

## LECTURES 23-25

### 6. The Lebesgue measure

In this section we apply various results from the previous sections to a very basic example: the *Lebesgue measure on  $\mathbb{R}^n$* .

NOTATIONS. We fix an integer  $n \geq 1$ . In Section 21 we introduced the semiring of “half-open boxes” in  $\mathbb{R}^n$ :

$$\mathcal{J}_n = \{\emptyset\} \cup \left\{ \prod_{j=1}^n [a_j, b_j) : a_1 < b_1, \dots, a_n < b_n \right\} \subset \mathcal{P}(\mathbb{R}^n).$$

For a non-empty box  $A = [a_1, b_1) \times \dots \times [a_n, b_n) \in \mathcal{J}_n$ , we defined its  $n$ -dimensional volume by

$$\text{vol}_n(A) = \prod_{k=1}^n (b_k - a_k).$$

We also defined  $\text{vol}_n(\emptyset) = 0$ .

By Theorem 4.2, we know that  $\text{vol}_n$  is a *finite measure on  $\mathcal{J}_n$* .

DEFINITIONS. The maximal outer extension of  $\text{vol}_n$  is called the  *$n$ -dimensional outer Lebesgue measure*, and is denoted by  $\lambda_n^*$ .

The  $\lambda_n^*$ -measurable sets in  $\mathbb{R}^n$  will be called  *$n$ -Lebesgue measurable*. The  $\sigma$ -algebra  $\mathcal{M}_{\lambda_n^*}(\mathbb{R}^n)$  will be denoted simply by  $\mathcal{M}(\mathbb{R}^n)$ . The measure  $\lambda_n^*|_{\mathcal{M}(\mathbb{R}^n)}$  is simply denoted by  $\lambda_n$ , and is called the  *$n$ -dimensional Lebesgue measure*. Although this notation may appear to be confusing, it turns out (see Proposition 5.3) that  $\lambda_n^*$  is indeed the maximal outer extension of  $\lambda_n$ . In the case when  $n = 1$ , the subscript will be omitted.

We know (see Section 21) that

$$\mathbf{S}(\mathcal{J}_n) = \Sigma(\mathcal{J}_n) = \text{Bor}(\mathbb{R}^n).$$

Using the fact that the semiring  $\mathcal{J}_n$  is  $\sigma$ -total in  $\mathbb{R}^n$ , by the definition of the outer Lebesgue measure, we have

$$(1) \quad \lambda_n^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}_n(B_k) : (B_k)_{k=1}^{\infty} \subset \mathcal{J}_n, \bigcup_{k=1}^{\infty} B_k \supset A \right\}, \quad \forall A \subset \mathbb{R}^n$$

Using Corollary 5.2, we have the equality

$$\mathcal{M}(\mathbb{R}^n) = \overline{\text{Bor}(\mathbb{R}^n)},$$

where  $\overline{\text{Bor}(\mathbb{R}^n)}$  is the completion of  $\text{Bor}(\mathbb{R}^n)$  with respect to the measure  $\lambda_n|_{\text{Bor}(\mathbb{R}^n)}$ . This means that a subset  $A \subset \mathbb{R}^n$  is Lebesgue measurable, if and only if there exists a Borel set  $B$  and a negligible set  $N$  such that  $A = B \cup N$ . (The fact  $N$  is

negligible means that  $\lambda_n^*(N) = 0$ , and is equivalent to the existence of a Borel set  $C \supset N$  with  $\lambda_n(C) = 0$ .)

*Exercise 1.* Let  $A = [a_1, b_1) \times \cdots \times [a_n, b_n)$  be a half-open box in  $\mathbb{R}^n$ . Assume  $A \neq \emptyset$  (which means that  $a_1 < b_1, \dots, a_n < b_n$ ). Consider the open box  $\text{Int}(A)$  and the closed box  $\overline{A}$ , which are given by

$$\text{Int}(A) = (a_1, b_1) \times \cdots \times (a_n, b_n) \text{ and } \overline{A} = [a_1, b_1] \times \cdots \times [a_n, b_n].$$

Prove the equalities

$$\lambda_n(\text{Int}(A)) = \lambda_n(\overline{A}) = \text{vol}_n(A).$$

REMARKS 6.1. If  $D \subset \mathbb{R}^n$  is a non-empty open set, then  $\lambda_n(D) > 0$ . This is a consequence of the above exercise, combined with the fact that  $D$  contains at least one non-empty open box.

The Lebesgue measure of a countable subset  $C \subset \mathbb{R}^n$  is zero. Using  $\sigma$ -additivity, it suffices to prove this only in the case of singletons  $C = \{x\}$ . If we write  $x$  in coordinates  $x = (x_1, \dots, x_n)$ , and if we consider half-open boxes of the form

$$J_\varepsilon = [x_1, x_1 + \varepsilon) \times \cdots \times [x_n, x_n + \varepsilon),$$

then the obvious inclusion  $\{x\} \subset J_\varepsilon$  will force

$$0 \leq \lambda_n(\{x\}) \leq \lambda_n(J_\varepsilon) = \varepsilon^n,$$

so taking the limit as  $\varepsilon \rightarrow 0$ , we indeed get  $\lambda_n(\{x\}) = 0$ .

The (outer) Lebesgue measure is completely determined by its values on open sets. More explicitly, one has the following result.

PROPOSITION 6.1. *Let  $n \geq 1$  be an integer. For every subset  $A \subset \mathbb{R}^n$  one has:*

$$(2) \quad \lambda_n^*(A) = \inf\{\lambda_n(D) : D \text{ open subset of } \mathbb{R}^n, \text{ with } D \supset A\}.$$

PROOF. Throughout the proof the set  $A$  will be fixed. Let us denote, for simplicity, the right hand side of (2) by  $\nu(A)$ . First of all, since every open set is Lebesgue measurable (being Borel), we have  $\lambda_n(D) = \lambda_n^*(D)$ , for all open sets  $D$ , so by the monotonicity of  $\lambda_n^*$ , we get the inequality

$$\lambda_n^*(A) \leq \nu(A).$$

We now prove the inequality  $\lambda_n^*(A) \geq \nu(A)$ . Fix for the moment some  $\varepsilon > 0$ , and use (1). to get the existence of a sequence  $(B_k)_{k=1}^\infty \subset \mathcal{J}_n$ , such that  $\bigcup_{k=1}^\infty B_k \supset A$ , and

$$\sum_{k=1}^\infty \text{vol}_n(B_k) < \lambda_n^*(A) + \varepsilon.$$

For every  $k \geq 1$ , we write

$$B_k = [a_1^{(k)}, b_1^{(k)}) \times \cdots \times [a_n^{(k)}, b_n^{(k)}),$$

so that  $\text{vol}_n(B_k) = \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)})$ . Using the obvious continuity of the map

$$\mathbb{R} \ni t \longmapsto \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)} - t) \in \mathbb{R},$$

we can find, for each  $k \geq 1$  some numbers  $c_1^{(k)} < a_1^{(k)}, \dots, c_n^{(k)} < a_n^{(k)}$ , with

$$(3) \quad \prod_{j=1}^n (b_j^{(k)} - c_j^{(k)}) < \frac{\varepsilon}{2^k} + \prod_{j=1}^n (b_j^{(k)} - a_j^{(k)}).$$

Notice that, if we define the half-open boxes

$$E_k = [c_1^{(k)}, b_1^{(k)}) \times \cdots \times [c_n^{(k)}, b_n^{(k)}),$$

then for every  $k \geq 1$ , we clearly have  $B_k \subset \text{Int}(E_k)$ , and by Exercise 1, combined with (3), we also have the inequality

$$\lambda_n(\text{Int}(E_k)) = \text{vol}_n(E_k) < \frac{\varepsilon}{2^k} + \text{vol}_n(B_k).$$

Summing up we then get

$$(4) \quad \sum_{k=1}^{\infty} \lambda_n(\text{Int}(E_k)) < \sum_{k=1}^{\infty} \left[ \frac{\varepsilon}{2^k} + \text{vol}_n(B_k) \right] = \varepsilon + \sum_{k=1}^{\infty} \text{vol}_n(B_k) < 2\varepsilon + \lambda_n^*(A).$$

Now we observe that by  $\sigma$ -sub-additivity we have

$$\lambda_n\left(\bigcup_{k=1}^{\infty} \text{Int}(E_k)\right) \leq \sum_{k=1}^{\infty} \lambda_n(\text{Int}(E_k)),$$

so if we define the open set  $D = \bigcup_{k=1}^{\infty} \text{Int}(E_k)$ , then using (4) we get

$$(5) \quad \lambda_n(D) < 2\varepsilon + \lambda_n^*(A).$$

It is clear that we have the inclusions

$$A \subset \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{k=1}^{\infty} \text{Int}(E_k) = D,$$

so by the definition of  $\nu(A)$ , combined with (5), we finally get

$$\nu(A) \leq \lambda_n(D) < 2\varepsilon + \lambda_n^*(A).$$

Up to this moment  $\varepsilon > 0$  was fixed. Since the inequality  $\nu(A) < 2\varepsilon + \lambda_n^*(A)$  holds for any  $\varepsilon > 0$  however, we finally get the desired inequality  $\nu(A) \leq \lambda_n^*(A)$ .  $\square$

The Lebesgue measure can also be recovered from its values on compact sets.

