Lecture 22

5. Outer measures

Although measures can be defined on arbitrary collections of sets, the most natural domain of a measure is a σ-ring. In the previous section we dealt however only with (semi)rings. Therefore it is natural to ask the following

Question 1: *Given a measure μ on a (semi)ring $\mathcal{J}$, is it possible to extend it to a measure on the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$?*

As a particular case of the above question, we can specifically ask if there exists a measure on $Bor(\mathbb{R}^n)$, which agrees with vol_n on "half-open boxes."

As a consequence of a remarkably clever construction, due to Caratheodory, we will be able to answer the above general question in the affirmative. Caratheodory's approach is based on the following concept.

Definition. Given a non-empty set X, an *outer measure on X* is simply a map $\nu : \mathcal{P}(X) \to [0, \infty]$ with the following properties.

(0) $\nu(\varnothing) = 0$.

(M) If $A, B \in \mathcal{P}(X)$ are such that $A \subset B$, then $\nu(A) \le \nu(B)$.

(ADD$_\sigma^-$) ν is σ-sub-additive, i.e. whenever $A \in \mathcal{P}(X)$, and $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{P}(X)$ with $A \subset \bigcup_{n=1}^\infty A_n$, it follows that $\mu(A) \le \sum_{n=1}^\infty \mu(A_n)$.

The property (M) is called *monotonicity*.

Remark that ν is automatically sub-additive, in the sense that, whenever $A, A_1, \dots, A_n \in \mathcal{P}(X)$ are such that $A \subset A_1 \cup \dots \cup A_n$, it follows that $\nu(A) \le \nu(A_1) + \dots + \nu(A_n)$.

The following result explains how a measure on a semiring can be naturally extended to an outer measure on the ambient space.

Proposition 5.1. *Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X, and let $\mu : \mathcal{J} \to [0, \infty]$ be a measure on $\mathcal{J}$. Consider the collection*

$$\mathcal{P}_\sigma^{\mathcal{J}}(X) = \Big\{ A \subset X \,:\, \text{there exists } (B_n)_{n=1}^\infty \subset \mathcal{J}, \text{ with } A \subset \bigcup_{n=1}^\infty B_n \Big\}.$$

Define the map $\bar{\mu} : \mathcal{P}_\sigma^{\mathcal{J}}(X) \to [0, \infty]$ by

$$\bar{\mu}(A) = \inf \left\{ \sum_{n=1}^\infty \mu(B_n) \,:\, (B_n)_{n=1}^\infty \subset \mathcal{J},\ A \subset \bigcup_{n=1}^\infty B_n \right\},\ \ \forall\, A \in \mathcal{P}_\sigma^{\mathcal{J}}(X).$$

Then the map $\mu^* : \mathcal{P}(X) \to [0, \infty]$, defined by

$$\mu^*(A) = \begin{cases} \bar{\mu}(A) & \text{if } A \in \mathcal{P}_\sigma^{\mathcal{J}}(X) \\ \infty & \text{if } A \notin \mathcal{P}_\sigma^{\mathcal{J}}(X) \end{cases}$$

is an outer measure on X, and $\mu^*\big|_{\mathcal{J}} = \mu$.

Proof. It is obvious that $\mu^*(\varnothing) = 0$. It is also clear that μ^* is monotone. To prove that μ^* is σ-sub-additive, start with $A \in \mathcal{P}(X)$ and a sequence $(A_n)_{n=1}^\infty \in \mathcal{P}(X)$, such that $A \subset \bigcup_{n=1}^\infty A_n$, and let us prove the inequality $\mu^*(A) \leq \sum_{n=1}^\infty \mu^*(A_n)$. If there exists some n with $A_n \notin \mathcal{P}_\sigma^{\mathcal{J}}(X)$, there is nothing to prove. Assume $A_n \in \mathcal{P}_\sigma^{\mathcal{J}}(X)$, for all n. Then it is clear that $A \in \mathcal{P}_\sigma^{\mathcal{J}}(X)$. Fix for the moment some $\varepsilon > 0$. For every $n \in \mathbb{N}$ choose a sequence $(B_k^n)_{k=1}^\infty \subset \mathcal{J}$, such that

$$\sum_{k=1}^\infty \mu(B_k^n) < \frac{\varepsilon}{2^n} + \bar{\mu}(A_n).$$

It is clear that, if we list the countable family $(B_k^n)_{n,k=1}^\infty$ as a sequence $(D_m)_{m=1}^\infty$, then $A \subset \bigcup_{m=1}^\infty D_m$, and

$$\bar{\mu}(A) \leq \sum_{m=1}^\infty \mu(D_m) = \sum_{n=1}^\infty \sum_{k=1}^\infty \mu(B_k^n) \leq \sum_{n=1}^\infty \left[\frac{\varepsilon}{2^n} + \bar{\mu}(A_n)\right] = \varepsilon + \sum_{n=1}^\infty \bar{\mu}(A_n).$$

Since the above inequality holds for all $\varepsilon > 0$, we conclude that

$$\mu^*(A) = \bar{\mu}(A) \leq \sum_{n=1}^\infty \bar{\mu}(A_n) = \sum_{n=1}^\infty \mu^*(A_n),$$

so μ^* is indeed σ-sub-additive.

Finally, we must show that $\mu^*\big|_{\mathcal{J}} = \mu$. Start with some $A \in \mathcal{J}$. On the one hand, since μ is a measure on $\mathcal{J}$, we know that μ is σ-subadditive (see Theorem 4.2). This means that, for any sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n$, we have $\sum_{n=1}^\infty \mu(B_n) \geq \mu(A)$. Since A obviously belongs to $\mathcal{P}_\sigma^{\mathcal{J}}(X)$, this will force

$$\mu^*(A) = \bar{\mu}(A) \geq \mu(A).$$

On the other hand, if we consider the sequence $B_1 = A$, $B_2 = B_3 = \cdots = \varnothing$, then we clearly have $\sum_{n=1}^\infty \mu(B_n) = \mu(A)$, which gives $\bar{\mu}(A) \leq \mu(A)$, so in fact we must have equality $\bar{\mu}(A) = \mu(A)$. □

Definition. The outer measure μ^*, defined in the above result, is called the *maximal outer extension of* μ. This terminology is justified by the following.

Exercise 1. Let $\mathcal{J}$ be a semiring on X, and let μ be a measure on $\mathcal{J}$. Prove that any outer measure ν on X, with $\nu\big|_{\mathcal{J}} = \mu$, then $\nu \leq \mu^*$, in the sense that

$$\nu(A) \leq \mu^*(A), \ \ \forall\, A \subset X.$$

Exercise 2. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be semirings on X with $\mathcal{J}_1 \subset \mathcal{J}_2$, and let μ_1, μ_2 be respectively measures on $\mathcal{J}_1$, $\mathcal{J}_2$, such that $\mu_2\big|_{\mathcal{J}_1} \leq \mu_1$. Let μ_1^*, μ_2^* respectively be the maximal outer extensions of μ_1, μ_2. Prove the inequality $\mu_2^* \leq \mu_1^*$.

Given a measure μ on a semiring $\mathcal{J}$ on X, one can ask whether there exists a unique outer measure on X, which extends μ. The answer is no, even in the most trivial cases.

Example 5.1. Work on the set $X = \{1, 2\}$. Take the semiring $\mathcal{J} = \{\varnothing, X\}$ and define a measure μ on $\mathcal{J}$ by $\mu(\varnothing) = 0$ and $\mu(X) = 1$. Choose now any number $a \in (0, 1)$ and define $\nu_a : \mathcal{P}(X) \to [0, 1]$ by $\nu_a(A) = a\varkappa_A(1) + (1-a)\varkappa_A(2)$. Then ν_a is an outer measure on X - in fact ν_a is a measure on $\mathcal{P}(X)$ - and $\nu_a\big|_{\mathcal{J}} = \mu$. It is obvious that $\mu^*(\{1\}) = 1 \neq a = \nu_a(\{1\})$ and $\mu^*(\{2\}) = 1 \neq 1 - a = \nu_a(\{2\})$.

We introduce now another concept, which is very important in our analysis.

Definition. Let ν be an outer measure on a non-empty set X. A subset $A \subset X$ is said to be *ν-measurable*, if it satisfies the condition

($\mathcal{m}$) $\nu(S) = \nu(S \cap A) + \nu(S \smallsetminus A), \forall\, S \subset X.$

For a given S, it is useful to think the equality $\nu(S) = \nu(S \cap A) + \nu(S \smallsetminus A)$ in unorthodox terms as "A sharply cuts S," so that saying that A is ν-measurable means that "A sharply cut every set $S \subset X$."

Remarks 5.1. Let ν be an outer measure on X.

A. Since ν is (finitely) sub-additive, for any two sets $A, S \subset X$, one always has the inequality $\nu(S) \le \nu(S \cap A) + \nu(S \smallsetminus A)$. Therefore, a set $A \subset X$ is ν-measurable, if and only if

$$\nu(S) \ge \nu(S \cap A) + \nu(S \smallsetminus A), \ \ \forall\, S \subset X.$$

B. Any subset $N \subset X$, with $\nu(N) = 0$, is ν-measurable. Indeed, from the monotonicity of ν, we see that for every $S \subset X$, we have

$$\nu(S \cap N) + \nu(S \smallsetminus N) \le \nu(N) + \nu(S) = \nu(S),$$

so by the preceding remark, N is indeed ν-measurable. Such a set N is called *ν-negligeable.*

The first key result in this section is the following.

Theorem 5.1. *Let ν be an outer measure on a non-empty set X. Then the collection*

$$\mathcal{m}_\nu(X) = \big\{ A \subset X \,:\, A \text{ } \nu\text{-measurable} \big\}$$

is a σ-algebra on X. Moreover, the restriction

$$\nu\big|_{\mathcal{m}_\nu(X)} : \mathcal{m}_\nu(X) \to [0, \infty]$$

is a measure on $\mathcal{m}_\nu(X)$.

Proof. The proof will be carried on in several steps.

Step 1: *If $A \in \mathcal{m}_\nu(X)$, then $X \smallsetminus A \in \mathcal{m}_\nu(X)$.*

This is trivial, since for every $S \subset X$, one has the equalities

$$S \cap (X \smallsetminus A) = S \smallsetminus A \text{ and } S \smallsetminus (X \smallsetminus A) = S \cap A.$$

Step 2: *If $A, B \in \mathcal{m}_\nu(X)$, then $A \cap B \in \mathcal{m}_\nu(X)$.*

Start with some arbitrary $S \subset X$. Since B is ν-measurable, it "shaprply cuts the set $S \smallsetminus (A \cap B)$," which means that

$$\nu\big(S \smallsetminus (A \cap B)\big) = \nu\big([S \smallsetminus (A \cap B)] \cap B\big) + \nu\big([S \smallsetminus (A \cap B)] \smallsetminus B\big).$$

Since we clearly have $[S \smallsetminus (A \cap B)] \cap B = (S \cap B) \smallsetminus A$, and $[S \smallsetminus (A \cap B)] \smallsetminus B = S \smallsetminus B$, the above equality gives

$$\nu\big(S \smallsetminus (A \cap B)\big) = \nu\big((S \cap B) \smallsetminus A\big) + \nu(S \smallsetminus B).$$

Adding $\nu\big((S \cap B) \cap A\big)$, and using the fact that A "sharply cuts $S \cap B$," we now get

$$\nu(\big(S \cap (A \cap B)\big) + \nu\big(S \smallsetminus (A \cap B)\big) =$$
$$= \nu\big((S \cap B) \cap A\big) + \nu\big((S \cap B) \smallsetminus A\big) + \nu(S \smallsetminus B) = \nu(S \cap B) + \nu(S \smallsetminus B).$$

Finally, using the fact that B "sharply cuts S," we get

$$\nu(\big(S \cap (A \cap B)\big) + \nu\big(S \smallsetminus (A \cap B)\big) = \nu(S \cap B) + \nu(S \smallsetminus B) = \nu(S),$$

so $A \cap B$ is indeed ν-measurable.

