Lecture 21

4. The concept of measure

Definition. Let X be a non-empty set, and let $\mathcal{E}$ be an arbitrary collection of subsets of X. Assume $\varnothing \in \mathcal{E}$. A *measure on* $\mathcal{E}$ is a map $\mu : \mathcal{E} \to [0, 1]$ with the following properties

(0) $\mu(\varnothing) = 0$.

(ADD_σ) *Whenever* $(E_n)_{n=1}^{\infty} \subset \mathcal{E}$ *is a pair-wise disjoint sequence, with* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$, *it follows that we have the equality*

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \mu(E_n).$$

Property (ADD_σ) is called *σ-additivity*.

Convention. For a sequence $(\alpha_n)_{n=1}^{\infty} \subset [0, \infty]$ we define

$$\sum_{n=1}^{\infty} \alpha_n = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \alpha_n & \text{if } \alpha_n \in [0, \infty),\ \forall\, n \in \mathbb{N} \\ \infty & \text{if there exists } n \in \mathbb{N} \text{ with } \alpha_n = \infty. \end{cases}$$

(Of course, in the first case, it is still possible to have $\sum_{n=1}^{\infty} \alpha_n = \infty$.)

Remark 4.1. If μ is a measure on $\mathcal{E}$, then μ is *additive*, i.e.

(ADD) *Whenever* $(E_n)_{n=1}^{N} \subset \mathcal{E}$ *is a finite pair-wise disjoint system, such that* $E_1 \cup \cdots \cup E_N \in \mathcal{E}$, *it follows that we have the equality*

$$\mu\big(E_1 \cup \cdots \cup E_N\big) = \mu(E_1) + \cdots + \mu(E_N).$$

This follows from (ADD_σ) (0), after completing the sequence $E_1, \dots, E_N$ to an infinite sequence by taking $E_n = \varnothing$, $\forall\, n > N$.

Comment. The most natural setting for measures is the one when $\mathcal{E}$ is a σ-ring. In this case, the stipulation that $\bigcup_{n=1}^{\infty} E_n \in \mathcal{E}$, which appears in the definition, is superfluous.

The purpose of this section is to study measures on more rudimentary collections.

Examples 4.1. Let X be a non-empty set.

A. If we take $\mathcal{E} = \{\varnothing, X\}$ and we define $\mu(\varnothing) = 0$ and $\mu(X)$ to be any element in $[0, \infty]$, then μ is obviously a measure on $\{\varnothing, X\}$.

B. If we take $\mathcal{E} = \mathcal{P}(X)$ and we define

$$\mu(E) = \begin{cases} 0 & \text{if } E = \varnothing \\ \infty & \text{if } E \neq \varnothing \end{cases}$$

then μ is a measure on $\mathcal{P}(X)$.

C. If we take $\mathcal{E} = \mathcal{P}(X)$ and we define

$$\mu(E) = \begin{cases} \operatorname{card} E & \text{if } E \text{ is finite} \\ \infty & \text{if } E \text{ is infinite} \end{cases}$$

then μ is a measure on $\mathcal{P}(X)$. This is called the *counting measure*.

Exercise 1. Let X_1, X_2 be non-empty spaces, let $\mathcal{E}_k \subset \mathcal{P}(X_k)$ be arbitrary collections with $\varnothing \in \mathcal{E}_k$, $k = 1, 2$. Let μ_1 be a measure on $\mathcal{E}_1$ and μ_2 be a measure on $\mathcal{E}_2$. Consider the collections

$$f_*\mathcal{E}_1 = \{A \subset X_2 \, : \, f^{-1}(A) \in \mathcal{E}_1\} \subset \mathcal{P}(X_2);$$
$$f^*\mathcal{E}_2 = \{f^{-1}(A) \, : \, A \in \mathcal{E}_2\} \subset \mathcal{P}(X_1).$$

A. Prove that the map $f_*\mu_1 : f_*\mathcal{E}_1 \to [0, \infty]$, defined by

$$(f_*\mu_1)(A) = \mu_1\big(f^{-1}(A)\big), \ \forall A \in f_*\mathcal{E}_1,$$

is a measure on $f_*\mathcal{E}_1$.

B. If f is *surjective*, prove that the map $f^*\mu_2 : f^*\mathcal{E}_2 \to [0, \infty]$, defined by

$$(f^*\mu_2)\big(\big) = \mu_2\big(f(B)\big), \ \forall B \in f^*\mathcal{E}_2,$$

is a measure on $f^*\mathcal{E}_2$.

We now concentrate on the most rudimentary types of collections $\mathcal{E}$ on which measures can be somehow easily defined. Actually, what we have in mind is a set of easy conditions on a map $\mu : \mathcal{E} \to [0, \infty]$ which would guarrantee that μ is a measure.