**PROPOSITION 6.2.** *Let  $n \geq 1$  be an integer. For every Lebesgue measurable subset  $A \subset \mathbb{R}^n$  one has:*

$$(6) \quad \lambda_n(A) = \sup\{\lambda_n(K) : K \text{ compact subset of } \mathbb{R}^n, \text{ with } K \subset A\}.$$

**PROOF.** Let us denote, for simplicity, the right hand side of (6) by  $\mu(A)$ . First of all, by the monotonicity we clearly have the inequality

$$\lambda_n(A) \geq \mu(A).$$

To prove the inequality  $\lambda_n(A) \leq \mu(A)$ , we shall first use a reduction to the bounded case. For each integer  $k \geq 1$ , we define the compact box

$$B_k = [-k, k] \times \cdots \times [-k, k].$$

Notice that we have  $B_1 \subset B_2 \subset \cdots$ , with  $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^n$ . We then have

$$B_1 \cap A \subset B_2 \cap A \subset \cdots,$$

with  $\bigcup_{k=1}^{\infty} (B_k \cap A) = A$ , so using the Continuity Lemma 4.1, we have

$$(7) \quad \lambda_n(A) = \lim_{k \rightarrow \infty} \lambda_n(B_k \cap A) = \sup\{\lambda_n(B_k \cap A) : k \geq 1\}.$$

Fix for the moment some  $\varepsilon > 0$ , and use the (7) to find some  $k \geq 1$ , such that  $\lambda_n(A) \leq \lambda_n(B_k \cap A) + \varepsilon$ . Apply Proposition 6.1 to the set  $B_k \setminus A$ , to find an open set  $D$ , with  $D \supset B_k \setminus A$ , and  $\lambda_n(B_k \setminus A) \geq \lambda_n(D) - \varepsilon$ . On the one hand, we have

$$(8) \quad \begin{aligned} \lambda_n(B_k) &= \lambda_n(B_k \cap A) + \lambda_n(B_k \setminus A) \geq \lambda_n(B_k \cap A) + \lambda_n(D) - \varepsilon \geq \\ &\geq \lambda_n(B_k \cap A) + \lambda_n(B_k \cap D) - \varepsilon. \end{aligned}$$

On the other hand, we have

$$\lambda_n(B_k) = \lambda_n(B_k \setminus D) + \lambda_n(B_k \cap D),$$

so using (8) we get the inequality

$$\lambda_n(B_k \setminus D) + \lambda_n(B_k \cap D) \geq \lambda_n(B_k \cap A) + \lambda_n(B_k \cap D) - \varepsilon,$$

and since all numbers involved in the above inequality are finite, we conclude that

$$\lambda_n(B_k \setminus D) \geq \lambda_n(B_k \cap A) - \varepsilon \geq \lambda_n(A) - 2\varepsilon.$$

Obviously the set  $K = B_k \setminus D$  is compact, with  $K \subset B_k \cap A \subset A$ , so we have  $\mu(A) \geq \lambda_n(K)$ , hence we get the inequality

$$\mu(A) \geq \lambda_n(A) - 2\varepsilon.$$

Since this is true for all  $\varepsilon > 0$ , the desired inequality  $\mu(A) \geq \lambda_n(A)$  follows.  $\square$

**COROLLARY 6.1.** *For a set  $A \subset \mathbb{R}^n$ , the following are equivalent:*

- (i)  *$A$  is Lebesgue measurable;*
- (ii) *there exists a negligible set  $N$  and a sequence of  $(K_j)_{j=1}^{\infty}$  of compact subsets of  $\mathbb{R}^n$ , such that*

$$A = N \cup \bigcup_{j=1}^{\infty} K_j.$$

**PROOF.** (i)  $\Rightarrow$  (ii). Start by using the boxes

$$B_k = [-k, k] \times \cdots \times [-k, k]$$

which have the property that  $\bigcup_{k=1}^{\infty} B_k = \mathbb{R}^n$ , so we get  $A = \bigcup_{k=1}^{\infty} (B_k \cap A)$ . Fix for the moment  $k$ . Apply Proposition 6.2. to find a sequence  $(C_r^k)_{r=1}^{\infty}$  of compact subsets of  $B_k \cap A$ , such that  $\lim_{r \rightarrow \infty} \lambda_n(C_r^k) = \lambda_n(B_k \cap A)$ . Consider the countable family  $(C_r^k)_{k,r=1}^{\infty}$  of compact sets, and enumerate it as a sequence  $(K_j)_{j=1}^{\infty}$ , so that we have

$$\bigcup_{j=1}^{\infty} K_j = \bigcup_{k=1}^{\infty} \bigcup_{r=1}^{\infty} C_r^k.$$

If we define, for each  $k \geq 1$ , the sets  $E_k = \bigcup_{r=1}^{\infty} C_r^k \subset B_k \cap A$  and  $N_k = (B_k \cap A) \setminus E_k$ , then, because of the inclusion  $C_r^k \subset E_k \subset B_k \cap A$ , we have the inequalities

$$(9) \quad 0 \leq \lambda_n(N_k) = \lambda_n(B_k \cap A) - \lambda_n(E_k) \leq \lambda_n(B_k \cap A) - \lambda_n(C_r^k), \quad \forall r \geq 1.$$

Using the fact that

$$\lim_{r \rightarrow \infty} \lambda_n(C_r^k) = \lambda_n(B_k \cap A) \leq \lambda_n(B_k) < \infty,$$

the inequalities (9) force  $\lambda_n(N_k) = 0, \forall k \geq 1$ . Now if we define the set  $N = A \setminus (\bigcup_{j=1}^{\infty} K_j)$ , we have

$$\begin{aligned} N &= \bigcup_{k=1}^{\infty} \left[ (B_k \cap A) \setminus \left( \bigcup_{j=1}^{\infty} K_j \right) \right] = \bigcup_{k=1}^{\infty} \left[ (B_k \cap A) \setminus \left( \bigcup_{p=1}^{\infty} E_p \right) \right] \subset \\ &\subset \bigcup_{k=1}^{\infty} [(B_k \cap A) \setminus E_k] = \bigcup_{k=1}^{\infty} N_k, \end{aligned}$$

which proves that  $\lambda_n(N) = 0$ .

The implication (ii)  $\Rightarrow$  (i) is trivial.  $\square$

Proposition 6.2 does not hold if  $A \subset \mathbb{R}^n$  is non-measurable. In fact the equality (6), with  $\lambda_n$  replaced by  $\lambda_n^*$ , essentially forces  $A$  to be measurable, as shown by the following.

*Exercise 2.* Let  $A \subset \mathbb{R}^n$  be an arbitrary subset, with  $\lambda_n^*(A) < \infty$ . Prove that the following are equivalent:

- (i)  $A$  is Lebesgue measurable;
- (ii)  $\lambda_n^*(A) = \sup\{\lambda_n(K) : K \text{ compact subset of } \mathbb{R}^n, \text{ with } K \subset A\}$ .

Propositions 6.1 and 6.2 are *regularity properties*. The following terminology is useful:

DEFINITIONS. Suppose  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ , and  $\mu$  is a measure on  $\mathcal{A}$ . Suppose we have a sub-collection  $\mathcal{F} \subset \mathcal{A}$ .

- (i) We say that  $\mu$  is *regular from below, with respect to*  $\mathcal{F}$ , if

$$\mu(A) = \sup\{\mu(F) : F \subset A, F \in \mathcal{F}\}.$$

- (ii) We say that  $\mu$  is *regular from above, with respect to*  $\mathcal{F}$ , if

$$\mu(A) = \inf\{\mu(F) : F \supset A, F \in \mathcal{F}\}.$$

With this terminology, Proposition 6.1 gives the fact that the Lebesgue measure is regular from above with respect to open sets, while Proposition 6.2 gives the fact that the Lebesgue measure is regular from below with respect to compact sets.

*Exercise 3.* For a subset  $A \subset \mathbb{R}^n$ , prove that the following are equivalent:

- (i)  $A$  is Lebesgue measurable;
- (ii) There exist a sequence of compact sets  $(K_j)_{j=1}^{\infty}$ , and a sequence of open sets  $(D_j)_{j=1}^{\infty}$ , such that  $\bigcup_{j=1}^{\infty} K_j \subset A \subset \bigcap_{j=1}^{\infty} D_j$ , and the difference  $(\bigcap_{j=1}^{\infty} D_j) \setminus (\bigcup_{j=1}^{\infty} K_j)$  is negligible.

HINT: For the implication (i)  $\Rightarrow$  (ii) analyze first the case when  $\lambda^*(A) < \infty$ . Then write  $A$  as a countable union of sets of finite outer measure.

In the one-dimensional case  $n = 1$ , the Lebesgue measure of open sets can be computed with the aid of the following result.

PROPOSITION 6.3. *For every open set  $D \subset \mathbb{R}$ , there exists a countable (or finite) pair-wise disjoint collection  $\{J_i\}_{i \in I}$  of open intervals with  $D = \bigcup_{i \in I} J_i$ .*

PROOF. For every point  $x \in D$ , we define

$$a_x = \inf\{a < x : (a, x) \subset D\} \text{ and } b_x = \sup\{b > x : (x, b) \subset D\}.$$

(The fact that  $D$  is open guarantees the fact that both sets above are non-empty.) It is clear that, for every  $x \in D$ , the open interval  $J_x = (a_x, b_x)$  is contained in  $D$ , so

we have the equality  $D = \bigcup_{x \in D} J_x$ . The problem at this point is the fact that the collection  $\{J_x\}_{x \in D}$  is not pair-wise disjoint. What we need to find is a countable (or finite) subset  $X \subset D$ , such that the sub-collection  $\{J_x\}_{x \in X}$  is pair-wise disjoint, and we still have  $D = \bigcup_{x \in X} J_x$ . One way to do this is based on the following

*Claim: For two points  $x, y \in D$ , the following are equivalent:*

- (i)  $x \in J_y$ ;
- (ii)  $J_x \supset J_y$ ;
- (ii)  $J_x \cap J_y \neq \emptyset$ ;
- (iii)  $J_x = J_y$ .