So far, Steps 1 and 2 prove that $\mathcal{M}_\nu(X)$ is an *algebra on* X.

Step 3: *For any pair-wise disjoint finite sequence* $(A_n)_{n=1}^N \subset \mathcal{M}_\nu(X)$*, one has the equality*

$$\nu\big(S \cap \big[A_1 \cup \cdots \cup A_N\big]\big) = \sum_{n=1}^N \nu(S \cap A_n),\ \ \forall\, S \subset X.$$

Since $\mathcal{M}_\nu(X)$ is an algebra, it suffices to prove the aboove equalityonly for $N = 2$. (The case of arbitrary N follows immediately by induction.) To prove that $\nu\big(S \cap (A_1 \cup A_2)\big) = \nu(S \cap A_1) + \nu(S \cap A_2)$, we simply use the fact that A_1 "sharply cuts $S \cap (A_1 \cup A_2)$," which gives

$$\nu\big(S \cap (A_1 \cup A_2)\big) = \nu\big([S \cap (A_1 \cup A_2)] \cap A_1\big) + \nu\big([S \cap (A_1 \cup A_2)] \smallsetminus A_1\big).$$

The desired equality then immediately follows from the obvious equalities

$$[S \cap (A_1 \cup A_2)] \cap A_1 = S \cap A_1 \text{ and } [S \cap (A_1 \cup A_2)] \smallsetminus A_1 = S \cap A_2.$$

The preceding step can be in fact extended to infinite sequences.

Step 4: *For any pair-wise disjoint sequence* $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$*, one has the equality*

$$\nu\Big(S \cap \big[\bigcup_{n=1}^\infty A_n\big]\Big) = \sum_{n=1}^\infty \nu(S \cap A_n),\ \ \forall\, S \subset X.$$

To prove this fact, we fix a sequence $(A_n)_{n=1}^\infty$ as above, as well as $S \subset X$. By σ-sub-additivity, we already know that

$$\nu\Big(S \cap \big[\bigcup_{n=1}^\infty A_n\big]\Big) = \nu\Big(\bigcup_{n=1}^\infty [S \cap A_n]\Big) \leq \sum_{n=1}^\infty \nu(S \cap A_n),$$

so the only thing we have to show is the inequality

$$\sum_{n=1}^N \nu(S \cap A_n) \leq \nu\Big(S \cap \big[\bigcup_{n=1}^\infty A_n\big]\Big),\ \ \forall\, N \in \mathbb{N}.$$

This follows immediately from Step 3 and the monotonicity:

$$\sum_{n=1}^N \nu(S \cap A_n) = \nu\Big(S \cap \big[\bigcup_{n=1}^N A_n\big]\Big) \leq \nu\Big(S \cap \big[\bigcup_{n=1}^\infty A_n\big]\Big).$$

Step 5: $\mathcal{M}_\nu(X)$ *is a monotone class.*

We need to prove the properties:

(i) whenever $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ is a sequence with $A_n \subset A_{n+1}$, $\forall\, n \in \mathbb{N}$, it follows that $\bigcup_{n=1}^\infty A_n$ belongs to $\mathcal{M}_\nu(X)$;

(ii) whenever $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ is a sequence with $A_n \supset A_{n+1}$, $\forall\, n \in \mathbb{N}$, it follows that $\bigcap_{n=1}^\infty A_n$ belongs to $\mathcal{M}_\nu(X)$.

Since $\mathcal{M}_\nu(X)$ is an algebra, it suffices only to prove (i). Start with an arbitrary subset S, and a sequence $(A_n)_{n=1}^\infty \subset \mathcal{M}_\nu(X)$ with $A_n \subset A_{n+1}$, $\forall\, n \in \mathbb{N}$, and denote the union $\bigcup_{n=1}^\infty A_n$ simply by A. Define the sets $B_1 = A_1$ and $B_n = A_n \smallsetminus A_{n-1}$, $\forall\, n \geq 2$. It is obvious that $(B_n)_{n=1}^\infty$ is a pair-wise disjoint sequence. Since $\mathcal{M}_\nu(X)$

is an alegbra, all the B_n's belong to $\mathcal{M}_\nu(X)$. We have, $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n = A$, which, using Step 4 gives

$$(1) \qquad \nu(S\cap A) = \nu\Big(S\cap\big[\bigcup_{n=1}^\infty A_n\big]\Big) = \nu\Big(S\cap\big[\bigcup_{n=1}^\infty B_n\big]\Big) = \sum_{n=1}^\infty \nu(S\cap B_n).$$

Using Step 3, combined with the equality $\bigcup_{n=1}^N B_n = A_N$, we also have

$$\sum_{n=1}^N \nu(S\cap B_n) = \nu\Big(S\cap\big[\bigcup_{n=1}^N B_n\big]\Big) = \nu(S\cap A_N),\ \ \forall\, N\in\mathbb{N},$$

so by (1) we have

$$(2) \qquad \nu(S\cap A) = \sum_{n=1}^\infty \nu(S\cap B_n) = \lim_{N\to\infty}\nu(S\cap A_N).$$

Notice now that, using the fact that A_N "sharply cuts S," combined with the monotonicity of ν and the obvious inclusion $S\smallsetminus A\subset S\smallsetminus A_N$, we have

$$\nu(S\cap A_N) + \nu(S\smallsetminus A) \le \nu(S\cap A_N) + \nu(S\smallsetminus A_N) = \nu(S),\ \ \forall\, N\in\mathbb{N},$$

so using (2), we immediately get

$$\nu(S\cap A) + \nu(S\smallsetminus A)\le \nu(S).$$

Since the above inequality holds for all $S\subset X$, by Remark 5.1.A it follows that A indeed belongs to $\mathcal{M}_\nu(X)$.

By the results from Section 1, we know that the fact that $\mathcal{M}_\nu(X)$ is simultaneously an algebra, and a monotone class, implies the fact that $\mathcal{M}_\nu(X)$ is a σ-algebra.

We now show that $\nu\big|_{\mathcal{M}_\nu(X)}$ is a measure. If we start with a pair-wise disjoint sequence $(A_n)_{n=1}^\infty\subset\mathcal{M}_\nu(X)$, then the equality equality

$$\nu\big(\bigcup_{n=1}^\infty A_n\big) = \sum_{n=1}^\infty \nu(A_n)$$

is an immediate consequence of Step 4, applied to the set $S=\bigcup_{n=1}^\infty A_n$, which clearly satisfies $S\cap A_n = A_n$, $\forall\, n\in\mathbb{N}$. □

We are now in position to answer the Question 1.

Theorem 5.2. *Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X, let μ be a measure on $\mathcal{J}$, and let μ^* be the maximal outer extension of μ. Then $\mathcal{J}\subset\mathcal{M}_{\mu^*}(X)$. In particular, $\mathcal{M}_{\mu^*}(X)$ contains the σ-algebra $\mathbf{\Sigma}(\mathcal{J})$ on X, generated by $\mathcal{J}$, and $\mu^*\big|_{\mathbf{\Sigma}(\mathcal{J})}$ is a measure on $\mathbf{\Sigma}(\mathcal{J})$.*

Proof. What we need to prove is the fact that every set $A\in\mathcal{J}$ is μ^*-measurable. Start with an arbitrary set $S\subset X$. As noticed before (Remark 5.1.A), we only need to prove the inequality

$$(3) \qquad \mu^*(S\cap A) + \mu^*(S\smallsetminus A)\le \mu^*(S).$$

If $\mu^*(S)=\infty$, there is nothing to prove, so we can assume that $\mu^*(S)<\infty$. In particular this means that $S\in\mathcal{P}_\sigma^{\mathcal{J}}(X)$. Fix for the moment $\varepsilon>0$. By the definition of $\mu^*(S) = \bar{\mu}(S)$, there exists a sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$, such that $S \subset \bigcup_{n=1}^\infty B_n$, and

$$\sum_{n=1}^{\infty} \mu(B_n) \leq \mu^*(S) + \varepsilon. \tag{4}$$

Since $\mathcal{J}$ is a semiring, for each $n \in \mathbb{N}$, we can find some integer $p_n \geq 1$, and a sequence $(D_j^n)_{j=0}^{p_n} \subset \mathcal{J}$, such that

- $B_n \cap A = D_0^n \subset D_1^n \subset \cdots \subset D_{p_n}^n = B_n$,
- $D_j \smallsetminus D_{j-1} \in \mathcal{J}$, $\forall\, j \in \{1, \ldots, p_n\}$.

Define the numbers $k_0 = 0$, and $k_n = \sum_{j=1}^n p_j$, $\forall\, n \in \mathbb{N}$, and the sequence $(C_m)_{m=1}^\infty \subset \mathcal{J}$, by

$$C_m = D_{m-k_{n-1}}^n \smallsetminus D_{m-1-k_{n-1}}^n, \text{ if } k_{n-1} < m \leq k_n,\ \ n \in \mathbb{N}.$$

By construction, for each $n \in \mathbb{N}$, we have

$$\bigcup_{m=k_{n-1}+1}^{k_n} C_m = \bigcup_{j=1}^{p_n} (D_j^n \smallsetminus D_{j-1}^n) = B_n \smallsetminus A_n.$$

Moreover, for each $n \in \mathbb{N}$ the system

$$(D_0^n, C_{k_{n-1}+1}, C_{k_{n-1}+2}, \ldots, C_{k_n}) = (D_0^n, D_1^n \smallsetminus D_0^n, D_2^n \smallsetminus D_1^n, \ldots, D_{p_n}^n \smallsetminus D_{p_n-1}^n)$$

in $\mathcal{J}$ is pair-wise disjoint, and has

$$D_0^n \cup \bigcup_{m=k_{n-1}+1}^{k_n} C_m = B_n,$$

so we get the equality

$$\mu(D_0^n) + \sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) = \mu(B_n).$$

Using (4) we now get

$$\begin{aligned} &\sum_{n=1}^{\infty} \mu(D_0^n) + \sum_{m=1}^{\infty} \mu(C_m) = \sum_{n=1}^{\infty} \mu(D_0^n) + \sum_{n=1}^{\infty} \Big(\sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) \Big) = \\ &= \sum_{n=1}^{\infty} \Big(\mu(D_0^n) + \sum_{m=k_{n-1}+1}^{k_n} \mu(C_m) \Big) = \sum_{n=1}^{\infty} \mu(B_n) \leq \mu^*(S) + \varepsilon. \end{aligned} \tag{5}$$