Definition. Let X be a non-empty set. A collection $\mathcal{J} \subset \mathcal{P}(X)$ is called a *semiring*, if it satisfies the following properties:

- $\varnothing \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$, then $A \cap B \in \mathcal{J}$;
- if $A, B \in \mathcal{J}$ and $A \subset B$, then there exists an integer $n \geq 1$, and sets $D_0, D_1, \dots, D_n \in \mathcal{J}$, such that $A = D_0 \subset D_1 \subset \cdots \subset D_n = B$, and $D_k \smallsetminus D_{k-1} \in \mathcal{J}$, $\forall k \in \{1, \dots, n\}$.

Remark that every ring is a semiring.

Exercise 2. Prove that the semiring type is not consistent. Give an example of two semirings $\mathcal{J}_1, \mathcal{J}_2 \subset \mathcal{P}(X)$, such that $\mathcal{J}_1 \cap \mathcal{J}_2$ is not a semiring.

Hint: Use the set $X = \{1, 2, 3\}$.

Exercise 3. Let $X_1, \dots, X_n$ be non-empty sets, and let $\mathcal{J}_k \subset \mathcal{P}(X_k)$, $k = 1, \dots, n$, be semirings. Prove that

$$\mathcal{J} = \big\{A_1 \times \cdots \times A_n \, : \, A_1 \in \mathcal{J}_1, \dots, A_n \in \mathcal{J}_n\big\} \subset \mathcal{P}(X_1 \times \cdots \times X_n)$$

is a semiring.

Hint: First prove the case $n = 2$, and then use induction.

Example 4.2. Take $X = \mathbb{R}$. The collection

$$\mathcal{J} = \{\varnothing\} \cup \big\{[a, b) \, : \, a, b \in \mathbb{R}, \ a < b\big\} \subset \mathcal{P}(\mathbb{R})$$

is a semiring.

Indeed, the first two axioms are pretty clear. To prove the third axiom, we start with two intervals $A = [a, b)$ and $B = [c, d)$ with $A \subset B$. This means that $a \geq c$ and $b \leq d$. If $a = c$ or $b = d$, we set $D_0 = A$ and $D_1 = B$. If $a > c$ and $b < d$, we set $D_0 = A$, $D_1 = [a, d)$ and $D_2 = B$.

More generally, by Exercise 3, the collection of "half-open boxes"

$$\mathcal{J}_n = \{\varnothing\} \cup \Big\{ \prod_{j=1}^{n} [a_j, b_j) \,:\, a_1 < b_1, \dots, a_n < b_n \Big\} \subset \mathcal{P}(\mathbb{R}^n)$$

is a semiring.

Exercise 4. Let $\mathcal{J}_n \subset \mathcal{P}(\mathbb{R}^n)$ be the semiring defined above. Prove that the σ-ring $\mathbf{S}(\mathcal{J})$ generated by $\mathcal{J}_n$ coincides with $Bor(\mathbb{R}^n)$.

The ring generated by a semiring has a particularly nice description (compare to Proposition 2.1):

PROPOSITION 4.1. *Let $\mathcal{J}$ be a semiring on X. For a subset $A \subset X$, the following are equivalent:*

(i) *A belongs to $\mathbf{R}(\mathcal{J})$, the ring generated by $\mathcal{J}$;*
(ii) *There exists an integer $n \geq 1$, and a pair-wise disjoint system $(A_j)_{j=1}^n \subset \mathcal{J}$, such that $A = A_1 \cup \dots \cup A_n$.*

PROOF. Denote by $\mathcal{R}$ the collection of all subsets $A \subset X$ that satisfy condition (ii). It is obvious that

$$\mathcal{J} \subset \mathcal{R} \subset \mathbf{R}(\mathcal{J}),$$

so (see Section III.2) we only need to prove that $\mathcal{R}$ is a ring.

Let us first remark that we obviously have the property:

(I) if $A, B \in \mathcal{R}$, and $A \cap B = \varnothing$, then $A \cup B \in \mathcal{R}$.

Secondly, we remark that we have have the implication:

(II) $A, B \in \mathcal{J} \Rightarrow A \smallsetminus B \in \mathcal{R}$.

Indeed, since $A \cap B \in \mathcal{J}$, by the definition of a semiring, there exist $D_0, D_1, \dots, D_n \in \mathcal{J}$ with $A \cap B = D_0 \subset D_1 \subset \dots \subset D_n = A$, and $D_k \smallsetminus D_{k-1} \in \mathcal{J}$, $\forall\, k \in \{1, \dots, n\}$. Then the equality

$$A \smallsetminus = \bigcup_{k=1}^{n} (D_k \smallsetminus D_{k-1})$$

shows that $A \smallsetminus B$ indeed belongs to $\mathcal{R}$.