To prove the implication (i)  $\Rightarrow$  (ii) we observe that if  $x \in J_y$ , then  $a_y < x < b_y$ , so we have  $(a_y, x) \subset D$  and  $(x, b_y) \subset D$ , which means that  $a_x \leq a_y$  and  $b_x \geq b_y$ , therefore we have the inclusion  $J_x = (a_x, b_x) \supset (a_y, b_y) = J_y$ . The implication (ii)  $\Rightarrow$  (iii) is trivial. To prove (iii)  $\Rightarrow$  (iv), assume  $J_x \cap J_y \neq \emptyset$ , and pick a point  $z \in J_x \cap J_y$ . Using the implication (i)  $\Rightarrow$  (ii) we have the inclusions  $J_z \supset J_x$  and  $J_z \supset J_y$ . In particular we have  $x \in J_z$ , so again using the implication (i)  $\Rightarrow$  (ii) we get  $J_x \supset J_z$ , which means that we have in fact the equality  $J_x = J_z$ . Likewise we have the equality  $J_y = J_z$ , so (iv) follows. The implication (iv)  $\Rightarrow$  (i) is trivial.

Going back to the proof of the Proposition, we now see that, using the fact that any open interval contains a rational number, if we put  $X_0 = D \cap \mathbb{Q}$ , then for any  $y \in D$ , there exists  $x \in X_0$ , such that  $J_x = J_y$ . This gives the equality  $D = \bigcup_{x \in X_0} J_x$ , this time with the indexing set  $X_0$  countable. Finally, if we equip the set  $X_0$  with the equivalence relation

$$x \sim y \iff J_x = J_y,$$

and we choose  $X \subset X_0$  to be a list of all equivalence classes. This means that, for every  $y \in X_0$ , there exists a unique  $x \in X$  with  $J_x = J_y$ . It is clear now that we still have  $D = \bigcup_{x \in X} J_x$ , but now if  $x, x' \in X$  are such that  $x \neq x'$ , then  $x \not\sim x'$ , so we have  $J_x \neq J_{x'}$ , which by the Claim gives  $J_x \cap J_{x'} = \emptyset$ .  $\square$

COMMENTS. When we want to compute the Lebesgue measure of an open set  $D \subset \mathbb{R}$ , we should first try to write  $D = \bigcup_{i \in I} J_i$  with  $(J_i)_{i \in I}$  a countable (or finite) pair-wise collection of open intervals. If we succeed, then we would have

$$\lambda(D) = \sum_{i \in I} \lambda(J_i).$$

For intervals (open or not) the Lebesgue measure is the same as the length.

There are instances when we can manage only to write a given open set  $D$  as a union  $D = \bigcup_{k=1}^{\infty} J_k$ , with the  $J$ 's not necessarily disjoint. In that case we can only get the estimate

$$\lambda(D) \leq \sum_{k=1}^{\infty} \lambda(J_k).$$

EXAMPLE 6.1. Consider the ternary Cantor set  $K_3 \subset [0, 1]$ , discussed in III.3. We know (see Remarks 3.5) that one can find a pair-wise sequence  $(D_n)_{n=0}^{\infty}$  of open subsets of  $(0, 1)$  such that  $K_3 = [0, 1] \setminus \bigcup_{n=0}^{\infty} D_n$ , and such that, for each  $n \geq 0$ , the open set  $D_n$  is a disjoint union of  $2^n$  intervals of length  $1/3^{n+1}$ . In particular, this means that  $\lambda(D_n) = 2^n/3^{n+1}$ , so

$$\lambda(K_3) = \lambda([0, 1]) - \lambda\left(\bigcup_{n=0}^{\infty} D_n\right) = 1 - \sum_{n=0}^{\infty} \lambda(D_n) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = 0.$$

What is interesting here (see Remarks 3.5) is the fact that  $\text{card } K_3 = \mathfrak{c}$ .

REMARK 6.2. An interesting consequence of the above computation is the fact that *all subsets of  $K_3$  are Lebesgue measurable*, i.e. one has the inclusion  $\mathcal{P}(K_3) \subset \mathcal{M}(\mathbb{R})$ . This gives the inequality

$$\text{card } \mathcal{M}(\mathbb{R}) \geq \text{card } \mathcal{P}(K_3) = 2^{\text{card } K_3} = 2^{\mathfrak{c}}.$$

Since we also have  $\mathcal{M}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$ , we get

$$\text{card } \mathcal{M}(\mathbb{R}) \leq \text{card } \mathcal{P}(\mathbb{R}) = 2^{\text{card } \mathbb{R}} = 2^{\mathfrak{c}},$$

so using the Cantor-Bernstein Theorem we get the equality

$$\text{card } \mathcal{M}(\mathbb{R}) = 2^{\mathfrak{c}}.$$

We also know (see Corollary 2.5) that  $\text{card } \text{Bor}(\mathbb{R}) = \mathfrak{c}$ .

As a consequence of this difference in cardinalities, one gets the fact that we have a strict inclusion

$$(10) \quad \text{Bor}(\mathbb{R}) \subsetneq \mathcal{M}(\mathbb{R}).$$

Later on we shall construct (more or less) explicitly a Lebesgue measurable set which is not Borel.

*Exercise 4.* The strict inclusion (10) holds also if  $\mathbb{R}$  is replaced with  $\mathbb{R}^n$ , with  $n \geq 2$ . In this case, instead of using Cantor sets, one can proceed as follows. Consider the set  $S = \mathbb{R}^{n-1} \times \{0\}$ . Prove that  $\lambda_n(S) = 0$ . Conclude that  $\text{card } \mathcal{M}(\mathbb{R}^n) = 2^{\mathfrak{c}}$ .

One key feature of the Lebesgue (outer) measure is the *translation invariance property*, described in the following result. To formulate it we introduce the following notation. For an integer  $n \geq 1$ , a point  $x \in \mathbb{R}^n$ , and a subset  $A \subset \mathbb{R}^n$ , we define the set

$$A + x = \{a + x : a \in A\}.$$

Remark that the map  $\Theta_x : \mathbb{R}^n \ni a \mapsto a + x \in \mathbb{R}^n$  is a homeomorphism. In particular, both  $\Theta_x$  and  $\Theta_x^{-1} = \Theta_{-x}$  are Borel measurable, which means that, for a set  $A \subset \mathbb{R}^n$ , one has the equivalence

$$A \in \text{Bor}(\mathbb{R}^n) \iff A + x \in \text{Bor}(\mathbb{R}^n).$$

PROPOSITION 6.4. *Let  $n \geq 1$  be an integer. For any set  $A \subset \mathbb{R}^n$  one has the equality*

$$\lambda_n^*(A + x) = \lambda_n^*(A).$$

PROOF. Fix  $A$  and  $x$ . First remark that, for every half-open box  $B \in \mathcal{J}_n$ , its translation  $B + x$  is again a half-open box, and we have the equality

$$\text{vol}_n(B + x) = \text{vol}_n(B).$$

Fix for the moment  $\varepsilon > 0$ , and choose a sequence  $(B_k)_{k=1}^{\infty} \subset \mathcal{J}_n$ , such that  $A \subset \bigcup_{k=1}^{\infty} B_k$ , and

$$\sum_{k=1}^{\infty} \text{vol}_n(B_k) \leq \lambda_n^*(A) + \varepsilon.$$

Then, using the obvious inclusion  $A + x \subset \bigcup_{k=1}^{\infty} (B_k + x)$ , by the remark made at the beginning of the proof, combined with the monotonicity of the outer Lebesgue measure, we have

$$\begin{aligned} \lambda_n^*(A + x) &\leq \lambda_n^*\left(\bigcup_{k=1}^{\infty} (B_k + x)\right) \leq \sum_{k=1}^{\infty} \lambda_n^*(B_k + x) = \\ &= \sum_{k=1}^{\infty} \text{vol}_n(B_k + x) = \sum_{k=1}^{\infty} \text{vol}_n(B_k) \leq \lambda_n^*(A) + \varepsilon. \end{aligned}$$

Since the inequality  $\lambda_n^*(A + x) \leq \lambda_n^*(A) + \varepsilon$  holds for all  $\varepsilon > 0$ , we get

$$\lambda_n^*(A + x) \leq \lambda_n^*(A).$$

The other inequality follows from the above one applied to the set  $A + x$  and the translation by  $-x$ .  $\square$

**COROLLARY 6.2.** *For a subset  $A \subset \mathbb{R}^n$ , one has the equivalence*

$$A \in \mathcal{M}(\mathbb{R}^n) \iff A + x \in \mathcal{M}(\mathbb{R}^n).$$

**PROOF.** Write  $A = B \cup N$ , with  $B$  Borel, and  $N$  negligible. Then we have  $A + x = (B + x) \cup (N + x)$ . The set  $B + x$  is Borel. By the above result we have  $\lambda_n^*(N + x) = \lambda_n^*(N) = 0$ , i.e.  $N + x$  is negligible. Therefore  $A + x$  is Lebesgue measurable.  $\square$

As we have seen, the fact that there exist Lebesgue measurable sets that are not Borel is explained by the difference in cardinalities. Since  $\text{card } \mathcal{M}(\mathbb{R}^n) = 2^{\aleph_0} = \text{card } \mathcal{P}(\mathbb{R}^n)$ , it is legitimate to ask whether the inclusion  $\mathcal{M}(\mathbb{R}^n) \subset \mathcal{P}(\mathbb{R}^n)$  is strict. In other words, *do there exist sets that are not Lebesgue measurable?* The answer is affirmative, as discussed in the following.