On the one hand, we clearly have

$$\begin{aligned} \bigcup_{m=1}^{\infty} C_m &= \bigcup_{n=1}^{\infty} \Big(\bigcup_{m=k_{n-1}+1}^{k_n} C_m \Big) = \bigcup_{n=1}^{\infty} \Big(\bigcup_{j=1}^{p_n} (D_j^n \smallsetminus D_{j-1}^n) \Big) = \\ &= \bigcup_{n=1}^{\infty} (D_{p_n}^n \smallsetminus D_0^n) = \bigcup_{n=1}^{\infty} (B_n \smallsetminus A) = \big(\bigcup_{n=1}^{\infty} B_n \big) \smallsetminus A \supset S \smallsetminus A, \end{aligned}$$

which gives the inequality

$$\sum_{m=1}^{\infty} \mu(C_m) \geq \mu^*(S \smallsetminus A). \tag{6}$$

On the other hand, we also have

$$\bigcup_{n=1}^{\infty} D_0^n = \bigcup_{n=1}^{\infty} (B_n \cap A) = \big(\bigcup_{n=1}^{\infty} B_n\big) \cap A \supset S \cap A,$$

which gives the inequality

$$\sum_{n=1}^{\infty} \mu(D_0^n) \geq \mu^*(S \cap A). \tag{7}$$

Combining (6) and (7) with (5) immediately gives the desired inequality (3). □

The construction

$$\left\{ \begin{matrix} \mu \\ \text{measure on } \mathfrak{J} \end{matrix} \right\} \xrightarrow[]{\text{maximal outer extension}} \left\{ \begin{matrix} \mu^* \\ \text{outer measure on } X \end{matrix} \right\} \xrightarrow[]{\text{restriction}} \left\{ \begin{matrix} \mu^*\big|_{\mathbf{\Sigma}(\mathfrak{J})} \\ \text{measure on } \mathbf{\Sigma}(\mathfrak{J}) \end{matrix} \right\}$$

is referred to as the *Caratheodory construction.*

DEFINITIONS. Let $\mathfrak{J}$ be a semiring on X, and let μ be a measure on $\mathfrak{J}$. The Caratheodory construction provides us with two measures. The first measure - $\mu^*\big|_{\mathbf{S}(\mathfrak{J})}$ - is a measure on the σ-ring $\mathbf{S}(\mathfrak{J})$ generated by $\mathfrak{J}$, and is called the *maximal σ-ring extension of μ.* The second measure - $\mu^*\big|_{\mathbf{\Sigma}(\mathfrak{J})}$ - is a measure on the σ-algebra $\mathbf{\Sigma}(\mathfrak{J})$ generated by $\mathfrak{J}$, and is called the *maximal σ-algebra extension of μ.*

The above terminology is justified by the following result.

PROPOSITION 5.2. *Let $\mathfrak{J}$ be a semiring on X, and let μ be a measure on $\mathfrak{J}$.*

(i) *If ν is a measure on the σ-ring $\mathbf{S}(\mathfrak{J})$ generated by $\mathfrak{J}$, with $\nu\big|_{\mathfrak{J}} = \mu$, then $\nu \leq \mu^*\big|_{\mathbf{S}(\mathfrak{J})}$.*

(ii) *If ν is a measure on the σ-algebra $\mathbf{\Sigma}(\mathfrak{J})$ generated by $\mathfrak{J}$, with $\nu\big|_{\mathfrak{J}} = \mu$, then $\nu \leq \mu^*\big|_{\mathbf{\Sigma}(\mathfrak{J})}$.*

PROOF. We prove both statements simultaneously. Let $\mathfrak{J}_1$ denote either the σ-ring, or the σ-algebra generated by $\mathfrak{J}$. In particular $\mathfrak{J}_1$ is a semiring, and $\mathfrak{J} \subset \mathfrak{J}_1$. Since ν is a measure on $\mathfrak{J}_1$ with $\nu\big|_{\mathfrak{J}} = \mu$, if we denote by ν^* its maximal outer extension, then by Exercise 2 we know that $\nu^* \leq \mu^*$. In particular, by Proposition 5.1 and Theorem 5.2, we get $\nu = \nu^*\big|_{\mathfrak{J}_1} \leq \mu^*\big|_{\mathfrak{J}_1}$. □

We now discuss the uniqueness of extensions of a semiring measure. In order to clarify this matter, we have to introduce a technical condition, which turns out to be very helpful not only here, but in many other situations.

DEFINITIONS. Let $\mathfrak{J}$ be a semiring on X, and let μ be a measure on $\mathfrak{J}$.

A. We say that a subset $A \subset X$ is *$\mathfrak{J}$-μ-σ-finite*, if there exists a sequence $(B_n)_{n=1}^{\infty} \subset \mathfrak{J}$, such that $A \subset \bigcup_{n=1}^{\infty} B_n$, and $\mu(B_n) < \infty$, $\forall\, n \in \mathbb{N}$. (When there is no danger of confusion, we will use the terms "μ-σ-finite," or simply "σ-finite.")

B. We say that the measure μ is *σ-finite*, if every $A \in \mathfrak{J}$ is σ-finite.

C. We say that the measure μ is *finite*, if $\mu(A) < \infty$, $\forall\, A \in \mathfrak{J}$.

Clearly every finite measure on $\mathfrak{J}$ is σ-finite.

REMARK 5.2. Let $\mathfrak{J}$ be a semiring on X, let μ be a measure on $\mathfrak{J}$, and let A be a set which belongs to the σ-algebra $\mathbf{\Sigma}(\mathfrak{J})$ generated by $\mathfrak{J}$. If A if $\mathfrak{J}$-μ-σ-finite, then A in fact belongs to the semiring $\mathbf{S}(\mathfrak{J})$ generated by $\mathfrak{J}$. The only thing that is actually needed here is the existence of a sequence $(B_n)_{n=1}^{\infty} \subset \mathfrak{J}$ with $A \subset \bigcup_{n=1}^{\infty} B_n$. This gives the fact that A belongs to $\mathcal{P}_\sigma^{\mathcal{J}}(X)$, so by Proposition 2.3, the set A belongs to the intersection $\mathbf{\Sigma}(\mathcal{J}) \cap \mathcal{P}_\sigma^{\mathcal{J}}(X) = \mathbf{S}(\mathcal{J})$.

Using the above terminology, we have the following uniqueness result.

THEOREM 5.3. *Let $\mathcal{J}$ be a semiring on X, let μ be a measure on $\mathcal{J}$, let μ^* be the maximal outer extension of μ, and let ν be a measure on the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, with $\nu\big|_{\mathcal{J}} = \mu$. Then one has $\nu(A) = \mu^*(A)$, for all $\mathcal{J}$-μ-σ-finite sets $A \in \mathbf{S}(\mathcal{J})$.*

PROOF. Fix a $\mathcal{J}$-μ-σ-finite set $A \in \mathbf{S}(\mathcal{J})$.

Claim: *There exists a pair-wise disjoint sequence $(D_n)_{n=1}^\infty \subset \mathbf{S}(\mathcal{J})$ such that $A \subset \bigcup_{n=1}^\infty D_n$, and $\nu(D_n) = \mu^*(D_n) < \infty$, $\forall\, n \in \mathbb{N}$.*

To prove the above statement, start with a sequence $(B_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n$ and $\mu(B_n) < \infty$, $\forall\, n \in \mathbb{N}$. Define the sets D_n, $n \in \mathbb{N}$ by $D_1 = B_1$, and $D_n = B_n \smallsetminus (B_1 \cup \dots \cup B_{n-1})$, $\forall\, n \geq 2$. It is clear that the sequence $(D_n)_{n=1}^\infty$ is pair-wise disjoint, and

$$A \subset \bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty D_n.$$

Moreover, all the D_n's belong to the ring $\mathbf{R}(\mathcal{J})$ generated by $\mathcal{J}$. The inclusions $D_n \subset B_n$ then prove that

$$\mu^*(D_n) \leq \mu^*(B_n) = \mu(B_n) < \infty, \ \ \forall\, n \in \mathbb{N}.$$

Finally, since both $\mu^*\big|_{\mathbf{R}(\mathcal{J})}$ and $\nu\big|_{\mathbf{R}(\mathcal{J})}$ are measures on $\mathbf{R}(\mathcal{J})$, which have the same values on $\mathcal{J}$, using the Semiring-to-Ring Extension Theorem 4.1, it follows that

$$\mu^*\big|_{\mathbf{R}(\mathcal{J})} = \nu\big|_{\mathbf{R}(\mathcal{J})}. \tag{8}$$

In particular we have the equalities

$$\nu(D_n) = \mu^*(D_n), \ \ \forall\, n \in \mathbb{N}.$$

Having proven the Claim, we now show that $\nu(A) = \mu^*(A)$. We choose a sequence $(D_n)_{n=1}^\infty \subset \mathbf{S}(\mathcal{J})$ as in the Claim. On the one hand, since the D_n's are pair-wise disjoint, and both ν and $\mu^*\big|_{\mathbf{S}(\mathcal{J})}$ are measures on the σ-ring $\mathbf{S}(\mathcal{J})$, one has the equalities

$$\nu(A) = \sum_{n=1}^\infty \nu(A \cap D_n) \text{ and } \mu^*(A) = \sum_{n=1}^\infty \mu^*(A \cap D_n).$$

So, in order to prove the equality $\nu(A) = \mu^*(A)$, it suffices to prove that

$$\nu(A \cap D_n) = \mu^*(A \cap D_n), \ \ \forall\, n \in \mathbb{N}. \tag{9}$$

Fix $n \in \mathbb{N}$. On the one hand, by Proposition 5.2(i), we have the inequalities

$$\nu(A \cap D_n) \leq \mu^*(A \cap D_n) < \infty \text{ and } \nu(D_n \smallsetminus A) \leq \mu^*(D_n \smallsetminus A) < \infty. \tag{10}$$

On the other hand, we have

$$\nu(A \cap D_n) + \nu(D_n \smallsetminus A) = \nu(D_n) = \mu^*(D_n) = \mu^*(A \cap D_n) + \mu^*(D_n \smallsetminus A).$$

Now if we go back to (10), we see that none of the two inequalities can be strict, because in that case we would get $\nu(D_n) < \mu^*(D_n)$. (The assumption that $\mu^*(D_n) < \infty$ is essential here.) So we must have (9), and we are done. □

Corollary 5.1. *If μ is a σ-finite measure on a semiring $\mathcal{J}$, then there exists a unique measure ν on the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, such that $\nu\big|_{\mathcal{J}} = \mu$. Moreover, ν is σ-finite.*

Proof. The existence is given by the Caratheodory construction. The uniqueness follows from Theorem 5.3.

To prove σ-finiteness, start with some $A \in \mathbf{S}(\mathcal{J})$, and let us find a sequence $(B_n)_{n=1}^{\infty} \subset \mathbf{S}(\mathcal{J})$ with $A \subset \bigcup_{n=1}^{\infty} B_n$ and $\nu(B_n) < \infty$, $\forall\, n \in \mathbb{N}$. First of all, since $\mathcal{P}_{\sigma}^{\mathcal{J}}(X)$ is a σ-ring which contains $\mathcal{J}$, it follows that $\mathbf{S}(\mathcal{J}) \subset \mathcal{P}_{\sigma}^{\mathcal{J}}(X)$. In particular, there exists $(D_n)_{n=1}^{\infty} \subset \mathcal{J}$ such that $A \subset \bigcup_{n=1}^{\infty} D_n$. Using the fact that μ is σ-finite, we see that for each n we can find a sequence $(D_k^n)_{k=1}^{\infty} \subset \mathcal{J}$, with $D_n \subset \bigcup_{k=1}^{\infty} D_k^n$ and $\mu(D_k^n) < \infty$, $\forall\, k \in \mathbb{N}$. If we list all the sets D_k^n, $k, n \in \mathbb{N}$ as a sequence $(B_m)_{m=1}^{\infty}$, then we are done. □

In the absence of the σ-finitess condition the uniqueness of the σ-ring extension fails, as illustrated by the following.