Thirdly, we prove the implication:

(III) $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$.

Write $A = A_1 \cup \dots \cup A_m$ and $B = B_1 \cup \dots \cup B_n$, with $(A_i)_{i=1}^m$, $(B_k)_{k=1}^n \subset \mathcal{J}$ pair-wise disjoint systems. If we define the sets $D_{ik} = A_j \cap B_k \in \mathcal{J}$, $(i,k) \in \{1, \dots, m\} \times \{1, \dots, n\}$ then it is obvious that

$$A \cap B = \bigcup_{i=1}^{m} \bigcup_{k=1}^{n} D_{ik},$$

and $(D_{ik})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \subset \mathcal{J}$ is a pair-wise disjoint system, therefore $A \cap B$ indeed belongs to $\mathcal{R}$.

Finally, we show the implication:

(IV) if $A, B \in \mathcal{R}$ and $A \supset B$, then $A \smallsetminus B \in \mathcal{R}$.

Write $A = A_1 \cup \dots \cup A_m$, with $(A_i)_{i=1}^m \subset \mathcal{J}$ a pair-wise disjoint system. Notice that

$$A \smallsetminus B = \bigcup_{i=1}^{m} (A_i \smallsetminus B),$$

with $(A_i \smallsetminus B)_{i=1}^m$ a pair-wise disjoint system, so by (I) it suffices to show that $A_i \smallsetminus B \in \mathcal{R}$, $\forall\, i \in \{1,\dots,m\}$. To prove this, we fix i and we write $B = B_1 \cup \cdots \cup B_n$, with $(B_k)_{k=1}^n \subset \mathcal{J}$ a pair-wise disjoint system. Then

$$A_i \smallsetminus B = (A_i \smallsetminus B_1) \cap \cdots \cap (A_i \smallsetminus B_n),$$

and the fact that $A_i \smallsetminus B$ belongs to $\mathcal{R}$ follows from (II) and (III).

Having proven (I)-(IV), it we now prove that $\mathcal{R}$ is a ring. By (III), we only need to prove the implication

$(*)$ $A, B \in \mathcal{R} \Rightarrow A \triangle B \in \mathcal{R}$.

On the one hand, using (IV), it follows that the sets $A \smallsetminus B = A \smallsetminus (A \cap B)$ and $B \smallsetminus A = B \smallsetminus (A \cap B)$ both belong to $\mathcal{R}$. Since $A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A)$, and $(A \smallsetminus B) \cap (B \smallsetminus A) = \varnothing$, by (I) is follows that $A \triangle B$ indeed belongs to $\mathcal{R}$. □

THEOREM 4.1 (Semiring-to-ring extension). *Let $\mathcal{J}$ be a semiring on X, and let $\mu : \mathcal{J} \to [0,\infty]$ be an additive map with $\mu(\varnothing) = 0$.*

(i) *There exists a unique additive map $\bar{\mu} : \mathbf{R}(\mathcal{J}) \to [0,\infty]$, such that $\bar{\mu}\big|_{\mathcal{J}} = \mu$.*

(ii) *If μ is σ-additive, then so is $\bar{\mu}$.*

PROOF. The key step is contained in the following

Claim: *If $(A_i)_{i=1}^m \subset \mathcal{J}$ and $(B_j)_{j=1}^n \subset \mathcal{J}$ are pair-wise disjoint systems, with*

$$A_1 \cup \cdots \cup A_m = B_1 \cup \cdots \cup B_n,$$

then $\mu(A_1) + \cdots + \mu(A_m) = \mu(B_1) + \cdots + \mu(B_n)$.

To prove this fact, we define the pair-wise disjoint system $(D_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ by $D_{ij} = A_i \cap B_j$, $\forall\,(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$. Since

$$\bigcup_{j=1}^n D_{ij} = A_i,\ \ \forall\, i \in \{1,\dots,m\},$$
$$\bigcup_{i=1}^m D_{ij} = B_j,\ \ \forall\, j \in \{1,\dots,n\},$$

using additivity, we have the equalities

$$\sum_{j=1}^n \mu(D_{ij}) = \mu(A_i),\ \ \forall\, i \in \{1,\dots,m\},$$
$$\sum_{i=1}^m \mu(D_{ij}) = \mu(B_j),\ \ \forall\, j \in \{1,\dots,n\},$$

and then we get

$$\sum_{i=1}^m \mu(A_i) = \sum_{i=1}^m \Big[\sum_{j=1}^m \mu(D_{ij})\Big] = \sum_{j=1}^n \Big[\sum_{i=1}^n \mu(D_{ij})\Big] = \sum_{j=1}^n \mu(B_j).$$

To prove (i), for any set $A \in \mathbf{R}(\mathcal{J})$ we choose (use Proposition 4.1) a finite pair-wise disjoint system $(A_i)_{i=1}^n \subset \mathcal{J}$, with $A = A_1 \cup \cdots \cup A_n$, and we define

$$\bar{\mu}(A) = \mu(A_1) + \cdots + \mu(A_n). \tag{1}$$

By the above Claim, *the number $\bar{\mu}(A)$ is independent of the particular choice of the pair-wise disjoint system $(A_i)_{i=1}^n$*. Also, it is clear that $\bar{\mu}\big|_{\mathcal{J}} = \mu$, and $\bar{\mu}$ is additive.