**EXAMPLE 6.2.** Equip  $\mathbb{R}$  with the equivalence relation

$$x \sim y \iff x - y \in \mathbb{Q}.$$

Denote by  $\mathbb{R}/\mathbb{Q}$  the quotient space (this is in fact the quotient group of  $(\mathbb{R}, +)$  with respect to the subgroup  $\mathbb{Q}$ ), and denote by  $\pi : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Q}$  the quotient map. Since for every  $x \in \mathbb{R}$ , one can find some  $y \sim x$ , with  $y \in [0, 1)$ , it follows that the map  $\pi|_{[0,1)} : [0, 1) \rightarrow \mathbb{R}/\mathbb{Q}$  is surjective. Choose then a map  $\phi : \mathbb{R}/\mathbb{Q} \rightarrow [0, 1)$ , such that  $\phi \circ \pi = \text{Id}$ , and put  $E = \phi(\mathbb{R}/\mathbb{Q})$ . The set  $E$  is a complete set of representatives for the equivalence relation  $\sim$ . In other words,  $E \subset [0, 1)$  has the property that, for every  $x \in \mathbb{R}$ , there exists exactly one element  $y \in E$ , with  $x \sim y$ . In particular, the collection of sets  $(E + q)_{q \in \mathbb{Q}}$  is pair-wise disjoint, and satisfies  $\bigcup_{q \in \mathbb{Q}} (E + q) = \mathbb{R}$ . Using  $\sigma$ -sub-additivity, we get

$$\infty = \lambda(\mathbb{R}) \leq \sum_{q \in \mathbb{Q}} \lambda^*(E + q).$$

Since (by Proposition 6.5) we have  $\lambda^*(E + q) = \lambda^*(E)$ , the above inequality forces  $\lambda^*(E) > 0$ .

**Claim:** *The set  $E$  is not Lebesgue measurable*

Assume  $E$  is Lebesgue measurable. If we define the set  $X = \mathbb{Q} \cap [0, 1)$ , then the sets  $E + q$ ,  $q \in X$  are pair-wise disjoint. On the one hand, the measurability of  $E$ , combined with the Corollary 6.2 would imply the measurability of the set  $S = \bigcup_{q \in X} (E + q)$ . On the other hand, the equalities  $\lambda(E + q) = \lambda(E) > 0$  will

force  $\lambda(S) = \infty$ . But this is impossible, since we obviously have  $S \subset [0, 2]$ , which forces  $\lambda(S) \leq 2$ .

*Exercise 5.* Let  $E \in \mathcal{M}(\mathbb{R}^n)$ . Prove that the map

$$\mathbb{R}^n \ni x \longmapsto \lambda(E \cup (E + x)) \in [0, \infty]$$

is continuous.

HINT: Analyze first the case when  $E$  is compact. In this particular case, show that for every  $x_0 \in \mathbb{R}^n$  and every open set  $D \supset E \cup (E + x_0)$ , there exists some neighborhood  $V$  of  $x_0$ , such that

$$D \supset E \cup (E + x), \quad \forall x \in V.$$

Use then regularity from above, combined with the inequality<sup>1</sup>

$$|\lambda(A) - \lambda(B)| \leq \lambda(A \Delta B), \text{ for all } A, B \in \mathcal{M}(\mathbb{R}^n), \text{ with } \lambda(A), \lambda(B) < \infty.$$

In the general case, use regularity from below. (The case  $\lambda(E) = \infty$  is trivial.)

*Exercise 6.* Let  $E \in \mathcal{M}(\mathbb{R}^n)$ , be such that  $\lambda_n(E) > 0$ . Prove that the set

$$E - E = \{x - y : x, y \in E\}$$

is a neighborhood of 0.

HINT: Assume the contrary, which means that there exists a sequence  $(x_p)_{p=1}^\infty \subset \mathbb{R}^n \setminus (E - E)$ , with  $\lim_{p \rightarrow \infty} x_p = 0$ . This will force  $E \cap (E + x_p) = \emptyset, \forall p \geq 1$ . Use the preceding Exercise to get a contradiction.

We are now in position to construct a Lebesgue measurable set which is not Borel.

EXAMPLE 6.3. In Section 3 we discussed the compact space  $T = \{0, 1\}^{\aleph_0}$  and the maps

$$\phi_r : T \ni (\alpha_n)_{n=1}^\infty \longmapsto (r-1) \sum_{n=1}^\infty \frac{\alpha_n}{r^n} \in [0, 1].$$

For each  $r \geq 2$  the map  $\phi_r : T \rightarrow [0, 1]$  is continuous so the set  $K_r = \phi_r(T)$  is compact. We have  $K_2 = [0, 1]$ , and  $K_3$  is the ternary Cantor set. We also know (see Theorem 3.5) that, for a set  $A \subset T$ , one has the equivalence

$$(11) \quad A \in \text{Bor}(T) \iff \phi_r(A) \in \text{Bor}(K_r).$$

Choose now a set  $E \subset [0, 1]$  which is not Lebesgue measurable. In particular,  $E$  is not Borel, so  $E \notin \text{Bor}([0, 1])$ . Since  $\phi_2 : T \rightarrow [0, 1]$  is surjective, by (11) it follows that the set  $A = \phi_2^{-1}(E)$  is not in  $\text{Bor}(T)$ . Again, by (11) it follows that the set  $S = \phi_3(A)$  is not in  $\text{Bor}(K_3)$ . Since

$$\text{Bor}(K_3) = \text{Bor}(\mathbb{R})|_{K_3}$$

this gives  $S \notin \text{Bor}(\mathbb{R})$ . Notice however that since  $S \subset K_3$ , it follows that  $S$  is Lebesgue measurable.

COMMENT. When one wants to prove that a Lebesgue measurable set  $M \subset \mathbb{R}$  has positive measure, a sufficient condition for this property is that  $\text{Int}(M) \neq \emptyset$  (see Remark 6.1). It turns out however that this condition is not always necessary, as seen from the following:

<sup>1</sup> This inequality holds for any additive map defined on a ring.

*Exercise 7.* Start with an arbitrary interval  $[0, 1]$ , and list all rational numbers in  $[0, 1]$  as a sequence  $\mathbb{Q} \cap [0, 1] = \{x_n\}_{n=1}^{\infty}$ . Fix some  $\varepsilon > 0$ , and consider the open set

$$D = \bigcup_{n=1}^{\infty} \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right).$$

Consider the compact set  $K = [0, 1] \setminus D$ .

- (i) Prove that  $\lambda(D) \leq \varepsilon$ .
- (ii) Prove that  $\lambda(K) \geq 1 - \varepsilon$ .
- (iii) Prove that  $\text{Int}(K) = \emptyset$ .

HINT: For (iii) use the fact that  $K \cap \mathbb{Q} = \emptyset$ .

*Exercise 8\*.* Prove that, for every non-empty open set  $D \subset \mathbb{R}$ , and any two positive numbers  $\alpha, \beta$  with  $\alpha + \beta < \lambda(D)$ , there exist compact sets  $A, B \subset D$ , with  $\lambda(A) > \alpha$ ,  $\lambda(B) > \beta$ , such that  $A \cap B = \emptyset$  and  $(A \cup B) \cap \mathbb{Q} = \emptyset$ .

HINT: Write  $D$  as a union of a pair-wise disjoint sequence  $(J_n)_{n=1}^{\infty}$  of open intervals, so that  $\lambda(D) = \sum_{n=1}^{\infty} \lambda(J_n)$ . Find then two sequences  $(\alpha_n)_{n=1}^{\infty}$  and  $(\beta_n)_{n=1}^{\infty}$  of positive numbers, such that  $\sum_{n=1}^{\infty} \alpha_n > \alpha$ ,  $\sum_{n=1}^{\infty} \beta_n > \beta$ , and  $\alpha_n + \beta_n < \lambda(J_n)$ , for all  $n \geq 1$ . This reduces essentially the problem to the case when  $D$  is an open interval, for which one can use the construction outlined in Exercise 7.

*Exercise 9\*.* Construct a Borel set  $A \subset \mathbb{R}$ , such that, for every open interval  $I \subset \mathbb{R}$  one has  $\lambda(I \cap A) > 0$  and  $\lambda(I \setminus A) > 0$ .

HINTS: List all open intervals with rational endpoints as a sequence  $(I_n)_{n=1}^{\infty}$ . Start (use exercise 8) off by choosing two compact sets  $A_1, B_1 \subset I_1$ , with  $A_1 \cap B_1 = \emptyset$ ,  $(A_1 \cup B_1) \cap \mathbb{Q} = \emptyset$ , and  $\lambda(A_1), \lambda(B_1) > 0$ . Use Exercise 5 to construct two sequences  $(A_n)_{n=1}^{\infty}$  and  $(B_n)_{n=1}^{\infty}$  of compact sets, such that, for all  $n \geq 1$  we have: (i)  $A_n \cap B_n = \emptyset$ ; (ii)  $(A_n \cup B_n) \cap \mathbb{Q} = \emptyset$ ; (iii)  $\lambda(A_n), \lambda(B_n) > 0$ ; (iv)  $A_{n+1} \cup B_{n+1} \subset I_{n+1} \setminus [\bigcup_{k=1}^n (A_k \cup B_k)]$ . Put  $A = \bigcup_{n=1}^{\infty} A_n$  and  $B = \bigcup_{n=1}^{\infty} B_n$ . Notice that  $A \cap B = \emptyset$ ,  $\lambda(A), \lambda(B) > 0$ , and  $\lambda(A \cap I_n), \lambda(B \cap I_n) > 0, \forall n \geq 1$ .