Example 5.2. Consider the set $X = \mathbb{Q}$, and the semiring of rational half-open intervals

$$\mathcal{J}_1 = \big\{\varnothing\big\} \cup \big\{[a,b) \cap \mathbb{Q} \,:\, a, b \in \mathbb{R},\ a < b\big\}.$$

We equipp $\mathcal{J}_1$ with the measure μ defined by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{if } A \neq \varnothing \end{cases}$$

Notice that, if we look at the inclusion $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$, then $\mathcal{J}_1 = \mathcal{J}\big|_{\mathbb{Q}}$, where $\mathcal{J}$ is the semiring of half-open intervals in $\mathbb{R}$. By the Generating Theorem we then have

$$\mathbf{S}(\mathcal{J}_1) = \mathbf{S}(\mathcal{J}\big|_{\mathbb{Q}}) = \mathbf{S}(\mathcal{J})\big|_{\mathbb{Q}} = Bor(\mathbb{R})\big|_{\mathbb{Q}} = \mathcal{P}(\mathbb{Q}).$$

Define now the measures $\nu_1, \nu_2 : \mathbf{S}(\mathcal{J}) \to [0, \infty]$ by

$$\nu_1(A) = \begin{cases} \text{card}\, A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases} \qquad \nu_2(A) = \begin{cases} 2 \cdot \text{card}\, A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite} \end{cases}$$

It is obvious that both ν_1 and ν_2 satisfy $\nu_1\big|_{\mathcal{J}_1} = \nu_2\big|_{\mathcal{J}_1} = \mu$, but obviously ν_1 and ν_2 are not equal.

Comment. In connection with the Caratheodory construction, it is legitimate to ask the following.

> *Question 2*: *What happens if we do the Caratheodory construction twice?*

This problem has in fact two aspects.

> *Question 2A*: *Suppose ω is an outer measure on X. Take $\mathcal{I} = \mathcal{M}_{\omega}(X)$ and $\nu = \omega\big|_{\mathcal{I}}$, so that $\mathcal{I}$ is a semiring (in fact it is a σ-algebra) on X, and ν is a measure on $\mathcal{I}$. Let ν^* be the maximal outer extension of ν. Is it true that $\nu^* = \omega$?*

By Exercise 2, we always have $\omega \leq \nu^*$. In general the answer to Question 2A in negative, as shown in Exercise ??? below. One can ask however the following

> *Question 2B*: *Same question as 2A, but suppose $\omega = \mu^*$, the maximal outer extension of a measure μ on a semiring $\mathcal{J}$.*

The following result shows that Question 2B always has an affirmative answer.

PROPOSITION 5.3. *Let X be a non-empty set, let $\mathcal{J}$ be a semiring on X, and let μ be a measure on $\mathcal{J}$. Let μ^* be the maximal outer exetension of ν. Let $\mathcal{I}$ be a semiring, with $\mathcal{I} \supset \mathcal{J}$ Consider the measure $\nu = \mu^*\big|_{\mathcal{I}}$, and let ν^* be the maximal outer extension of ν. Then $\nu^* = \mu^*$.*

PROOF. First of all, since $\nu\big|_{\mathcal{J}} = \mu^*\big|_{\mathcal{J}} = \mu$, by Exercise 2, we have the inequality $\nu^* \leq \mu^*$.

To prove the other inequality, we start with an arbitrary set $A \subset X$, and we prove that $\mu^*(A) \leq \nu^*(A)$. If $\nu^*(A) = \infty$, there is nothing to prove, so we may assume $\nu^*(A) < \infty$. In particular, $A \in \mathcal{P}_\sigma^{\mathcal{I}}(X)$, i.e. there exists at least one sequence $(B_n)_{n=1}^\infty \subset \mathcal{I}$, with $A \subset \bigcup_{n=1}^\infty B_n$, and we have

$$\nu^*(A) = \inf\left\{ \sum_{n=1}^\infty \nu(B_n) \,:\, (B_n)_{n=1}^\infty \subset \mathcal{I},\;\; A \subset \bigcup_{n=1}^\infty B_n \right\}.$$

Fix for the moment a some $\varepsilon > 0$, and choose a sequence $(B_n^\varepsilon)_{n=1}^\infty \subset \mathcal{I}$, such that

$$A \subset \bigcup_{n=1}^\infty B_n^\varepsilon \text{ and } \sum_{n=1}^\infty \nu(B_n^\varepsilon) \leq \nu^*(A) + \varepsilon. \tag{11}$$

By σ-subadditivity of μ^*, we have

$$\mu^*(A) \leq \sum_{n=1}^\infty \mu^*(B_n^\varepsilon).$$

Using the fact that $\nu = \mu^*\big|_{\mathcal{I}}$, the above inequality, combined with (11) yields

$$\mu^*(A) \leq \nu^*(A) + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, it forces the inequality $\mu^*(A) \leq \nu^*(A)$. □

Exercise 3. Let X be an uncountable set, and define $\omega : \mathcal{P}(X) \to [0, \infty]$ by

$$\omega(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ 1 & \text{if } 0 < \text{card}\, A \leq \aleph_0 \\ 2 & \text{if } A \text{ is uncountable} \end{cases}$$

(i) Prove that ω is an outer measure on X.
(ii) Take $\mathcal{I} = \mathcal{M}_\omega(X)$. Prove that $\mathcal{I} = \{\varnothing, X\}$.
(iii) Consider the measure $\nu = \omega\big|_{\mathcal{I}}$, and let ν^* be the maximal outer extension of ν. Prove that there are sets $A \subset X$, with $\omega(A) < \nu^*(A)$.

HINTS: For (ii) start with some A with $\varnothing \subsetneq A \subsetneq X$. Prove that A is not ω-measurable, by showing that A does not "sharply cut" sets of the form $\{a, b\}$ with $a \in A$ and $b \in X \smallsetminus A$.

COMMENT. Suppose $\mathcal{J}$ is a semiring on X, and μ is a measure on $\mathcal{J}$. We have used the maximal outer extension μ^* as a tool in defining measures on the σ-ring $\mathbf{S}(\mathcal{J})$ and the σ-algebra $\mathbf{\Sigma}(\mathcal{J})$ generated by $\mathcal{J}$, by employing the Caratheodory construction, which uses the σ-algebra $\mathcal{M}_{\mu^*}(X)$ of μ^*-measurable sets. A legitimate question is then

Question 3: *Is the inclusion $\mathbf{\Sigma}(\mathcal{J}) \subset \mathcal{M}_{\mu^*}(X)$ strict?*

In most cases this inclusion is indeed strict (see Examples ?? below, or the discussion in the next section). This can be seen by looking at μ^*-negligeable sets $N \subset X$, which are automatically μ^*-measurable. The following result gives some useful information.

Proposition 5.4. *Suppose $\mathcal{J}$ is a semiring on X, and μ is a measure on $\mathcal{J}$. Let μ^* be the maximal outer extension of μ. For any set $A \in \mathcal{P}_\sigma^{\mathcal{J}}(X)$, there exists some set B in the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, such that $A \subset B$, and $\mu^*(A) = \mu^*(B)$.*

In particular, a subset $N \subset X$ is μ^*-neglijeable, i.e. $\mu^*(N) = 0$, if and only if there exists a μ^*-negligeable set $B \in \mathbf{S}(\mathcal{J})$, such that $N \subset B$.

Proof. Since $A \in \mathcal{P}_\sigma^{\mathcal{J}}(X)$, there exists a sequence $(D_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty D_n$. Moreover, we have

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^\infty \mu(D_n) \,:\, (D_n)_{n=1}^\infty \subset \mathcal{J},\ A \subset \bigcup_{n=1}^\infty D_n \right\}.$$

For each integer $k \geq 1$, we can then choose a sequence $(B_n^k)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty B_n^k$ and $\sum_{n=1}^\infty \mu(B_n^k) \leq \mu^*(A) + 1/k$. For each integer $k \geq 1$, we define the set $B_k = \bigcup_{n=1}^\infty B_n^k$. It is clear that $A_k \in \mathbf{S}(\mathcal{J})$, and $B_k \supset A$, for all $k \in \mathbb{N}$. Moreover, by σ-sub-additivity of μ^*, and the equality $\mu^*\big|_{\mathcal{J}} = \mu$, we have the inequalities

$$\mu^*(B_k) \leq \sum_{n=1}^\infty \mu^*(B_n^k) = \sum_{n=1}^\infty \mu(B_n^k) \leq \mu^*(A) + \frac{1}{k},\ \ \forall\, k \in \mathbb{N}.$$

If we then form $B = \bigcap_{k=1}^\infty B_k$, then B still belongs to $\mathbf{S}(\mathcal{J})$, and we have $A \subset B \subset B_k$, which gives

$$\mu^*(A) \leq \mu^*(B) \leq \mu^*(B_k) \leq \mu^*(A) + \frac{1}{k}\ \ \forall\, k \in \mathbb{N},$$

thus forcing $\mu^*(B) = \mu^*(A)$.

To prove the second assertion, we see that the "only if" part is a particular case of the first part. The "if" part is trivial, since the inclusion $N \subset B$ forces the inequality $\mu^*(N) \leq \mu^*(B)$. □

In connection to Question 3, it is useful to introduce the following terminology.

Definition. Let X be a non-empty set, and let $\mathcal{J}$ be a semiring on X. A measure μ on $\mathcal{J}$ is said to be *complete*, if it satisfies the condition

(c) *whenever $N \in \mathcal{J}$ has $\mu(N) = 0$, it follows that $\mathcal{J}$ contains all the subsets of N.*

Remarks 5.3. A. Given an outer measure ν on a set X, the measure $\nu\big|_{\mathcal{M}_\nu(X)} : \mathcal{M}_\nu(X) \to [0, \infty]$ is always complete, as a consequence of monotonicity, and of Remark 5.1.B.

B. Given a semiring $\mathcal{J}$ on X, and a measure μ on $\mathcal{J}$, we now see that a sufficient condition, for having a strict inclusion $\mathbf{\Sigma}(\mathcal{S}) \subsetneq \mathcal{M}_{\mu^*}(X)$, is the lack of completeness for the measure $\mu^*\big|_{\mathbf{\Sigma}(\mathcal{J})}$. Later on (see Corollary 5.2) we shall see that in the case of σ-finite measures, defined on σ-total semirings, this condition is also necessary.

The lack of completeness of a σ-ring measure can be compensated by the following result.

Theorem 5.4. *Let X be a non-empty set, let $\mathcal{S}$ be a σ-ring on X, and let ν be a measure on $\mathcal{S}$.*

(*i*) *The collection*

$$\mathcal{N}(\mathcal{S}, \nu) = \big\{ N \subset X \,:\, \text{there exists } D \in \mathcal{S} \text{ with } N \subset D \text{ and } \nu(D) = 0 \big\}$$

is a σ-ring on X. Moreover, if $N \in \mathcal{N}(\mathcal{S},\nu)$, then $\mathcal{N}(\mathcal{S},\nu)$ contains all subsets of N.