The uniqueness is also clear, because the equality $\bar{\mu}\big|_{\mathcal{J}} = \mu$ and additivity of $\bar{\mu}$ force (1)

(ii). Assume now that μ is σ-additive, and let us prove that $\bar{\mu}$ is again σ-additive. Start with a pair-wise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathbf{R}(\mathcal{J})$, with $\bigcup_{n=1}^\infty A_n \in \mathbf{R}(\mathcal{J})$, and let us prove the equality

$$\bar{\mu}\big(\bigcup_{n=1}^\infty A_n\big) = \sum_{n=1}^\infty \bar{\mu}(A_n). \tag{2}$$

Since $\bigcup_{n=1}^\infty A_n \in \mathbf{R}$, there exists a finite pair-wise disjoint system $(B_i)_{i=1}^p \subset \mathcal{J}$, such that $\bigcup_{n=1}^\infty A_n = B_1 \cup \cdots \cup B_p$. With this choice we have

$$\bar{\mu}\big(\bigcup_{n=1}^\infty A_n\big) = \sum_{i=1}^p \mu(B_i). \tag{3}$$

For each $i \in \{1, \dots, p\}$, we have $B_i = \bigcup_{n=1}^\infty (B_i \cap A_n)$. Fix for the moment a pair $(n, i) \in \mathbb{N} \times \{1, \dots, p\}$. Since $B_i \cap A_n \in \mathbf{R}(\mathcal{J})$, it follows that there exist an integer $N_{ni} \geq 1$ and a finite pair-wise disjoint system $(C_k^{ni})_{k=1}^{N_{ni}} \subset \mathcal{J}$, such that $B_i \cap A_n = \bigcup_{k=1}^{N_{ni}} C_k^{ni}$.

Since, for each $i \in \{1, \dots, p\}$, the countable system $(C_k^{ni})_{\substack{n\in\mathbb{N}\\ 1\leq k\leq N_{ni}}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{n=1}^\infty \bigcup_{k=1}^{N_{ni}} C_k^{ni} = \bigcup_{n=1}^\infty (B_i \cap A_n) = B_i \in \mathcal{J},$$

by the σ-additivity of μ, we have

$$\mu(B_i) = \sum_{n=1}^\infty \sum_{k=1}^{N^{ni}} \mu(C_k^{ni}), \ \ \forall\, i \in \{1, \dots, p\}. \tag{4}$$

Since, for each $n \in \mathbb{N}$, the finite system $(C_k^{ni})_{\substack{1\leq i\leq p\\ 1\leq k\leq N_{ni}}} \subset \mathcal{J}$ is pair-wise disjoint, and we have the equality

$$\bigcup_{i=1}^p \bigcup_{k=1}^{N_{ni}} C_k^{ni} = \bigcup_{i=1}^\infty (B_i \cap A_n) = A_n \in \mathcal{J},$$

by the definition of $\bar{\mu}$, we have

$$\bar{\mu}(A_n) = \sum_{i=1}^p \sum_{k=1}^{N^{ni}} \mu(C_k^{ni}), \ \ \forall\, i \in \{1, \dots, p\}.$$

Combining this with (4) yields

$$\sum_{n=1}^\infty \bar{\mu}(A_n) = \sum_{n=1}^\infty \sum_{i=1}^p \sum_{k=1}^{N_{ni}} \mu(C_k^{ni}) = \sum_{i=1}^p \mu(B_i),$$

and the equality (2) follows from (3). □

Definition. Let X be a non-empty set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets. We say that a map $\mu : \mathcal{E} \to [0, \infty]$ is *sub-additive*, if

(ADD⁻) *whenever $A \in \mathcal{E}$, and $(A_n)_{k=1}^n$ is a finite sequence in $\mathcal{E}$ with $A \subset \bigcup_{k=1}^n A_k$, it follows that $\mu(A) \leq \sum_{k=1}^n \mu(A_k)$.*

Note that we do not require the A_k's to be pair-wise disjoint. With this terminology, Theorem 4.1 has the following.