In the remainder of this section we discuss some applications of the Lebesgue measure to the theory of Riemann integration. The following technical result will be very useful.

LEMMA 6.1. *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a non-negative Riemann integrable function, let  $A, B \subset [a, b]$  be two disjoint sets, with  $A \cup B = [a, b]$ . Then one has the estimates*

$$\lambda^*(A) \cdot \inf_{z \in A} f(z) \leq \int_a^b f(t) dt \leq (b-a) \cdot \sup_{x \in A} f(x) + \lambda^*(B) \cdot \sup_{y \in B} f(y).$$

PROOF. Define the numbers

$$\alpha = \sup_{x \in A} f(x), \quad \beta = \sup_{y \in B} f(y), \quad \text{and} \quad \gamma = \inf_{z \in A} f(z).$$

Recall first that, if for each partition  $\Delta = (a = x_0 < x_1 < \cdots < x_n = b)$  of  $[a, b]$ , we define the lower and the upper Darboux sums of  $f$  with respect to  $\Delta$ :

$$L(\Delta, f) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \inf_{t \in [x_{k-1}, x_k]} f(t),$$

$$U(\Delta, f) = \sum_{k=1}^n (x_k - x_{k-1}) \cdot \sup_{t \in [x_{k-1}, x_k]} f(t),$$

then one has the equalities

$$(12) \quad \int_a^b f(t) dt = \sup \{L(\Delta, f) : \Delta \text{ partition of } [a, b]\} = \\ = \inf \{U(\Delta, f) : \Delta \text{ partition of } [a, b]\}.$$

Fix now a partition  $\Delta = (a = x_0 < x_1 < \dots < x_n = b)$  of  $[a, b]$ , and define the set

$$S = \{k \in \{1, \dots, n\} : [x_{k-1}, x_k] \cap A \neq \emptyset\}.$$

It is clear that

$$\inf_{x \in [x_{k-1}, x_k]} f(x) \leq \alpha, \quad \sup_{x \in [x_{k-1}, x_k]} f(x) \geq \gamma, \quad \forall k \in S, \\ \inf_{y \in [x_{k-1}, x_k]} f(y) \leq \beta, \quad \sup_{y \in [x_{k-1}, x_k]} f(y) \geq 0, \quad \forall k \in \{1, \dots, n\} \setminus S,$$

so we get

$$(13) \quad L(\Delta, f) \leq \left[ \sum_{k \in S} (x_k - x_{k-1}) \right] \cdot \alpha + \left[ \sum_{k \notin S} (x_k - x_{k-1}) \right] \cdot \beta$$

$$(14) \quad U(\Delta, f) \geq \left[ \sum_{k \in S} (x_k - x_{k-1}) \right] \cdot \gamma$$

Consider now the sets

$$M = \bigcup_{k \in S} [x_{k-1}, x_k] \text{ and } N = \bigcup_{k \notin S} [x_{k-1}, x_k].$$

Since the intervals involved in both  $M$  and  $N$  have at most singleton overlaps, it follows that we have the equalities

$$\sum_{k \in S} (x_k - x_{k-1}) = \lambda(M) \text{ and } \sum_{k \notin S} (x_k - x_{k-1}) = \lambda(N),$$

so the estimates (13) and (14) read

$$(15) \quad L(\Delta, f) \leq \lambda(M) \cdot \alpha + \lambda(N) \cdot \beta$$

$$(16) \quad U(\Delta, f) \geq \lambda(M) \cdot \gamma$$

Since we clearly have  $A \subset M \subset [a, b]$  and  $N \subset B$ , we have the inequalities

$$\lambda^*(A) \leq \lambda(M) \leq b - a \text{ and } \lambda(N) \leq \lambda^*(B),$$

so the inequalities (15) and (16) give

$$L(\Delta, f) \leq (b - a) \cdot \alpha + \lambda^*(B) \cdot \beta \text{ and } U(\Delta, f) \geq \lambda^*(A) \cdot \gamma.$$

Since  $\Delta$  is arbitrary, the desired inequality then follows from (12). □

One application of the above result is the following.

**PROPOSITION 6.5.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, and the set*

$$N = \{x \in [a, b] : f(x) \neq 0\}$$

*is negligible, then*

$$(17) \quad \int_a^b f(x) dx = 0.$$

PROOF. Since  $f$  is bounded, there exists some constant  $C > 0$ , such that the Riemann integrable functions  $C+f$  and  $C-f$  are both non-negative. Apply Lemma 6.1 to these two functions with  $A = [a, b] \setminus N$  and  $B = N$ . Since  $f|_{[a, b] \setminus N} = 0$ , we get  $(C \pm f)|_{[a, b] \setminus N} = C$ , so we get

$$\int_a^b [C \pm f(x)] dx \leq (b-a) \cdot C,$$

which yields

$$\pm \int_a^b f(x) dx = \int_a^b \{[C \pm f(x)] - C\} dx = \int_a^b [C \pm f(x)] dx - (b-a) \cdot C \leq 0,$$

from which (17) immediately follows.  $\square$

In order to make the exposition a bit easier to follow, it will be helpful to introduce the following

CONVENTION. Given two functions  $f_1, f_2 : [a, b] \rightarrow \mathbb{R}$ , and a relation  $R$  on  $\mathbb{R}$  (in our case  $R$  will be either “=,” or “ $\geq$ ,” or “ $\leq$ ”), we write

$$f_1 R f_2, \text{ a.e.}$$

if the set

$$A = \{x \in [a, b] : f_1(x) R f_2(x)\}$$

has negligible complement in  $[a, b]$ , i.e.  $\lambda^*([a, b] \setminus A) = 0$ . The abbreviation “a.e.” stands for “almost everywhere.”

For example, using this convention, Proposition 6.6 reads: if  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, and  $f = 0$ , a.e., then  $\int_a^b f(x) dx = 0$ .

*Exercise 10.* A. Prove that “= a.e.” is an equivalence relation, and “ $\geq$  a.e.” and “ $\leq$  a.e.” are transitive relations on the collection of all function  $[a, b] \rightarrow \mathbb{R}$ .

B. Prove that  $f_1 \geq f_2$ , a.e. and  $f_1 \leq f_2$ , a.e. imply  $f_1 = f_2$ , a.e.

C. Prove that these relations are compatible with the arithmetic operations, in the exact way as their “honest” versions. For example, if  $R$  is one of “=,” or “ $\geq$ ,” or “ $\leq$ ”, and if  $f_1 R f_2$ , a.e. and  $g_1 R g_2$ , a.e., then  $(f_1 + g_1) R (f_2 + g_2)$ , a.e.

*Exercise 11.* Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous functions, such that  $f \geq g$ , a.e. Prove that  $f \geq g$ .

*Exercise 12.* Let  $f : [a, b] \rightarrow \mathbb{R}$  be a non-negative Riemann integrable function, with  $\int_a^b f(x) dx = 0$ . Prove that  $f = 0$ , a.e.

COMMENT. Riemann integrability is quite a rigid condition. For example the characteristic function  $\chi_{\mathbb{Q} \cap [a, b]}$  of the set of rational numbers in  $[a, b]$  is not Riemann integrable. By the above result however, we can introduce a slightly weaker notion, which will make such functions integrable, in a weaker sense. This will be a first “improvement” of the Riemann integration theory. Eventually (see Chapter IV), a more sophisticated theory - the Lebesgue integral - will emerge.

DEFINITION. We say that a function  $f : [a, b] \rightarrow \mathbb{R}$  is *almost Riemann integrable*, if there exists a Riemann integrable function  $g : [a, b] \rightarrow \mathbb{R}$ , with  $f = g$ , a.e. Of course, such a  $g$  is not unique. Notice however that, if  $h : [a, b] \rightarrow \mathbb{R}$  is another

Riemann integrable function, with  $f = h$ , a.e., then  $g = h$ , a.e., so by Proposition 6.6, we immediately get the equality

$$\int_a^b g(x) dx = \int_a^b h(x) dx.$$

This observation shows that we can unambiguously define

$$\int_a^b f(x) dx = \int_a^b g(x) dx.$$

EXAMPLE 6.4. Consider the function  $f = \chi_{\mathbb{Q} \cap [a, b]}$ . Since  $\mathbb{Q} \cap [a, b]$  is negligible, we have  $f = 0$ , a.e. So  $f$  is almost Riemann integrable (although it is not Riemann integrable), and we have

$$\int_a^b f(x) dx = 0.$$

We now focus our attention to (honest) Riemann integrability, with an eye on the role played by continuity. For a function  $f : [a, b] \rightarrow \mathbb{R}$  we define the set

$$D_f = \{x \in [a, b] : f \text{ not continuous at } x\}.$$

It is well-known that continuous functions are Riemann integrable. There are discontinuous functions which are still Riemann integrable, for instance we know that

$$(18) \quad D_f \text{ finite} \implies f \text{ Riemann integrable.}$$

NOTATIONS. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Suppose  $\Delta = (a = x_0 < x_1 < \dots < x_n = b)$  is a partition. For each  $k \in \{1, \dots, n\}$  we consider the numbers

$$M_k = \sup_{t \in [x_{k-1}, x_k]} f(t) \text{ and } m_k = \inf_{t \in [x_{k-1}, x_k]} f(t),$$

and we define the functions

$$\begin{aligned} f_\Delta &= m_1 \cdot \chi_{[x_0, x_1]} + m_2 \cdot \chi_{(x_1, x_2]} + \dots + m_n \cdot \chi_{(x_{n-1}, x_n]}, \\ f^\Delta &= M_1 \cdot \chi_{[x_0, x_1]} + M_2 \cdot \chi_{(x_1, x_2]} + \dots + M_n \cdot \chi_{(x_{n-1}, x_n]}. \end{aligned}$$

Clearly the functions  $f_\Delta$  and  $f^\Delta$  have only finitely many points of discontinuity, so they are Riemann integrable.