(*ii*) *For a subset $A \subset X$, the following are equivalent:*
  (*a*) *there exists $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S},\nu)$, such that $A = B \smallsetminus N$;*
  (*b*) *there exists $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S},\nu)$, such that $A = F \cup M$.*

(iii) *The collection $\bar{\mathcal{S}}$ of all subsets $A \subset X$, satisfying the equivalent conditions in (ii), is a σ-ring. We have the equality*

$$\bar{\mathcal{S}} = \mathbf{S}\big(\mathcal{N}(\mathcal{S},\nu) \cup \mathcal{S}\big).$$

(iv) *There exists a unique measure $\bar{\nu}$ on $\bar{\mathcal{S}}$, such that $\bar{\nu}\big|_{\mathcal{N}(\mathcal{S},\nu)} = 0$ and $\bar{\nu}\big|_{\mathcal{S}} = \nu$. The measure $\bar{\nu}$ is complete.*

(v) *If $\mathcal{E}$ is a σ-ring with $\mathcal{E} \supset \mathcal{S}$, and if λ is a complete measure on $\mathcal{E}$ with $\lambda\big|_{\mathcal{S}} = \nu$, then $\mathcal{E} \supset \bar{\mathcal{S}}$ and $\lambda\big|_{\bar{\mathcal{S}}} = \bar{\nu}$.*

Proof. (i). This is pretty clear. In fact, if one takes $\mathcal{E} = \{B \in \mathcal{S} : \nu(B) = 0\}$, then one has the equality $\mathcal{N}(\mathcal{S},\nu) = \mathcal{P}^{\mathcal{E}}_{\sigma}(X)$.

(ii). (*a*) $\Rightarrow$ (*b*). Assume $A = B \smallsetminus N$ with $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S},\nu)$. Choose $D \in \mathcal{S}$ with $\nu(D) = 0$ and $N \subset D$. We now have

$$B \smallsetminus D \subset B \smallsetminus N = A,$$

so if we put $F = B \smallsetminus D$, we have the equality $A = F \cup M$, where

$$M = A \smallsetminus F = (B \smallsetminus N) \smallsetminus (B \smallsetminus D) \subset D.$$

Notice that $F \in \mathcal{S}$, while the inclusion $M \subset D$ shows that $M \in \mathcal{N}(\mathcal{S},\nu)$.

(*b*) $\Rightarrow$ (*a*). Assume $A = F \cup M$ with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S},\nu)$. Choose $D \in \mathcal{S}$ with $M \subset D$ and $\nu(D) = 0$. Define $B = F \cup D$. It is clear that $B \in \mathcal{S}$, and $A \subset B$. Define $N = B \smallsetminus A$, so we clearly have $A = B \smallsetminus N$. We have

$$N = (F \cup D) \smallsetminus (F \cup M) \subset D \smallsetminus M \subset D,$$

so N clearely belongs to $\mathcal{N}(\mathcal{S},\nu)$.

(iii). We need to prove the following properties:

($*$) whenever A_1, A_2 are sets in $\bar{\mathcal{S}}$, it follows that the difference $A_1 \smallsetminus A_2$ also belongs to $\bar{\mathcal{S}}$;

($**$) whenever $(A_n)_{n=1}^{\infty}1$ is a sequence of sets in $\bar{\mathcal{S}}$, it follows that the union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\bar{\mathcal{S}}$.

To prove ($*$), we write $A_1 = B \smallsetminus N$ and $A_2 = F \cup M$, with $B, F \in \mathcal{S}$ and $M, N \in \mathcal{N}(\mathcal{S},\nu)$. Then we have

$$A_1 \smallsetminus A_2 = (B \smallsetminus N) \smallsetminus (F \cup M) = B \smallsetminus (F \cup M \cup N) = (B \smallsetminus F) \smallsetminus (M \cup N).$$

The difference $B \smallsetminus F$ belongs to $\mathcal{S}$, and, using (i), the union $N \cup M$ belongs to $\mathcal{N}(\mathcal{S},\nu)$. By (ii) it follows that $A_1 \smallsetminus A_2$ belongs to $\bar{\mathcal{S}}$.

To prove ($**$), we write, for each $n \in \mathbb{N}$, the set A_n as $A_n = F_n \cup \smallsetminus M_n$ with $F_n \in \mathcal{S}$ and $M_n \in \mathcal{N}(\mathcal{S},\nu)$. Then

$$\bigcup_{n=1}^{\infty} A_n = \big(\bigcup_{n=1}^{\infty} F_n\big) \cup \big(\bigcup_{n=1}^{\infty} M_n\big).$$

The union $\bigcup_{n=1}^{\infty} F_n$ belongs to $\mathcal{S}$, and, using (i), the union $\bigcup_{n=1}^{\infty} M_n$ belongs to $\mathcal{N}(\mathcal{S},\nu)$. By (ii), the union $\bigcup_{n=1}^{\infty} A_n$ belongs to $\bar{\mathcal{S}}$.

Since $\bar{\mathcal{S}}$ is a σ-ring, which clearly contains both $\mathcal{N}(\mathcal{S},\nu)$ and $\mathcal{S}$, it follows that $\bar{\mathcal{S}} \supset \mathbf{S}\big(\mathcal{N}(\mathcal{S},\nu)\cup\mathcal{S}\big)$. The other inclusion $\bar{\mathcal{S}} \subset \mathbf{S}\big(\mathcal{N}(\mathcal{S},\nu)\cup\mathcal{S}\big)$ is trivial, by the definition of $\bar{\mathcal{S}}$.

(iv). To prove the existence, we consider the maximal outer extension ν^*. When restricted to the σ-algebra $\mathcal{M}_{\nu^*}(X)$ of all ν^*-measurable sets, then we get a measure. Notice that $\nu^*(N) = 0$, $\forall N \in \mathcal{N}(\mathcal{S},\nu)$, which gives the inclusion $\mathcal{N}(\mathcal{S},\nu) \subset \mathcal{M}_{\nu^*}(X)$. In particular, since $\mathcal{M}_{\nu^*}(X)$ is a σ-algebra, which contains both $\mathcal{N}(\mathcal{S},\nu)$ and $\mathcal{S}$, it follows that

$$\mathcal{M}_{\nu^*}(X) \supset \mathbf{S}\big(\mathcal{N}(\mathcal{S},\nu)\cup\mathcal{S}\big) = \bar{\mathcal{S}}.$$

In particular, $\bar{\nu} = \nu^*\big|_{\bar{\mathcal{S}}}$ is a measure on $\bar{\mathcal{S}}$, which clearly satisfies the required properties.

To prove uniqueness, let μ be another measure on $\bar{\mathcal{S}}$, such that $\mu\big|_{\mathcal{N}(\mathcal{S},\nu)} = 0$ and $\mu\big|_{\mathcal{S}} = \nu$. It we start with an arbitrary set $A \in \bar{\mathcal{S}}$, and we write it as $A = F \cup M$, with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S},\nu)$, then using the fact that $A \smallsetminus F \subset M$, we see that $A \smallsetminus F$ belongs to $\mathcal{N}(\mathcal{S},\nu)$, so we have

$$\mu(A) = \mu(F) + \nu(A \smallsetminus F) = \mu(F) = \nu(F) = \bar{\nu}(F) = \bar{\nu}(F) + \bar{\nu}(A \smallsetminus F) = \bar{\nu}(A).$$

Finally, we prove that the measure $\bar{\nu}$ is complete. Let $A \in \bar{\mathcal{S}}$ be a set with $\bar{\nu}(A) = 0$, and let U be an arbitrary subset of A. Using (ii) we write $A = F \cup M$, with $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S},\nu)$. Notice that we have

$$0 \le \nu(F) = \bar{\nu}(F) \le \bar{\nu}(F \cup M) = \bar{\nu}(A) = 0,$$

which forces $F \in \mathcal{N}(\mathcal{S},\nu)$, so using (i), we see that A itself belongs to $\mathcal{N}(\mathcal{S},\nu)$. By (i), it follows that $U \in \mathcal{N}(\mathcal{S},\nu) \subset \bar{\mathcal{S}}$.

(v) Let $\mathcal{E}$ and λ be as in indicated. In order to prove the inclusion $\mathcal{E} \supset \bar{\mathcal{S}}$, it suffices to prove the inclusion $\mathcal{N}(\mathcal{S},\nu) \subset \mathcal{E}$. But this inclusion is pretty obvious. If we start with some $N \in \mathcal{N}(\mathcal{S},\nu)$, then there exists $A \in \mathcal{S}$ with $N \subset A$ and $\nu(A) = 0$. In particular, we have $A \in \mathcal{E}$ and $\lambda(E) = 0$, and then the completeness of λ forces $N \in \mathcal{E}$. Notice that this also forces $\lambda(N) = \bar{\nu}(N) = 0$. Using (iv) it then follows that $\lambda|_{\bar{\mathcal{S}}} = \bar{\nu}$. □

Definition. Using the notations above, the σ-ring $\bar{\mathcal{S}}$ is called the *completion of $\mathcal{S}$ with respect to ν*. The correspondence $(\mathcal{S},\nu) \longmapsto (\bar{\mathcal{S}},\bar{\nu})$ is referred to as the *measure completion.* Remark that, if ν is already complete, then $\bar{\mathcal{S}} = \mathcal{S}$ and $\bar{\nu} = \nu$.

Exercise 4. Using the notations from Theorem 5.4, prove that for a set $A \subset X$, the condition $A \in \bar{\mathcal{S}}$ is equivalent to any of the following:

- (a') there exists $B \in \mathcal{S}$ and $N \in \mathcal{N}(\mathcal{S},\mu)$, with $A = B \smallsetminus N$, and $N \subset B$;
- (b') there exists $F \in \mathcal{S}$ and $M \in \mathcal{N}(\mathcal{S},\nu)$, such that $A = F \cup M$ and $F \cap M = \varnothing$;
- (c) there exists $E \in \mathcal{S}$ and $Z \in \mathcal{N}(\mathcal{S},\nu)$, such that $A = E \triangle Z$.
- (d) there exist $B, F \in \mathcal{S}$ such that $F \subset A \subset B$, and $\mu(B \smallsetminus F) = 0$.

The μ^*-measurable sets of a special type can be completely characterized using μ^*-negligeable ones.

Theorem 5.5. *Suppose $\mathcal{J}$ is a semiring on X, and μ is a measure on $\mathcal{J}$. Let μ^* be the maximal outer extension of μ. For a $\mathcal{J}$-μ-σ-finite subset $A \subset X$, the following are equivalent;*

(i) *A is μ^*-measurable;*

(ii) *there exists B in the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}$, and a μ^*-neglijeable set $N \subset X$, such that $A = B \smallsetminus N$.*

Proof. $(i) \Rightarrow (ii)$. Start by choosing a sequence $(D_n)_{n=1}^\infty \subset \mathcal{J}$ with $A \subset \bigcup_{n=1}^\infty D_n$ and $\mu(D_n) < \infty$, $\forall\, n \in \mathbb{N}$. Since $\mathcal{M}_{\mu^*}(X)$ is an algebra, which contains $\mathcal{J}$, it follows that all the intersections $A_n = A \cap D_n$, $n \in \mathbb{N}$, belong to $\mathcal{M}_{\mu^*}(X)$. For each $n \in \mathbb{N}$, we use the previous result to find some set $B_n \in \mathbf{S}(\mathcal{J})$ such that $A_n \subset B_n$, and $\mu^*(B_n) = \mu^*(A_n)$. On the one hand, if we put $V_n = B_n \smallsetminus A_n$, then $V_n \in \mathcal{M}_{\mu^*}(X)$, so we will have

$$\mu^*(B_n) = \mu^*(A_n) + \mu^*(V_n).$$

On the other hand, we know that $\mu^*(B_n) = \mu^*(A_n) \leq \mu^*(D_n) < \infty$, so the above equality forces $\mu^*(V_n) = 0$.