COROLLARY 4.1. *Let X be a non-empty set X, and let $\mathcal{J} \subset \mathcal{P}(X)$ be a semiring. Then any additive map $\mu : \mathcal{J} \to [0,\infty]$ is sub-additive.*

PROOF. Let $\bar{\mu} : \mathbf{R}(\mathcal{J}) \to [0,\infty]$ be the additive extension of μ to the ring generated by $\mathcal{J}$. It suffices to prove that $\bar{\mu}$ is sub-additive. Start with sets $A, A_1, \dots, A_n \in \mathbf{R}(\mathcal{J})$ such that $A \subset A_1 \cup \dots A_n$. Define the sets $B_1 = A_1$, and

$$B_k = A_k \smallsetminus (A_1 \cup \dots \cup A_{k-1}), \;\; forall\, k \in \{1, \dots, n\},\, k \geq 2.$$

Since we work in a ring, the sets B_k, $B_k \cap A$, $B_k \smallsetminus A$, and $A_n \smallsetminus B_n$, $n \in \mathbb{N}$, all belong to $\mathbf{R}(\mathcal{J})$. Moreover, the sequence $(B_k)_{k=1}^n$ is pair-wise disjoint and it satisfies

- $B_k \subset A_k$, $\forall\, k \in \{1, \dots, n\}$,
- $\bigcup_{k=1}^n B_k = \bigcup_{k=1}^n A_k \supset A$,

so by the additivity of $\bar{\mu}$, we get

$$\sum_{k=1}^n \bar{\mu}(A_k) = \sum_{k=1}^n \bar{\mu}\big((A_k \smallsetminus B_k) \cup B_k\big) = \sum_{k=1}^n \big[\bar{\mu}(A_k \smallsetminus B_k) + \bar{\mu}(B_k)\big] \geq$$
$$\geq \sum_{k=1}^n \bar{\mu}(B_k) = \sum_{k=1}^n \bar{\mu}\big((B_k \smallsetminus A) \cup (B_k \cap A)\big) = \sum_{k=1}^n \big[\bar{\mu}(B_k \smallsetminus A) + \bar{\mu}(B_k \cap A)\big] \geq$$
$$\geq \sum_{k=1}^n \bar{\mu}(B_k \cap A) = \bar{\mu}\big(\bigcup_{k=1}^n [B_k \cap A]\big) = \bar{\mu}(A). \qquad \square$$

*Exercise 5**. Let X_1, X_2 be non-empty sets, let $\mathcal{J}_k \subset \mathcal{P}(X_k)$, $k = 1, 2$, be semirings, and let $\mu_k : \mathcal{J}_k \to [0,\infty]$ be additive maps. Consider the semiring (see Exercise 3)

$$\mathcal{J} = \big\{A_1 \times A_2 \,:\, A_1 \in \mathcal{J}_1,\, A_2 \in \mathcal{J}_2\big\} \subset \mathcal{P}(X_1 \times X_2).$$

Then the map $\mu : \mathcal{J} \to [0,\infty]$ defined by

$$\mu(A_1 \times A_2) = \mu_1(A)_1 \cdot \mu_2(A_1)$$

is additive. Here we use the convention $0 \cdot \infty = \infty \cdot 0 = 0$.

HINTS: One wants to show that, whenever $A_1 \times A_2 \in \mathcal{J}$ is written as aunion

$$A_1 \times A_2 = \bigcup_{k=1}^n (A_1^k \times A_2^k),$$

with $(A_1^k \times A_2^k)_{k=1}^n \subset \mathcal{J}$ pair-wise disjoint, it follows that

$$\mu_1(A_1) \cdot \mu_2(A_2) = \sum_{k=1}^n \mu_1(A_1^k) \cdot \mu_2(A_2^k).$$

Analyze first the case of "strips," that is, when $A_1^1 = \dots = A_1^n = A_1$ or $A_1^2 = \dots = A_2^n = A_2$. In the general case, use induction, by picking some k such that $A_1^k \subsetneq A_1$ and splitting $A_1 \times A_2$ into "strips" of the form $B_\ell \times A_2$, where $B_1, \dots, B_m \in \mathcal{J}_1$ are pairwise disjoint, with $B_1 = A_1^k$ and $B_1 \cup \dots \cup B_m = A_1$.

COMMENT. In connection with the above exercise, one can as the following

> *Question*: *With the notations above, is it true that, if both μ_1 and μ_2 are measures, then μ is also a measure?*

As we shall see a bit later in the course, that the answer is is "yes."

Definition. Let X be a non-empty set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of sets. We say that a map $\mu : \mathcal{E} \to [0, \infty]$ is *σ-sub-additive*, if

(ADD$_\sigma^-$) *whenever $A \in \mathcal{E}$, and $(A_n)_{n=1}^\infty$ is a sequence in $\mathcal{E}$ with $A \subset \bigcup_{n=1}^\infty A_n$, it follows that $\mu(A) \leq \sum_{n=1}^\infty \mu(A_n)$.*

Note that we do not require the A_n's to be pair-wise disjoint.