With these notations we have the following

PROPOSITION 6.6. *For a bounded function  $f : [a, b] \rightarrow \mathbb{R}$ , the following are equivalent:*

- (i)  $f$  is Riemann integrable;
- (ii)  $\inf \left\{ \int_a^b [f^\Delta(x) - f_\Delta(x)] dx : \Delta \text{ partition of } [a, b] \right\} = 0$ ;
- (iii) there exists a sequence  $(\Delta_p)_{p=1}^\infty$  of partitions of  $[a, b]$ , with  $\Delta_1 \subset \Delta_2 \subset \dots$ , and  $\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0$ .

PROOF. From the definition of Riemann integrability, we know that (i) is equivalent to any of the following two conditions

- (ii')  $\inf \{U(\Delta, f) - L(\Delta, f) : \Delta \text{ partition of } [a, b]\} = 0$ ;
- (iii') there exists a sequence  $(\Delta_p)_{p=1}^\infty$  of partitions of  $[a, b]$ , with  $\Delta_1 \subset \Delta_2 \subset \dots$ , and  $\lim_{p \rightarrow \infty} [U(\Delta_p, f) - L(\Delta_p, f)] = 0$ .

Then the Proposition follows immediately from the fact that, for every partition  $\Delta$  one has the equalities

$$\int_a^b f_{\Delta}(x) dx = L(\Delta, f) \text{ and } \int_a^b f^{\Delta}(x) dx = U(\Delta, f). \quad \square$$

The following result gives a complete description of the relationship between Riemann integrability and continuity.

**THEOREM 6.1** (Lebesgue's criterion for Riemann integrability). *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The following are equivalent:*

- (i)  $f$  is Riemann integrable;
- (ii) the discontinuity set  $D_f$  is negligible.

**PROOF.** (i)  $\Rightarrow$  (ii). Assume  $f$  is Riemann integrable. Using Proposition 6.7, there exists a sequence  $(\Delta_p)_{p=1}^{\infty}$  of partitions of  $[a, b]$ , such that  $\Delta_1 \subset \Delta_2 \subset \dots$  and

$$\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0.$$

Notice that

$$(19) \quad f^{\Delta_1} \geq f^{\Delta_2} \geq f^{\Delta_3} \geq \dots \geq f \geq \dots \geq f_{\Delta_3} \geq f_{\Delta_2} \geq f_{\Delta_1}.$$

Define the Riemann integrable functions  $h_p = f^{\Delta_p} - f_{\Delta_p}$ ,  $p \in \mathbb{N}$ . We then clearly have

- ( $\alpha$ )  $h_p \geq h_{p+1} \geq 0$ ,  $\forall p \in \mathbb{N}$ ;
- ( $\beta$ )  $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$ .

Using ( $\alpha$ ) we can define the function  $h : [a, b] \rightarrow \mathbb{R}$  by

$$h(x) = \lim_{p \rightarrow \infty} h_p(x), \quad \forall x \in [a, b].$$

*Claim 1: The set  $N = \{x \in [a, b] : h(x) \neq 0\}$  is negligible.*

First of all, the functions  $h_p$  are all Lebesgue measurable. Secondly, since  $h$  is a point-wise limit of a sequence of Lebesgue measurable functions, it follows (see Theorem 3.2) that  $h$  itself is Lebesgue measurable. In particular  $N$  is Lebesgue measurable. For every integer  $j \geq 1$ , define

$$N_j = \left\{x \in [a, b] : h(x) > \frac{1}{j}\right\},$$

so that the sets  $N_j$ ,  $j \geq 1$  are again Lebesgue measurable, and  $N = \bigcup_{j=1}^{\infty} N_j$ . In order to prove that  $N$  is negligible, it then suffices to prove that  $\lambda(N_j) = 0$ , for all  $j \geq 1$ . Fix for the moment  $j \geq 1$ . Since  $h_p \geq h \geq 0$ , it follows that

$$\inf_{x \in N_j} h_p(x) \geq \frac{1}{j}, \quad \forall p \geq 1,$$

so by Lemma 6.1 we get the inequality

$$\frac{\lambda(N_j)}{j} \leq \int_a^b h_p(x) dx, \quad \forall p \geq 1,$$

so by ( $\beta$ ) we indeed get  $\lambda(N_j) = 0$ .

Define the set  $S = \bigcup_{p=1}^{\infty} \Delta_p$ .

*Claim 2: If  $y \in [a, b] \setminus (N \cup S)$ , then  $f$  is continuous at  $y$ .*

Fix  $y \in [a, b] \setminus (N \cup S)$ . In order to prove that  $f$  is continuous at  $y$ , we must find, for every  $\varepsilon > 0$ , some open interval  $J_\varepsilon \ni y$ , such that

$$(20) \quad |f(z) - f(y)| < \varepsilon, \quad \forall z \in J_\varepsilon \cap [a, b].$$

Since  $y \notin N$ , we have  $\lim_{p \rightarrow \infty} h_p(y) = 0$ . Fix  $\varepsilon$  and choose  $p \geq 1$ , such that  $0 \leq h_p(y) < \varepsilon$ . Write the partition  $\Delta_p$  as

$$\Delta_p = (a = x_0 < x_1 < \cdots < x_n = b).$$

Using the fact that  $y \notin \Delta_p$ , if we define  $k = \min \{j \in \{1, \dots, n\} : y < x_j\}$ , we have  $y \in (x_{k-1}, x_k)$ . In particular, we get

$$f^{\Delta_p}(y) = \sup_{t \in [x_{k-1}, x_k]} f(t) \quad \text{and} \quad f_{\Delta_p}(y) = \inf_{s \in [x_{k-1}, x_k]} f(s),$$

so the inequality  $0 \leq h_p(y) < \varepsilon$  gives

$$\left[ \sup_{t \in [x_{k-1}, x_k]} f(t) \right] - \left[ \inf_{s \in [x_{k-1}, x_k]} f(s) \right] < \varepsilon,$$

so if we choose  $J_\varepsilon = (x_{k-1}, x_k)$ , we clearly have (20).

Now we are done, because using the fact that  $S$  is countable, it follows that  $S$  is negligible, so  $N \cup S$  is also negligible. Since by Claim 2, we have  $D_f \subset N \cup S$ , it follows that  $D_f$  itself is negligible.

(ii)  $\Rightarrow$  (i). Assume now the discontinuity set  $D_f$  is negligible, and let us prove that  $f$  is Riemann integrable. Fix a sequence  $(\Delta_p)_{p=1}^\infty$  of partitions of  $[a, b]$ , with  $\Delta_1 \subset \Delta_2 \subset \dots$ , and<sup>2</sup>  $\lim_{p \rightarrow \infty} |\Delta_p| = 0$ . As before, we define the set  $S = \bigcup_{p=1}^\infty \Delta_p$ .

*Claim 3: For any point  $y \in [a, b] \setminus (D_f \cup S)$ , one has the equalities*

$$\lim_{p \rightarrow \infty} f^{\Delta_p}(y) = \lim_{p \rightarrow \infty} f_{\Delta_p}(y) = f(y).$$

Fix for the moment  $\varepsilon > 0$ . Since  $f$  is continuous at  $y$ , there exists some  $\delta_\varepsilon > 0$ , such that

$$(21) \quad |f(z) - f(y)| < \varepsilon, \quad \forall z \in (y - \delta_\varepsilon, y + \delta_\varepsilon) \cap [a, b].$$

Choose now  $q \geq 1$ , such that  $|\Delta_q| < \delta_\varepsilon$ . Write  $\Delta_q = (a = x_0 < x_1 < \cdots < x_n = b)$ . Using the fact that  $y \notin \Delta_q$ , we can find  $k \in \{1, \dots, n\}$  such that  $y \in (x_{k-1}, x_k)$ . Since  $x_k - x_{k-1} < \delta_\varepsilon$ , we have the inclusion  $[x_{k-1}, x_k] \subset (y - \delta_\varepsilon, y + \delta_\varepsilon)$ , so by (21) we immediately get

$$\begin{aligned} f(y) &\leq f^{\Delta_q}(y) = \sup_{z \in [x_{k-1}, x_k]} f(z) \leq f(y) + \varepsilon; \\ f(y) &\geq f_{\Delta_q}(y) = \inf_{z \in [x_{k-1}, x_k]} f(z) \geq f(y) - \varepsilon. \end{aligned}$$

Since the sequence  $(f^{\Delta_p}(y))_{p=1}^\infty$  is non-increasing, and the sequence  $(f_{\Delta_p}(y))_{p=1}^\infty$  is non-decreasing, the above inequalities give

$$|f^{\Delta_p}(y) - f(y)| \leq \varepsilon \quad \text{and} \quad |f_{\Delta_p}(y) - f(y)| \leq \varepsilon, \quad \text{for all } p \geq q,$$

and the Claim follows.