Since we have $B_n = A_n \cup V_n$, $\forall\, n \in \mathbb{N}$, we will get

$$\bigcup_{n=1}^\infty B_n = \big(\bigcup_{n=1}^\infty A_n\big) \cup \big(\bigcup_{n=1}^\infty V_n\big) = A \cup \big(\bigcup_{n=1}^\infty V_n\big),$$

so if we define $B = \bigcup_{n=1}^\infty E_n$ and $V = \bigcup_{n=1}^\infty V_n$, then B belongs to $\mathbf{S}(\mathcal{J})$, we have the equality $B = A \cup V$, and V is μ^*-negligeable, because of the inequalities

$$\mu^*(V) \leq \sum_{n=1}^\infty \mu^*(V_n).$$

The set $N = B \smallsetminus A \subset V$ is clearly μ^*-negligeable, because $\mu^*(N) \leq \mu^*(V)$. Now we are done because $B \smallsetminus N = A$.

$(ii) \Rightarrow (i)$. This part is trivial, since $\mathcal{M}_{\mu^*}(X)$ is an algebra. □

Remarks 5.4. A. The implication $(ii) \Rightarrow (i)$ holds without the assumption that A is $\mathcal{J}$-μ-σ-finite. In fact, for any $A \subset X$, one has the implications $(ii) \Rightarrow (ii') \Rightarrow (i)$, where

(ii') *there exists B in the σ-algebra $\mathbf{\Sigma}(\mathcal{J})$ generated by $\mathcal{J}$, and a μ^*-neglijeable set $N \subset X$, such that $A = B \smallsetminus N$.*

B. Consider the measure $\mu^*\big|_{\mathbf{S}(\mathcal{J})}$ on the σ-ring $\mathbf{S}(\mathcal{J})$. Using the notations from Theorem 5.4, by Proposition 5.3, we clearly have the equality

$$\big\{N \subset X \,:\, N \ \mu^*\text{-negligeable}\big\} = \mathcal{N}\big(\mathbf{S}(\mathcal{J}), \mu^*\big|_{\mathbf{S}(\mathcal{J})}\big).$$

So, if we denote by $\overline{\mathbf{S}(\mathcal{J})}$ the completion of $\mathbf{S}(\mathcal{J})$ with respect to $\mu^*\big|_{\mathbf{S}(\mathcal{J})}$, condition (ii) from Theorem 5.5 reads: $A \in \overline{\mathbf{S}(\mathcal{J})}$. Similarly, if we denote by $\overline{\mathbf{\Sigma}(\mathcal{J})}$ the completion of $\mathbf{\Sigma}(\mathcal{J})$ with respect to the measure $\mu^*\big|_{\mathbf{\Sigma}(\mathcal{J})}$, condition (ii') above reads: $A \in \overline{\mathbf{\Sigma}(\mathcal{J})}$. With these notations, we have the inclusions

$$\overline{\mathbf{S}(\mathcal{J})} \subset \overline{\mathbf{\Sigma}(\mathcal{J})} \subset \mathcal{M}_{\mu^*}(X). \tag{12}$$

With these notations, Theorem 5.5 states that

$$\overline{\mathbf{S}(\mathcal{J})} \cap \big\{A \subset X \,:\, A \ \mathcal{J}\text{-}\mu\text{-}\sigma\text{-finite}\big\} = \mathcal{M}_{\mu^*}(X) \cap \big\{A \subset X \,:\, A \ \mathcal{J}\text{-}\mu\text{-}\sigma\text{-finite}\big\}. \tag{13}$$

Theorem 5.5, written in the form (13) has the following.

Corollary 5.2. *If the semiring $\mathcal{J}$ is σ-total in X, and μ is a σ-finite measure on $\mathcal{J}$, then one has the equalities*

$$\overline{\mathbf{S}(\mathcal{J})} = \overline{\mathbf{\Sigma}(\mathcal{J})} = \mathcal{M}_{\mu^*}(X). \tag{14}$$

PROOF. Indeed, under the given assumptions on $\mathcal{J}$ and μ, it follows that *every set* $A \subset X$ *is* $\mathcal{J}$-μ-σ-*finite.* □

EXAMPLES 5.3. A. The implication $(i) \Rightarrow (ii)$ from Theorem 5.5 may fail, if A is not σ-finite. Start with an arbitrary set X, consider the semiring $\mathcal{J} = \{\varnothing, X\}$ and the measure μ on $\mathcal{J}$ defined by $\mu(\varnothing) = 0$ and $\mu(X) = \infty$. Notice that $\mathcal{J}$ is a σ-algebra, so it is trivial that $\mathcal{J}$ is σ-total in X. The maximal outer extension μ^* of μ is defined by

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{if } A \neq \varnothing \end{cases}$$

It is clear that, since μ^* is a measure on $\mathcal{P}(X)$, we have the equality $\mathcal{M}_{\mu^*}(X) = \mathcal{P}(X)$, but the only μ^*-neglijeable set is the empty set $\varnothing$. This means that the sets satisfying condition (ii) in Theorem 5.4 are only the sets $\varnothing$ and X, so, if $\varnothing \neq A \subsetneq X$, the implication $(i) \Rightarrow (ii)$ fails, although $\mathcal{J}$ is σ-total in X. What occurs here is the total lack of $\mathcal{J}$-μ-σ-finite sets.

B. Let X be an uncountable set, and let $\mathcal{J}$ be the semiring of all finite subsets of X. We have

$$\mathbf{S}(\mathcal{J}) = \{A \subset X : \text{card}\, A \leq \aleph_0\},$$
$$\mathbf{\Sigma}(\mathcal{J}) = \{A \subset X : \text{ either card}\, A \leq \aleph_0, \text{ or card}(X \smallsetminus A) \leq \aleph_0\}.$$

Equipp $\mathcal{J}$ with the trivial measure $\mu(A) = 0$, $\forall\, A \in \mathcal{J}$. The maximal outer extension μ^* is then defined by

$$\mu^*(A) = \begin{cases} 0 & \text{if card}\, A \leq \aleph_0 \\ \infty & \text{if } A \text{ is uncountable} \end{cases}$$

It is clear that μ^* is a measure on $\mathcal{P}(X)$, so we have $\mathcal{M}_{\mu^*}(X) = \mathcal{P}(X) \supsetneq \mathcal{J}$. Notice that both measures $\mu^*\big|_{\mathbf{S}(\mathcal{J})}$ and $\mu^*\big|_{\mathbf{\Sigma}(\mathcal{J})}$ are complete, so using the notations from Remark 5.4.B, we have the equalities

$$\overline{\mathbf{S}(\mathcal{J})} = \mathbf{S}(\mathcal{J}) \text{ and } \overline{\mathbf{\Sigma}(\mathcal{J})} = \mathbf{\Sigma}(\mathcal{J}).$$

It is clear however that both inclusions in (12) are strict, although μ is finite. What happens here is the fact that $\mathcal{J}$ is not σ-total in X.

C. In the same setting as in Example B, if we take $\mathcal{I} = \mathbf{\Sigma}(\mathcal{J})$, and $\nu = \mu^*\big|_{\mathcal{I}}$, then $\mathcal{I}$ is σ-total in X, simply because $\mathcal{I}$ is a σ-algebra. In this case, by Proposition 5.3, the maximal outer extension ν^* of ν coincides with μ^*. We have

$$\mathcal{I} = \mathbf{S}(\mathcal{I}) = \overline{\mathbf{S}(\mathcal{I})} = \mathbf{\Sigma}(\mathcal{I}) = \overline{\mathbf{\Sigma}(\mathcal{I})} \subsetneq \mathcal{M}_{\nu^*}(X),$$

the reason for the strict inclusion being this time the fact that ν is not σ-finite.

COMMENT. In the remainder of this section we take another look Question 3, trying to generalize the answer given by Corollary 5.2. To simplify matters a little bit, we start with a σ-algebra $\mathcal{B}$ on X (which is clearly σ-total in X), and a measure μ on $\mathcal{B}$. It we take μ^* to be the maximal outer extension of μ, and consider the completion $\bar{\mathcal{B}}$, we have the inclusion

$$\bar{\mathcal{B}} \subset \mathcal{M}_{\mu^*}(X), \tag{15}$$

so we can ask whether this inclusion is strict. Of course, if μ is σ-finite, then by Corollary 5.2 the inclusion (15) is not strict. As Example 5.3.C suggests, in the absence of the σ-finiteness assumption, the inclusion (15) may indeed be strict. As it turns out, the fact that the inclusion (15) is strict in Example 5.3.C is a consequence of the fact that there are "new" measurable sets which are not necessarily of the form $B \smallsetminus N$ with $B \in \mathcal{B}$ and N neglijeable. The existence of such sets is suggested by the following.

Remark 5.5. Suppose ν is an outer measure on X. For a set $A \subset X$, the following are equivalent:

(i) A is ν-measurable;
(ii) $\nu(S) \geq \nu(S \cap A) + \nu(S \smallsetminus A)$, for all $S \subset X$ with $\nu(S) < \infty$;

The implication $(i) \Rightarrow (ii)$ is trivial. To prove the converse, by Remark 5.1.A, we need to show that

$$\nu(S) \geq \nu(S \cap A) + \nu(S \smallsetminus A), \ \ \forall\, S \subset X.$$

But this is trivial, when $\nu(S) = \infty$. If $\nu(S) < \infty$, then this is exactly condition (ii).

The "new" sets, that were mentioned above, are of a type covered by the following.

Definition. Let ν be an outer measure on X. A subset $N \subset X$ is said to be *locally ν-neglijeable*, if

$$\nu(N \cap A) = 0, \text{ for all } A \subset X \text{ with } \nu(A) < \infty.$$

It is clear that every subset of N is also locally ν-neglijeable.

The above observation shows that *every locally ν-neglijeable set is ν-measurable.*

The term "local" will be used in connection with properties that hold when the subject set is cut down by sets of finite measure. For example, one can formulate the following.

Definitions. Let $\mathcal{B}$ be a σ-algebra on X, and μ be a measure on $\mathcal{B}$. We say that a set $N \in \mathcal{B}$ is *locally μ-null*, if

$$\mu(F \cap N) = 0, \text{ for all } F \in \mathcal{B}, \text{ with } \mu(F) < \infty. \tag{16}$$

Remark that locally μ-null sets do not necessarily have zero measure (see Example 5.3.C)

We say that μ is *locally complete*, if it satisfies the condition

(lc) *whenever $N \in \mathcal{B}$ is a locally μ-null set, it follows that $\mathcal{B}$ contains all subsets of N.*

Remarks 5.6. Use the notations above.

A. If the measure μ is σ-finite, the local completeness of μ is equivalent to completeness. The reason is the fact that, in the σ-finite case, condition (16) is equivalent to $\mu(N) = 0$.

B. Given an outer measure ν on X, the measure $\nu\big|_{\mathcal{M}_\nu(X)}$ is locally complete.

Comment. If we look at Example 5.3.C, we now see that although the measure ν on $\mathcal{I}$ is complete, it is not locally complete, thus giving another explanation for the strict inclusion $\mathcal{I} \subsetneq \mathcal{M}_{\nu^*}(X)$.

We are now in position to analyze Question 3, in the simplified given setting. The following fact will be helpful.