Proposition 4.2 (characterization of semiring measures). *Let X be a non-empty set, let $\mathcal{J} \subset \mathcal{P}(X)$ be a semiring, and let $\mu : \mathcal{J} \to [0, \infty]$ be a map with $\mu(\varnothing) = 0$. The following are equivalent:*

(i) *μ is a measure on $\mathcal{J}$;*
(ii) *μ is additive, and σ-sub-additive.*

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume μ is a measure on $\mathcal{J}$. It is clear that μ is additive, so we only need to prove σ-sub-additivity. Use Theorem 4.1 to find a measure $\bar{\mu}$ on the ring $\mathbf{R}(\mathcal{J})$ generated by $\mathcal{J}$, such that

$$\bar{\mu}(A) = \mu(A), \ \ \forall\, A \in \mathcal{J}.$$

Then it suffices to show that $\bar{\mu}$ is σ-sub-additive. Start with a set $A \in \mathbf{R}(\mathcal{J})$, and a sequence $(A_n)_{n=1}^\infty \subset \mathbf{R}(\mathcal{J})$, such that $A \subset \bigcup_{n=1}^\infty A_n$. Define the sets $B_1 = A_1$, and

$$B_n = A_n \smallsetminus (A_1 \cup \dots \cup A_{n-1}), \ \ for all\, n \geq 2.$$

Since we work in a ring, the sets B_n, $B_n \cap A$, $B_n \smallsetminus A$, and $A_n \smallsetminus B_n$, $n \in \mathbb{N}$, all belong to $\mathbf{R}(\mathcal{J})$. Moreover, the sequence $(B_n)_{n=1}^\infty$ is pair-wise disjoint and it satisfies

- $B_n \subset A_n$, $\forall\, n \in \mathbb{N}$,
- $\bigcup_{n=1}^\infty B_n = \bigcup_{n=1}^\infty A_n \supset A$,

so by σ-additivity of $\bar{\mu}$, we get

$$\sum_{n=1}^\infty \bar{\mu}(A_n) = \sum_{n=1}^\infty \bar{\mu}\big((A_n \smallsetminus B_n) \cup B_n\big) = \sum_{n=1}^\infty \big[\bar{\mu}(A_n \smallsetminus B_n) + \bar{\mu}(B_n)\big] \geq$$
$$\geq \sum_{n=1}^\infty \bar{\mu}(B_n) = \sum_{n=1}^\infty \bar{\mu}\big((B_n \smallsetminus A) \cup (B_n \cap A)\big) = \sum_{n=1}^\infty \big[\bar{\mu}(B_n \smallsetminus A) + \bar{\mu}(B_n \cap A)\big] \geq$$
$$\geq \sum_{n=1}^\infty \bar{\mu}(B_n \cap A) = \bar{\mu}\big(\bigcup_{n=1}^\infty [B_n \cap A]\big) = \bar{\mu}(A).$$

(ii) $\Rightarrow$ *(i)*. Assume $\mu : \mathcal{J} \to [0, \infty]$ is additive and σ-sub-additive, and let us show that μ is σ-additive. We again use Theorem 4.1, to find an additive map $\bar{\mu} : \mathbf{R}(\mathcal{J}) \to [0, \infty]$, such that $\bar{\mu}\big|_{\mathcal{J}} = \mu$. Start with a pair-wise disjoint sequence $(A_n)_{n=1}^\infty \subset \mathcal{J}$, such that the union $A = \bigcup_{n=1}^\infty A_n$ belongs to $\mathcal{J}$. On the one hand, by σ-sub-additivity, we have the inequality

$$\mu(A) \leq \sum_{n=1}^\infty \mu(A_n). \tag{5}$$

On the other hand, for any integer $N \geq 1$, we have

$$\mu(A) = \bar{\mu}(A) = \bar{\mu}\Big(\big[\bigcup_{n=1}^{N} A_n \big] \cup \big(A \smallsetminus \big[\bigcup_{n=1}^{N} A_n \big] \big) \Big) \geq$$

$$\geq \bar{\mu}\big(\bigcup_{n=1}^{N} A_n \big) = \sum_{n=1}^{N} \bar{\mu}(A_n) = \sum_{n=1}^{N} \mu(A_n),$$

which then gives

$$\mu(A) \geq \sup_{N\in\mathbb{N}} \sum_{n=1}^{N} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n),$$

so using (5) we immediately get $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$. □

The following technical result will be often employed in subsequent sections.