Going back to the proof of the Theorem, we will now prove that  $f$  satisfies condition (iii) in Proposition 6.6. Fix  $\varepsilon > 0$ . Since  $D_f \cup S$  is also negligible, using regularity from above with respect to open sets, we can find an open set  $E \subset \mathbb{R}$

<sup>2</sup> Recall that, for a partition  $\Delta = (a = x_0 < \cdots < x_n = b)$ , the number  $|\Delta|$  is defined as  $|\Delta| = \max \{x_k - x_{k-1} : 1 \leq k \leq n\}$ .

such that  $E \supset D_f \cup S$ , and  $\lambda(E) < \varepsilon$ . Define the compact set  $A = [a, b] \setminus E$ , and put  $B = [a, b] \cap E$ . We clearly have

$$(22) \quad \lambda(B) \leq \lambda(E) < \varepsilon.$$

Define the sequence  $(h_p)_{p=1}^\infty$  by  $h_p = f^{\Delta_p} - f_{\Delta_p}$ . Since  $A \cap \Delta_p = \emptyset$ , it follows that  $h_p|_A$  is continuous, for each  $p \geq 1$ . Since  $A \cap (D_f \cup S) = \emptyset$ , by Claim 3, we know that  $\lim_{p \rightarrow \infty} h_p(y) = 0$ ,  $\forall y \in A$ . Since  $(h_p)_{p=1}^\infty$  is monotone, by Dini's Theorem (see ??) it follows that

$$\lim_{p \rightarrow \infty} [\max_{y \in A} h_p(y)] = 0.$$

In particular, there exists  $p_\varepsilon \geq 1$ , such that

$$(23) \quad h_{p_\varepsilon}(y) \leq \varepsilon, \quad \forall y \in A.$$

Let

$$M = \sup_{x \in [a, b]} f(x) \text{ and } m = \inf_{x \in [a, b]} f(x).$$

Using Lemma 6.1 for  $h_{p_\varepsilon}$  and the sets  $A$  and  $B$ , combined with (22), we have

$$\begin{aligned} \int_a^b h_{p_\varepsilon}(x) dx &\leq (b-a) \cdot \sup_{y \in A} h_{p_\varepsilon}(y) + \lambda^*(B) \cdot \sup_{z \in B} h_{p_\varepsilon}(z) \leq \\ &\leq \varepsilon(b-a) + \lambda^*(B)(M-m) \leq \varepsilon(b-a + M-m). \end{aligned}$$

Since  $h_{p_\varepsilon} \geq h_p \geq 0$ , for all  $p \geq p_\varepsilon$ , we get the inequalities

$$0 \leq \int_a^b h_p(x) dx \leq \varepsilon(b-a + M-m), \quad \forall p \geq p_\varepsilon.$$

The above argument proves that  $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$ , i.e.

$$\lim_{p \rightarrow \infty} \int_a^b [f^{\Delta_p}(x) - f_{\Delta_p}(x)] dx = 0.$$

By Proposition 6.6, it follows that  $f$  is Riemann integrable.  $\square$

*Exercise 13.* Prove that a Riemann integrable function  $f : [a, b] \rightarrow \mathbb{R}$  is Lebesgue measurable.

HINT: Use a sequence of partitions  $(\Delta_p)_{p=1}^\infty$ , with  $\Delta_1 \subset \Delta_2 \subset \dots$ , and  $\lim_{p \rightarrow \infty} |\Delta_p| = 0$ . Use the arguments given in the proof of the implication (ii)  $\Rightarrow$  (i), to find a negligible set  $N \subset [a, b]$ , such that

$$\lim_{p \rightarrow \infty} f_{\Delta_p}(x) = f(x), \quad \forall x \in [a, b] \setminus N.$$

The sequence  $(f_{\Delta_p})_{p=1}^\infty$  is non-decreasing, so it has a point-wise limit, say  $g$ , which is Lebesgue measurable. Use the fact that

$$f(x) = g(x) \quad \forall x \in [a, b] \setminus N,$$

to show that  $f$  itself is Lebesgue measurable.

*Exercise 14.* Let  $K \subset [0, 1]$  be a compact set with  $K \cap \mathbb{Q} = \emptyset$ , and  $\lambda(K) > 0$  (see Exercise 7 for the existence of such sets). Prove that the characteristic function  $\chi_K : [0, 1] \rightarrow \mathbb{R}$  is not Riemann integrable. In fact,  $f$  cannot be almost Riemann integrable either.

HINT: Examine the discontinuity set  $D_f$ , and prove that  $K \subset D_f$ .

*Exercise 15.* Let  $f_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \geq 1$  be a sequence of Riemann integrable functions. Consider the product space  $P = \prod_{n=1}^{\infty} \text{Ran } f_n$ , equipped with the product topology (the sets  $\text{Ran } f_n$ ,  $n \geq 1$ , are equipped with the topology induced from  $\mathbb{R}$ ), and the function  $F : [a, b] \rightarrow P$ , defined by  $F(x) = (f_n(x))_{n=1}^{\infty}$ . Prove that, for every bounded continuous function  $g : P \rightarrow \mathbb{R}$ , the composition  $g \circ F : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable. In other words, *the result of a bounded continuous operation, involving a sequence of Riemann integrable functions, is again a Riemann integrable function.*

HINT: Examine the relationship between the discontinuity set  $D_{g \circ F}$  and the discontinuity sets  $D_{f_n}$ ,  $n \geq 1$ .

*Exercise 16.* Let  $M$  be an arbitrary subset of  $[a, b]$ , and let  $f : [a, b] \rightarrow \mathbb{R}$  be a Riemann integrable function, such that  $f \leq \varkappa_M$ . Prove the inequality

$$\int_a^b f(x) dx \leq \lambda^*(M).$$

HINT: Consider the function  $g : [a, b] \rightarrow \mathbb{R}$  defined by  $g(x) = \max\{f(x), 1\}$ . Then  $f \geq g \geq \varkappa_M$ , and  $g$  is still Riemann integrable. Apply Lemma 6.1 (the first inequality) to the function  $1 - g$ .

*Exercise 17\*.* Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Prove that the following are equivalent:

- (i)  $f$  is Riemann integrable;
- (ii) for every  $\varepsilon > 0$ , there exist continuous functions  $g, h : [a, b] \rightarrow \mathbb{R}$  with  $g \geq f \geq h$ , and  $\int_a^b [g(x) - h(x)] dx < \varepsilon$ ;
- (iii) for every  $\varepsilon > 0$ , there exist Riemann integrable functions  $g, h : [a, b] \rightarrow \mathbb{R}$  with  $g \geq f \geq h$ , and  $\int_a^b [g(x) - h(x)] dx < \varepsilon$ .

HINTS: For the implication (i)  $\Rightarrow$  (ii) analyze first the particular case when  $f = \varkappa_J$ , with  $J$  a sub-interval of  $[a, b]$ . Then analyze the functions of the type  $f^\Delta$  and  $f_\Delta$ . For the implication (iii)  $\Rightarrow$  (i), analyze the relationship among lower/upper Darboux sums of  $f$ ,  $g$  and  $h$ .

COMMENT. The statement of Theorem 6.1 shows that, apart from trivial cases, the problem of checking that a function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable, is a rather difficult one. The main difficulty arises from the fact that, if  $N \subset [a, b]$  is a negligible set, and  $f|_{[a, b] \setminus N}$  is continuous, then  $f$  need not be continuous at all points in  $[a, b] \setminus N$ . For instance, if we consider the characteristic function  $f = \varkappa_{\mathbb{Q} \cap [a, b]}$  of the rationals in  $[a, b]$ , and  $N = \mathbb{Q} \cap [a, b]$ , then clearly  $N$  is negligible,  $f|_{[a, b] \setminus N}$  is continuous (because it is constant zero), but  $D_f = [a, b]$ .

As earlier suggested, in the hope that such an anomaly can be eliminated, it is reasonable to consider the slightly weaker notion of almost Riemann integrability. In the remainder of this section, we take a closer look at this notion, and we will eventually show (see Theorem 6.2) that this indeed removes the above anomaly.

We begin with an “almost” version of Exercise 17.

LEMMA 6.2. *For a function  $f : [a, b] \rightarrow \mathbb{R}$ , the following are equivalent:*

- (i)  $f$  is almost Riemann integrable;
- (ii) for every  $\varepsilon > 0$ , there exist continuous functions  $g, h : [a, b] \rightarrow \mathbb{R}$  with  $g \geq f \geq h$  a.e., and  $\int_a^b [g(x) - h(x)] dx < \varepsilon$ ;
- (iii) for every  $\varepsilon > 0$ , there exist Riemann integrable functions  $g, h : [a, b] \rightarrow \mathbb{R}$  with  $g \geq f \geq h$  a.e., and  $\int_a^b [g(x) - h(x)] dx < \varepsilon$ .

PROOF. The implication (i)  $\Rightarrow$  (iii) is trivial.

The implication (iii)  $\Rightarrow$  (ii) follows from Exercise 17.