Lemma 5.1. *Let $\mathcal{B}$ be a σ-algebra on X, let μ be a measure on $\mathcal{B}$, and let μ^* be the maximal outer extension of μ. Then, for every subset $S \subset X$, one has the equality*

$$\mu^*(S) = \inf \big\{ \mu(B) \,:\, B \in \mathcal{B},\ B \supset S \big\}. \tag{17}$$

PROOF. Since $\mathcal{B}$ is σ-total in X, by definition we have

$$\mu^*(S) = \inf\Big\{\sum_{n=1}^{\infty}\mu(B_n) \,:\, (B_n)_{n=1}^{\infty}\subset\mathcal{B},\ S\subset\bigcup_{n=1}^{\infty}B_n\Big\}. \tag{18}$$

If we denote the right hand side of (??) by $\nu(S)$, then using (18) we clearly have $\mu^*(S)\leq\nu(S)$. Conversely, if we start with any sequence $(B_n)_{n=1}^{\infty}\subset\mathcal{B}$ with $S\subset\bigcup_{n=1}^{\infty}B_n$, then we clearly have

$$\sum_{n=1}^{\infty}\mu(B_n)\geq\mu\Big(\bigcup_{n=1}^{\infty}B_n\Big)\geq\nu(S),$$

so taking the infimum yields $\mu^*(S)\geq\nu(S)$. □

PROPOSITION 5.5. *Let $\mathcal{B}$ be a σ-algebra on X, and let μ be a measure on $\mathcal{B}$. Define the collection*

$$\mathcal{B}_{\text{fin}} = \big\{F\in\mathcal{B} \,:\, \mu(F)<\infty\big\}.$$

For every $F\in\mathcal{B}_{\text{fin}}$, denote by $\mathfrak{M}_F$ the completion of the σ-algebra $\mathcal{B}\big|_F$ (on F) with respect to the measure[1] $\mu\big|_F$. Denote by μ^* the maximal outer extension of μ.

A. *For a subset $A\subset X$, the following are equivalent*
- (i) *A is μ^*-measurable;*
- (ii) *$A\cap F\in\mathfrak{M}_F$, for each $F\in\mathcal{B}_{\text{fin}}$;*
- (iii) *$A\cap F$ is μ^*-measurable, for each $F\in\mathcal{B}_{\text{fin}}$.*

B. *For a subset $N\subset X$, the following are equivalent*
- (i) *N is locally μ^*-neglijeable;*
- (ii) *$\mu^*(N\cap F)=0$, for all $F\in\mathcal{B}_{\text{fin}}$.*

PROOF. Let us fix some useful notations. By construction, for every $F\in\mathcal{B}_{\text{fin}}$, we have

$$\mathcal{B}\big|_F = \big\{A\cap F \,:\, A\in\mathcal{B}\big\} = \big\{B\in\mathcal{B} \,:\, B\subset F\big\}.$$

For each $F\in\mathcal{B}_{\text{fin}}$, we denote the σ-ring $\mathcal{N}\big(\mathcal{B}\big|_F,\mu\big|_F\big)$ simply by $\mathcal{N}_F$. With the above identification we have

$$\mathcal{N}_F = \big\{N\subset F \,:\, \text{ there exists } D\in\mathcal{B} \text{ with } N\subset D\subset F \text{ and } \mu(D)=0\big\},$$

so (see Theorem 5.4) the σ-algebra $\mathfrak{M}_F$ is given as

$$\mathfrak{M}_F = \big\{B\smallsetminus N \,:\, B\in\mathcal{B},\ B\subset F,\ N\in\mathcal{N}_F\big\}.$$

A. To prove the implication $(i)\Rightarrow(ii)$, start with a μ^*-measurable set A, and with some $F\in\mathcal{B}_{\text{fin}}$. Since F is μ^*-measurable, the intersection $A\cap F$ is μ^*-measurable. Since $\mu^*(A\cap F)\leq\mu^*(F)=\mu(F)<\infty$, by Theorem 5.5 there exist $B_0\in\mathcal{B}$ and $N_0\subset X$ with $\mu^*(N_0)=0$ and $A\cap F=B_0\smallsetminus N_0$. If we then define $B=B_0\cap F$ and $N=N_0\cap F$, then we clearly have $B\in\mathcal{B}\big|_F$, $N\in\mathcal{N}_F$, and $A\cap F=B\smallsetminus N$, so $A\cap F$ indeed belongs to $\mathfrak{M}_F$.

The implication $(ii)\Rightarrow(ii)$ is trivial, since every set in $\mathfrak{M}_F$ is clearly μ^*-measurable.

To prove the implication $(iii)\Rightarrow(i)$, assume A has property (iii), and let us show that A is μ^*-measurable. We are going to use Remark 5.5, which means that it suffices to prove the inequality

$$\mu^*(S)\geq\mu^*(S\cap A)+\mu^*(S\smallsetminus A), \tag{19}$$

[1] Here $\mu\big|_F$ denotes the restriction of μ to the σ-algebra $\mathcal{B}\big|_F$.

only for those subsets $S \subset X$ with $\mu^*(S) < \infty$. Fix such a subset S. Since $\mu^*(S) < \infty$, Lemma 5.1 gives

$$\mu^*(S) = \inf \{\mu(F) : F \in \mathcal{B}_{\text{fin}},\ F \supset S\}. \tag{20}$$

Start with some arbitrary $\varepsilon > 0$, and choose some $F \in \mathcal{B}_{\text{fin}}$ with $F \supset S$ and $\mu(F) \leq \mu^*(S) + \varepsilon$. By (iii) the set $A \cap F$ is μ^*-measurable, so we have

$$\mu^*(F) = \mu^*\big(F \cap [A \cap F]\big) + \mu^*\big(F \smallsetminus [A \cap F]\big) = \mu^*(F \cap A) + \mu^*(F \smallsetminus A).$$

Since $F \cap A \supset S \cap A$, and $F \smallsetminus A \supset S \smallsetminus A$, we have the inequalities $\mu^*(F \cap A) \geq \mu^*(S \cap A)$ and $\mu^*(F \smallsetminus A) \geq \mu^*(S \smallsetminus A)$, so the above inequality gives

$$\mu^*(F) \geq \mu^*(S \cap A) + \mu^*(S \smallsetminus A).$$

By the choice of F, this gives

$$\mu^*(S) + \varepsilon \geq \mu^*(S \cap A) + \mu^*(S \smallsetminus A).$$

Since this inequality holds for all $\varepsilon > 0$, we immediately get the desired inequality (19).

B. The condition (i) says that

$$\mu^*(N \cap S) = 0, \text{ for all } S \subset X \text{ with } \mu^*(S) = 0. \tag{21}$$

It is obvious that we have the implication $(i) \Rightarrow (ii)$. Conversely, suppose N satisfies (ii), and let us prove (21). Start with some arbitrary subset $S \subset X$ with $\mu^*(S) < \infty$. Using (20), there exists some $F \in \mathcal{B}_{\text{fin}}$ with $S \subset F$. By (ii), and the monotocity of μ^* we have $0 = \mu^*(N \cap F) \geq \mu^*(N \cap S)$, which clearly forces $\mu^*(N \cap S) = 0$. □

The above result suggests that the σ-algebra $\mathcal{M}_{\mu^*}(X)$ can be regarded as some sort of "local" completion of $\mathcal{B}$. To simplify the exposition a little bit, we introduce the following.

Notation. Let $\mathcal{B}$ be a σ-algebra on X, let μ be a measure on $\mathcal{B}$, and let μ^* be the maximal outer extension of μ. The σ-algebra $\mathcal{M}_{\mu^*}(X)$, of all μ^*-measurable subsets of X, will be denoted by $\mathfrak{M}_\mu(\mathcal{B})$ (or just $\mathfrak{M}_\mu$, when there is no danger oof confusion). The measure $\mu^*\big|_{\mathfrak{M}_\mu}$ will be denoted by $\tilde{\mu}$. The pair $(\mathfrak{M}_\mu, \tilde{\mu})$ will be called the *quasi-completion of $\mathcal{B}$ with respect to μ.*

Unfortunately, analogues of Theorem 5.4 are not available, unless some (otherwise natural) restrictions are imposed. The type of restrictions we have in mind also aimed at making the test conditions A.(iii) and B.(ii) easier to check. We would like to check them on a "small" sub-collection of $\mathcal{B}_{\text{fin}}$. This naturally suggests the following.

Definition. Let $\mathcal{B}$ be a σ-algebra on X, and let μ be a measure on $\mathcal{B}$. A *sufficient μ-finite $\mathcal{B}$-partition of X* is a collection $\mathcal{F}$ of non-empty subsets of X, with the following properties:

(i) $\mathcal{F}$ is pairwise disjoint, and $\bigcup_{F \in \mathcal{F}} F = X$;
(ii) $\mathcal{F} \subset \mathcal{B}$, and $\mu(F) < \infty$, for all $F \in \mathcal{F}$;
(iii) for every set $B \in \mathcal{B}$, with $\mu(B) < \infty$, one has the equality

$$\mu(B) = \sum_{F \in \mathcal{F}} \mu(B \cap F).$$

Condition (iii) uses the summation convention from II.2. (The sum is defined as the supprenum of all finite partial sums.)

REMARKS 5.7. A. Suppose $\mathcal{F}$ is a sufficient μ-finite $\mathcal{B}$-partition of X. For every set $A \in \mathcal{B}$, we define the collection

$$S^{\mu}_{\mathcal{F}}(A) = \{F \in \mathcal{F} : \mu(A \cap F) > 0\}.$$

If $\mu(A) < \infty$, then

(*a*) $S^{\mu}_{\mathcal{F}}(A)$ is at most countable, and
(*b*) $\mu\big(A \smallsetminus \big[\bigcup_{F \in S^{\mu}_{\mathcal{F}}(A)} (A \cap F)\big]\big) = 0.$

By condition (iii) in the definition, it follows that, the family $\big(\mu(A \cap F)\big)_{F \in S^{\mu}_{\mathcal{F}}(A)}$ is summable, and

$$\sum_{F \in S^{\mu}_{\mathcal{F}}(A)} \mu(A \cap F) = \mu(A). \tag{22}$$

Since $\mu(A \cap F) > 0$, $\forall F \in S^{\mu}_{\mathcal{F}}(A)$, property (*a*) follows from Proposition II.2.2. If we denote the union $\bigcup_{F \in S^{\mu}_{\mathcal{F}}(A)} (A \cap F)$ by A_0, then by the σ-additivity of μ (it is here where we use (*a*) in an essential way) the equality (22) gives

$$\mu(A_0) = \sum_{F \in S^{\mu}_{\mathcal{F}}(A)} \mu(A \cap F) = \mu(A),$$

which combined with $\mu(A) < \infty$ forces $\mu(A \smallsetminus A_0) = 0$.

B. The existence of a sufficient μ-finite $\mathcal{B}$-partition of X is a generalization of σ-finitess. In fact the following are equivalent ($\mathcal{B}$ is a σ-algebra on X):

- μ is σ-finite;
- there exists a *countable* sufficient μ-finite $\mathcal{B}$-partition of X.

In the presence of a sufficient μ-finite $\mathcal{B}$-partition, the properties that appear in Proposition 5.5 are simplified.

PROPOSITION 5.6. *Let $\mathcal{B}$ be a σ-algebra on X, let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ-finite $\mathcal{B}$-partition of X. Denote by μ^* the maximal outer extension of μ.*

A. *For a subset $A \subset X$, the following are equivalent*
  (i) *A is μ^*-measurable;*
  (ii) *$A \cap F$ is μ^*-measurable, for each $F \in \mathcal{F}$.*

B. *For a subset $N \subset X$, the following are equivalent*
  (i) *N is locally μ^*-neglijeable;*
  (ii) *$\mu^*(N \cap F) = 0$, for all $F \in \mathcal{F}$.*

C. *If $A \subset X$ is a subset with $\mu^*(A) < \infty$, then*

$$\mu^*(A) = \sum_{F \in \mathcal{F}} \mu^*(A \cap F). \tag{23}$$

PROOF. It will be useful to introduce the following notations (use also the notations from Proposition 5.5). For every $B \in \mathcal{B}_{\text{fin}}$, we define

$$B' = \bigcup_{F \in S^{\mu}_{\mathcal{F}}(B)} (B \cap F).$$

By Remark 5.7 we know that $\mu(B \smallsetminus B') = 0$.