LEMMA 4.1 (Continuity). *Let $\mathcal{J}$ be a semiring, and let μ be a measure on $\mathcal{J}$.*

(i) *If $(A_n)_{n=1}^{\infty} \subset \mathcal{J}$ is a sequence of sets, with $A_1 \subset A_2 \subset \dots$, and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{J}$, then*

$$\mu\big(\bigcup_{n=1}^{\infty} A_n \big) = \lim_{n\to\infty} \mu(A_n).$$

(ii) *If $(B_n)_{n=1}^{\infty} \subset \mathcal{J}$ is a sequence of sets, with $B_1 \supset B_2 \supset \dots$, and $\bigcap_{n=1}^{\infty} B_n \in \mathcal{J}$, and $\mu(B_1) < \infty$, then*

$$\mu\big(\bigcap_{n=1}^{\infty} B_n \big) = \lim_{n\to\infty} \mu(B_n).$$

PROOF. Using Theorem 4.1, we can assume that $\mathcal{J}$ is already a ring. (Otherwise we replace $\mathcal{J}$ by $\mathbf{R}(\mathcal{J})$, and μ by its extension $\bar{\mu}$.)

(i). Consider the sets $D_1 = A_1$, and $D_k = A_n \smallsetminus A_{k-1}$, $\forall\, k \geq 2$. It is clear that $(D_k)_{k=1}^{\infty}$ is a pairwise disjoint sequence in $\mathcal{J}$, and we have the equality

$$\bigcup_{k=1}^{n} D_k = A_n, \ \ \forall\, n \geq 1. \tag{6}$$

This gives of course the equality

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{n=1}^{\infty} A_n \in \mathcal{J}.$$

Using this equality, combined with the (σ-)additivity of μ, and with (6), we get

$$\mu\big(\bigcup_{n=1}^{\infty} A_n \big) = \sum_{k=1}^{\infty} \mu(D_k) = \lim_{n\to\infty} \big[\sum_{k=1}^{n} \mu(D_k) \big] = \lim_{n\to\infty} \mu\big(\bigcup_{k=1}^{n} D_k \big) = \lim_{n\to\infty} \mu(A_n).$$

(ii). Consider the sets $B = \bigcap_{n=1}^{\infty} B_n$, and $A_n = B_1 \smallsetminus B_n$, $\forall\, n \geq 1$. It is clear that $(A_n)_{n=1}^{\infty} \subset \mathcal{J}$, and we have $A_1 \subset A_2 \subset \dots$. Moreover, we have $\bigcup_{n=1}^{\infty} A_n = B_1 \smallsetminus B$, so by part (i), we get

$$\mu(B_1 \smallsetminus B) = \lim_{n\to\infty} \mu(B_1 \smallsetminus B_n). \tag{7}$$

Using the fact that $\mu(B_1) < \infty$, it follows that

$$\mu(B) \leq \mu(B_n) \leq \mu(B_1) < \infty, \ \ \forall\, n \geq 1.$$

This gives then the equalities

$$\mu(B_1 \smallsetminus B) = \mu(B_1) - \mu(B) \text{ and } \mu(B_1 \smallsetminus B_n) = \mu(B_1) - \mu(B_n), \ \ \forall\, n \geq 1,$$

so the equality (7) immediately gives $\mu(B) = \lim_{n\to\infty} \mu(B_n)$. □

The above result has a (minor) generalization, which we record for future use. To formulate it we introduce the following.

NOTATION. Let $\mathcal{R}$ be a ring, and let μ be a measure on $\mathcal{R}$. For two sets $A, B \in \mathcal{R}$, we write $A \underset{\mu}{\subset} B$, if $\mu(A \smallsetminus B) = 0$.

Using this notation, we have the following generalization of Lemma 4.1.

PROPOSITION 4.3. *Let $\mathcal{R}$ be a ring, and let μ be a measure on $\mathcal{R}$.*

(i) *If $(A_n)_{n=1}^\infty \subset \mathcal{R}$ is a sequence of sets, with $A_1 \underset{\mu}{\subset} A_2 \underset{\mu}{\subset} \dots$, and $\bigcup_{n=1}^\infty A_n \in \mathcal{R}$, then*

$$\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \lim_{n\to\infty} \mu(A_n).$$

(ii) *If $(B_n)_{n=1}^\infty \subset \mathcal{R}$ is a sequence of sets, with $B_1 \underset{\mu}{\supset} B_2 \underset{\mu}{\supset} \dots$, and $\bigcap_{n=1}^\infty B_n \in \mathcal{J}$, and $\mu(B_1) < \infty$, then*

$$\mu\Big(\bigcap_{n=1}^\infty B_n\Big) = \lim_{n\to\infty} \mu(B_n).$$

PROOF. (i). Define the sequence of sets $(E_n)_{n=1}^\infty \subset \mathcal{R}$, by $E_n = \bigcup_{k=1}^n A_k$, $\forall\, n \geq 1$. Notice that, $A_1 = E_1$, and for each $n \geq 2$, we have $A_n \subset E_n$, as well as the equality

$$E_n \smallsetminus A_n = \bigcup_{k=1}^{n-1} [A_n \smallsetminus A_k].$$

Using sub-additivity, it follows that

$$\mu(E_n \smallsetminus A_n) \leq \sum_{k=1}^{n-1} \mu(A_n \smallsetminus A_k),$$

which forces $\mu(E_n \smallsetminus A_n) = 0$. This gives

$$\mu(E_n) = \mu(A_n) + \mu(E_n \smallsetminus A_n) = \mu(A_n), \ \ \forall\, n \geq 1. \tag{8}$$

