box $S_0 = [0, 1)^n$, and its measure $\beta = \mu(S_0)$. Prove that for a half open box of the form

$$B = [a_1, b_1) \times \cdots \times [a_n, b_n)$$

with $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{Q}$, one has $\mu(B) = \beta\lambda_n(B)$. Conclude that if a subset $A \subset \mathbb{R}^n$ is contained in a hyperplane of the form

$$\Pi_k(a) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \,:\, x_k = a\},$$

then $\mu(A) = 0$. Use this to get $\beta = 1$, so

$$\mu(B) = \lambda_n(B),$$

for every "rational" half-open box. Prove that this equality holds for *all* half-open boxes. Use Corollary 5.1 to conclude that $\mu = \lambda_n$.

The following two exercises show how a Haar measure can be used to get some topological information.

Exercise 11. Let G be a locally compact group, and let μ be a Haar measure on G. Prove that $\mu(D) > 0$, for every open subset $D \subset G$.

HINT: Use the inequality $\mu(K) \leq [K : D] \cdot \mu(D)$, for all compact $K \subset G$.

*Exercise 12**. Let G be a locally compact group, and let μ be a Haar measure on G. Prove that the following are equivalent:

(i) G is compact;
(ii) $\mu(G) < \infty$.

HINT: For the implication $(ii) \Rightarrow (i)$, start with some compact neighborhood V of the identity, and choose a maximal subset $A \subset G$, such that the sets gV, $g \in A$ are disjoint. Prove that A is finite. Conclude that $G = \bigcup_{g \in A}(gV \cdot V^{-1})$, so G is a finite union of compact sets.