A. The implication (*i*) $\Rightarrow$ (*ii*) is trivial. To prove the implication (*ii*) $\Rightarrow$ (*i*), we start with a set $A \subset X$ satisfying condition (*ii*), and we show that A satisfies condition (*iii*) from Proposition 5.5.A. Start with some arbitrary set $B \in \mathcal{B}_{\text{fin}}$, and let us show that $A \cap B$ is μ^*-measurable. Using the above notation, and the monotonicity of μ^* we have

$$\mu^*\big(A \cap [B \smallsetminus B']\big) \leq \mu^*(B \smallsetminus B) = \mu(B \smallsetminus B') = 0,$$

which in particular shows that $A \cap [B \smallsetminus B']$ is μ^*-measurable. Since we have $A \cap B = (A \cap B') \cup (A \cap [B \smallsetminus B'])$, it then suffices to show that $A \cap B'$ is μ^*-measurable. Notice that

$$A \cap B' = \bigcup_{S^{\mu}_{\mathcal{F}}(B)} (A \cap F \cap B),$$

and since the indexing set $S^{\mu}_{\mathcal{F}}(B)$ is at most countable, it then suffices to show that $A \cap F \cap B$ is μ^*-measurable, for each F. But this is obvious, since $A \cap F$ is μ^*-measurable, by condition (ii), and $B \in \mathcal{B}$.

C. Let $A \subset X$ be a subset with $\mu^*(A)$. Using Lemma 5.1, we can find, for every $\varepsilon > 0$, some set $B_\varepsilon \in \mathcal{B}_{\text{fin}}$, such that $B_\varepsilon \supset A$, and $\mu(B_\varepsilon) \leq \mu^*(A) + \varepsilon$. Fix for the moment ε. Since the family $\big(\mu(B_\varepsilon \cap F)\big)_{F \in \mathcal{F}}$ is summable, and $\mu^*(A \cap F) \leq \mu(B_\varepsilon \cap F)$, $\forall\, F \in \mathcal{F}$, it follows that the family $\big(\mu^*(A \cap F)\big)_{F \in \mathcal{F}}$ is summable, and moreover one has the inequality

$$\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \sum_{F \in \mathcal{F}} \mu(B_\varepsilon \cap F) = \mu(B_\varepsilon) \leq \mu^*(A) + \varepsilon.$$

Since we have $\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \mu^*(A) + \varepsilon$, for all $\varepsilon > 0$, it follows that we have in fact the inequality

$$\sum_{F \in \mathcal{F}} \mu^*(A \cap F) \leq \mu^*(A).$$

To prove the reverse inequality, we fix $\varepsilon = 1$ and we define set

$$G = \bigcup_{F \in S^{\mu}_{\mathcal{F}}(B_1)} F.$$

Since $S^{\mu}_{\mathcal{F}}(B_1)$ is at most countable, the set G belongs to $\mathcal{B}$. With the above notation, we have the equality $B_1' = B_1 \cap G$, and by Remark 5.7.A, we have $\mu(B_1 \smallsetminus G) = \mu(B_1 \smallsetminus B_1') = 0$. Since $A \smallsetminus G \subset B_1 \smallsetminus G$, it follows that $\mu^*(A \smallsetminus G) = 0$. Since G is μ^*-measurable, we get

$$\mu^*(A) = \mu^*(A \cap G) + \mu^*(A \smallsetminus G) = \mu^*(A \cap G).$$

Since G is a countable union of F's, by the σ-subadditivity of μ^*, we have

$$\mu^*(A) = \mu^*(A \cap G) = \mu^*\Big(\bigcup_{F \in S^{\mu}_{\mathcal{F}}(B_1)} [A \cap F]\Big) \leq \sum_{F \in S^{\mu}_{\mathcal{F}}(B_1)} \mu^*(A \cap F) \leq \sum_{F \in \mathcal{F}} \mu^*(A \cap F).$$

B. The implication $(i) \Rightarrow (ii)$ is trivial. To prove the implication $(ii) \Rightarrow (i)$, we must show that condition (ii) implies

$$\mu^*(N \cap B) = 0, \ \ \forall\, B \in \mathcal{B}_{\text{fin}}.$$

But if we fix some $B \in \mathcal{B}_{\text{fin}}$, then of course we have $\mu^* N \cap B) \leq \mu^*(B) = \mu(B) < \infty$, so using part C, we have

$$\mu^*(N \cap B) = \sum_{F \in \mathcal{F}} \mu^*(N \cap B \cap F) \leq \sum_{F \in \mathcal{F}} \mu^*(N \cap F) = 0,$$

and we are done. $\square$

Comments. Let $\mathcal{B}$ be a σ-algebra on X, let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ-finite $\mathcal{B}$-partition of X.

By Proposition 5.6.C, it follows that $\mathcal{F}$ is also a sufficient $\tilde{\mu}$-finite $\mathfrak{M}_\mu$-partition of X.

We see naow that $\mathfrak{M}_\mu$ may contain more "new" sets, appart from the "natural candidates," which are of the form $B \smallsetminus N$, with $B \in \mathcal{B}$ and N locally μ^*-neglijeable. Such "new" sets are those which belong (see Section 2) to the σ-algebra $\bigvee_{F\in\mathcal{F}}\big(\mathcal{B}\big|_F\big)$. More precisely, we have the following.

Corollary 5.3. *Let $\mathcal{B}$ be a σ-algebra on X, let μ be a measure on $\mathcal{B}$. Assume $\mathcal{F}$ is a sufficient μ-finite $\mathcal{B}$-partition of X.*

A. *One has the equality*

$$\mathfrak{M}_\mu = \bigvee_{F\in\mathcal{F}}\big(\mathfrak{M}_\mu\big|_F\big).$$

B. *For a subset $A \subset X$, the following are equivalent*

(i) *$A \in \mathfrak{M}_\mu$;*

(ii) *there exist a set $B \in \bigvee_{F\in\mathcal{F}}\big(\mathcal{B}\big|_F\big)$, and a locally μ^*-neglijeable set $N \subset X$, such that $A = B \smallsetminus N$.*

Proof. A. This is exactly property A from Proposition 5.6.

B. $(i) \Rightarrow (ii)$. Assume $A \in \mathfrak{M}_\mu$, i.e. A is μ^* measurable. For every $F \in \mathcal{F}$, the set $A\cap F$ is μ^*-measurable. Since $\mu^*(A\cap F) < \infty$, by Theorem 5.5, it follows that $A\cap F = B_F \smallsetminus N_F$, with $B_F \in \mathcal{B}$ and $\mu^*(N_F) = 0$. Replacing B_F with $B_F \cap F$, and N_F with $N_F \cap F$, we can assume that $B_F, N_F \subset F$. Form then the sets $B = \bigcup_{F\in\mathcal{F}} B_F$ and $N = \bigcup_{F\in\mathcal{F}} N_F$. On the one hand, we have $B \cap B = B_F \in \mathcal{B}$, $\forall\, F \in \mathcal{F}$, which means precisely that $B \in \bigvee_{F\in\mathcal{F}}\big(\mathcal{B}\big|_F\big)$. On the other hand, we also have $N \cap F = N_F$, so we get $\mu^*(N\cap F) = 0$, $\forall\, F \in \mathcal{F}$. By Proposition 5.6.B, it follows that N is locally μ^*-neglijeable. We clearly have $A = B \smallsetminus N$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

There is yet another nicer consequence of Proposition 5.6, for which we are going to use the following terminology.

Definition. Let $\mathcal{A}$ be a σ-algebra on X, and let μ be a measure on $\mathcal{A}$. A family $\mathcal{F}$ is called a *μ-finite decomposition for $\mathcal{A}$*, if

(i) $\mathcal{F}$ is a sufficient μ-finite $\mathcal{A}$-partition of X, and

(ii) one has the equality $\bigvee_{F\in\mathcal{F}}\big(\mathcal{A}\big|_F\big) = \mathcal{A}$.

(Given a collection $\mathcal{F} \subset \mathcal{A}$, one always has the inclusion $\bigvee_{F\in\mathcal{F}}\big(\mathcal{A}\big|_F\big) \subset \mathcal{A}$.)

A measure μ on $\mathcal{A}$ is said to be *decomposable*, if there exists at least one μ-finite decomposition for $\mathcal{A}$.

Remark 5.8. Decomposability is a generalization of σ-finiteness. This follows from Remark 5.6.B, combined with the fact that whenever $\mathcal{F} \subset \mathcal{A}$ is a *countable* sub-collection, one always has the equality $\bigvee_{F\in\mathcal{F}}\big(\mathcal{A}\big|_F\big) = \mathcal{A}$.

With this terminology, Corollary 5.3 states that *if $\mathcal{F}$ is a sufficient μ-finite $\mathcal{B}$-partition of X*, then $\mathcal{F}$ is a $\tilde{\mu}$-finite decomposition for $\mathfrak{M}_\mu$.

With the above terminology, Corollary 5.2 has the following generalization

Theorem 5.6. *Let μ be a decomposable measure on the σ-algebra $\mathcal{B}$.*

A. *For a subset $A \subset X$, the following are equivalent*

(i) *A is μ^*-measurable;*

(ii) *there exist* $B \in \mathcal{B}$, *and some locally* μ^**-neglijeable set* N, *such that* $A = B \smallsetminus N$.

B. *For a subset* $N \subset X$, *the following are equivalent*

(i) N *is locally* μ^**-neglijeable;*

(ii) *there exists a locally* μ*-null set* $D \in \mathcal{B}$ *with* $N \subset D$.

Proof. A. This is clear, by Corollary 5.3.

B. The implication $(ii) \Rightarrow (i)$ is trivial, because any locally μ-null set D is locally μ^*-neglijeable, and so is every subset of D.

To prove the implication $(i) \Rightarrow (ii)$ start with a locally μ^*-neglijeable set N, and we fix $\mathcal{F}$ a μ-finite decomposition of $\mathcal{B}$. We know that $\mu^*(N \cap F) = 0$, $\forall F \in \mathcal{F}$. In particular, using Remark 5.4.B, for each $F \in \mathcal{F}$, there exists some set $E_F \in \mathcal{B}$, with $N \cap F \subset E_F$, and $\mu(E_F) = 0$. Consider now the set $D = \bigcup_{F \in \mathcal{F}}(E_F \cap F)$. By construction, we have $D \cap F = E_F \cap F \in \mathcal{B}$, $\forall F \in \mathcal{F}$, which means that $D \in \bigvee_{F \in \mathcal{F}} \left(\mathcal{B}\big|_F\right)$. It is here where we use condition (ii) in the definition of μ-finite decompositions, to conclude that D belongs to $\mathcal{B}$. Of course, we have

$$\mu(D \cap F) = \mu(E_F \cap F) \leq \mu(E_F) = 0, \ \ \forall F \in \mathcal{F},$$

which by Proposition 5.6 means that D is locally μ^*-neglijeable. This means that

$$\mu(D \cap B) = \mu^*(D \cap B) = 0, \ \ \forall B \in \mathcal{B}_{\text{fin}},$$

which means that D is locally μ-null. Since $N \cap F \subset E_F \cap F \subset D$, $\forall F \in \mathcal{F}$, and $\mathcal{F}$ is a partition of X, we get $N \subset D$. $\square$