We now prove (ii)  $\Rightarrow$  (i). Assume  $f$  has property (ii). For each integer  $n \geq 1$ , choose continuous functions  $g_n, h_n : [a, b] \rightarrow \mathbb{R}$ , such that  $g_n \geq f \geq h_n$ , a.e., and  $\int_a^b [g_n(x) - h_n(x)] dx \leq 1/n$ . Define the functions  $G_n, H_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \geq 1$ , by

$$G_n(x) = \min \{g_1(x), \dots, g_n(x)\},$$

$$H_n(x) = \max \{g_1(x), \dots, g_n(x)\}.$$

It is clear that

- ( $\alpha$ )  $G_m \geq f \geq H_n$ , a.e.,  $\forall m, n \geq 1$ ;
- ( $\beta$ )  $G_1 \geq G_2 \geq \dots$  and  $H_1 \leq H_2 \leq \dots$ ;
- ( $\gamma$ )  $\int_a^b [G_n(x) - H_n(x)] dx \leq \int_a^b [g_n(x) - h_n(x)] dx \leq 1/n$ ,  $\forall n \geq 1$ .

Notice that, since the  $G_m$ 's and the  $H_n$ 's are continuous, by Exercise ??, we also have

- ( $\alpha'$ )  $G_m \geq H_n$  (everywhere!),  $\forall m, n \geq 1$ .

Use ( $\beta$ ) to define the functions  $G, H : [a, b] \rightarrow \mathbb{R}$ , by

$$G(x) = \lim_{n \rightarrow \infty} G_n(x) \text{ and } H(x) = \lim_{n \rightarrow \infty} H_n(x), \quad \forall x \in [a, b],$$

so by ( $\alpha'$ ) we clearly have  $G_n \geq G \geq H \geq H_n$ ,  $\forall n \geq 1$ . Using then ( $\gamma$ ), by Exercise 17 it follows that both  $G$  and  $H$  are Riemann integrable. Moreover, we have  $G - H \geq 0$  and

$$0 \leq \int_a^b [G(x) - H(x)] dx \leq \int_a^b [G_n(x) - H_n(x)] dx \leq 1/n, \quad \forall n \geq 1,$$

Which forces  $\int_a^b [G(x) - H(x)] dx = 0$ , so by Exercise ??, we get  $G = H$ , a.e. By ( $\alpha$ ) it follows that  $f = G$ , a.e., so  $f$  is indeed almost Riemann integrable.  $\square$

We are now in position to prove the "almost" version of Theorem 6.1.

**THEOREM 6.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The following are equivalent:*

- (i)  $f$  is almost Riemann integrable;
- (ii) there exists a negligible set  $N \subset [a, b]$  such that  $f|_{[a, b] \setminus N}$  is continuous.

PROOF. (i)  $\Rightarrow$  (ii). Assume  $f$  is almost Riemann integrable, so there exists a Riemann integrable function  $g : [a, b] \rightarrow \mathbb{R}$ , such that  $f = g$ , a.e. By Theorem 6.1, the discontinuity set  $D_g$  is negligible. Take

$$M = \{x \in [a, b] : f(x) \neq g(x)\}.$$

Since  $f = g$ , a.e., the set  $M$  is negligible, and so is the set  $N = M \cup D_g$ . On the one hand, since  $D_g \subset N$ , the restriction  $g|_{[a, b] \setminus N}$  is continuous. On the other hand, since  $M \subset N$ , we have  $f|_{[a, b] \setminus N} = g|_{[a, b] \setminus N}$ , so (ii) follows.

(ii)  $\Rightarrow$  (i). We are going to imitate the proof of Theorem 6.1, with some minor modifications. Fix  $N \subset [a, b]$  negligible, such that  $f|_{[a, b] \setminus N}$  is continuous. Fix also a sequence  $(\Delta_p)_{p=1}^\infty$  of partitions, with  $\Delta_1 \subset \Delta_2 \subset \dots$ , and  $\lim_{p \rightarrow \infty} |\Delta_p| = 0$ . Put  $S = \bigcup_{p=1}^\infty \Delta_p$ . Since  $S$  is countable, the set  $N \cup S$  is still negligible. We put  $T = [a, b] \setminus (N \cup S)$ , and we define the analogues of the functions  $f^{\Delta_p}$  and  $f_{\Delta_p}$  as

follows. Write each partition as  $\Delta_p = (a = x_0^p < x_1^p < \cdots < x_{n_p}^p = b)$ , and define, for each  $k \in \{1, \dots, n_p\}$ , the numbers

$$M_k^p = \sup \{f(t) : t \in [x_{k-1}^p, x_k^p] \cap T\} \text{ and } m_k^p = \inf \{f(t) : t \in [x_{k-1}^p, x_k^p] \cap T\}.$$

We then define, for each  $p \geq 1$ , the functions

$$\begin{aligned} g_p &= m_1^p \cdot \chi_{[x_0^p, x_1^p]} + m_2^p \cdot \chi_{(x_1^p, x_2^p]} + \cdots + m_n^p \cdot \chi_{(x_{n-1}^p, x_n^p]}, \\ g^p &= M_1^p \cdot \chi_{[x_0^p, x_1^p]} + M_2^p \cdot \chi_{(x_1^p, x_2^p]} + \cdots + M_n^p \cdot \chi_{(x_{n-1}^p, x_n^p]}. \end{aligned}$$

Note that we have the inequalities  $g^p(x) \geq f(x) \geq g_p(x)$ ,  $\forall x \in T$ , which give

$$(24) \quad g^p \geq f \geq g_p, \text{ a.e., } \forall p \geq 1.$$

It is obvious that  $g^p$  and  $g_p$ ,  $p \geq 1$ , are all Riemann integrable. We are now going to estimate the integrals  $\int_a^b [g^p(x) - g_p(x)] dx$ . Put  $h_p = g^p - g_p$ ,  $p \geq 1$ . First we observe that, since  $f|_T$  is continuous, and  $T \cap \Delta_p = \emptyset$ ,  $\forall p \geq 1$ , we clearly have the equalities  $\lim_{p \rightarrow \infty} g^p(x) = \lim_{p \rightarrow \infty} g_p(x) = f(x)$ ,  $\forall x \in T$ , which give

$$(25) \quad \lim_{p \rightarrow \infty} h_p(x) = 0, \quad \forall x \in T.$$

Fix some  $\varepsilon > 0$ , and use regularity from above, to find an open set  $D$  with  $D \supset N \cup S$  and  $\lambda(D) < \varepsilon$ . Take the compact set  $A = [a, b] \setminus D$ . Note that  $f|_A$  is continuous, since  $A \subset [a, b] \setminus N$ . Note also that, since  $A \subset [a, b] \setminus S$ , the functions  $g^p|_A$  and  $g_p|_A$  are also continuous, and so will be  $h_p|_A$ , for every  $p \geq 1$ . Since  $(g^p(x))_{p=1}^\infty$  is non-increasing, and  $(g_p(x))_{p=1}^\infty$  is non-decreasing, for all  $x$ , it follows that the sequence  $(h_p)_{p=1}^\infty$  is monotone, so by Dini's Theorem, (25) gives

$$\lim_{p \rightarrow \infty} [\max_{x \in A} h_p(x)] = 0.$$

In particular, there exists some  $p_\varepsilon \geq 1$ , such that

$$(26) \quad h_p(x) \leq \varepsilon, \quad \forall p \geq p_\varepsilon, x \in A.$$

Put  $B = [a, b] \setminus A$ , and take  $M = \sup_{x \in [a, b]} f(x)$  and  $m = \inf_{x \in [a, b]} f(x)$ . Using the inclusion  $B \subset D$ , we get  $\lambda^*(B) \leq \lambda(D) \leq \varepsilon$ , so by Lemma 6.1, (the functions  $h_p$ ,  $p \geq 1$ , are clearly non-negative), combined with (26), we get

$$\begin{aligned} \int_a^b h_p(x) dx &\leq (b-a) \cdot \sup_{x \in A} h_p(x) + \lambda^*(B) \cdot \sup_{x \in B} h_p(x) \\ &\leq (b-a)\varepsilon + \lambda^*(B)(M-m) \leq \varepsilon(b-a + M-m), \quad \forall p \geq p_\varepsilon. \end{aligned}$$

This estimate then proves that  $\lim_{p \rightarrow \infty} \int_a^b h_p(x) dx = 0$ , i.e.

$$\lim_{p \rightarrow \infty} \int_a^b [g^p(x) - g_p(x)] dx = 0.$$

Combining this with (24), and applying Lemma 6.2, yields the fact that  $f$  is almost Riemann integrable.  $\square$

COMMENT. The hypothesis that  $f$  is bounded can be replaced with a slightly weaker one, which assumes that  $f$  is *almost* bounded, meaning that there exists a negligible set  $U \subset [a, b]$ , such that  $f|_{[a, b] \setminus U}$  is bounded.

*Exercise 18.* Let  $f_n : [a, b] \rightarrow \mathbb{R}$ ,  $n \geq 1$ , be almost Riemann integrable functions, such that

- (i)  $f_n \geq f_{n+1} \geq 0$ , a.e.,  $\forall n \geq 1$ ;
- (ii)  $\lim_{n \rightarrow \infty} f_n(x) = 0$ , for “almost all”  $x \in [a, b]$ , i.e. there exists a negligible set  $N \subset [a, b]$ , such that  $\lim_{n \rightarrow \infty} f_n(x) = 0$ ,  $\forall x \in [a, b] \setminus N$ .

Prove that

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = 0.$$