Since $\bigcup_{n=1}^\infty E_n = \bigcup_{n=1}^\infty A_n$, and we have the inclusions $E_1 \subset E_2 \subset \dots$, by Lemma 4.1, combined with (8), we get

$$\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \mu\Big(\bigcup_{n=1}^\infty E_n\Big) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(A_n).$$

Part (ii) is proven exactly as part (ii) from Lemma 4.1. □

Exercise 6. Let μ be a measure on a ring $\mathcal{R}$. Prove that, for $A, B \in \mathcal{R}$, one has the implication

$$A \underset{\mu}{\subset} B \Rightarrow \mu(A) \leq \mu(B).$$

EXAMPLE 4.3. Fix some integer $n \geq 1$. Consider the semiring of "half-open boxes" in $\mathbb{R}^n$

$$\mathcal{J}_n = \{\varnothing\} \cup \Big\{ \prod_{j=1}^{n} [a_j, b_j) \, : \, a_1 < b_1, \dots, a_n < b_n \Big\} \subset \mathcal{P}(\mathbb{R}^n).$$

For a non-empty box $A = [a_1, b_1) \times \cdots \times [a_n, b_n) \in \mathcal{J}_n$, we define

$$\text{vol}_n(A) = \prod_{k=1}^{n} (b_k - a_k).$$

We also define $\text{vol}_n(\varnothing) = 0$.

THEOREM 4.2. *With the above notations, the map* $\text{vol}_n : \mathcal{J} \to [0, \infty]$ *is a measure on* $\mathcal{J}_n$.

PROOF. First we prove additivity. Using Exercise ?? (and induction on n) it suffices to analyze only the case $n = 1$, i.e. the case of half-open intervals in $\mathbb{R}$. We need to show the implication

$$\left. \begin{array}{c} [a, b) = \bigcup_{k=1}^{p} [a_p, b_p) \\ \\ \big\{[a_k, b_k)\big\}_{k=1}^{p} \text{ pair-wise disjoint} \end{array} \right\} \Longrightarrow b - a = \sum_{k=1}^{p} (b_k - a_k). \tag{9}$$

We can prove this using induction on p. The case $p = 1$ is trivial. Assuming that the above fact holds for $p = N$, let us prove it for $p = N + 1$. Pick $k_1 \in \{1, \dots, N + 1\}$ such that $a_{k_1} = a$. Then we clearly have

$$\bigcup_{\substack{1 \leq k \leq N+1 \\ k \neq k_1}} [a_k, b_k) = [b_{k_1}, b),$$

so by the inductive hypothesis we get

$$b - b_{k_1} = \sum_{\substack{1 \leq k \leq N+1 \\ k \neq k_1}} (b_k - a_k),$$

so we get

$$\sum_{k=1}^{N+1} (b_k - a_k) = (b_{k_1} - a_{k_1}) + (b - b_{k_1}) = b - a_{k_1} = b - a,$$

and we are done.

We now prove that vol_n is σ-sub-additive. Suppose we have $A \in \mathcal{J}_n$ and a sequence $(A_k)_{k=1}^{\infty} \subset \mathcal{J}_n$, such that $A \subset \bigcup_{k=1}^{\infty} A_k$, and let us prove the inequality

$$\text{vol}_n(A) \leq \sum_{k=1}^{\infty} \text{vol}_n(A_k). \tag{10}$$

It will be helpfull to introduce the following notations. For every half-open box

$$B = [x_1, y_1) \times \cdots \times [x_n, y_n),$$

and every $\delta > 0$, we define the boxes boxes

$$B^{\delta} = [x_1 - \delta, y_1) \times \cdots \times [x_n - \delta, y_n) \text{ and } B_{\delta} = [x_1, y_1 - \delta) \times \cdots \times [x_n, y_n - \delta).$$

It is clear that, for any box $B \in \mathcal{J}_n$ we have

$$\overline{B_\delta} \subset B \subset \operatorname{Int}(B^\delta), \tag{11}$$

$$\operatorname{vol}_n(B) = \lim_{\delta \to 0^+} \operatorname{vol}_n(B^\delta) = \lim_{\delta \to 0^+} \operatorname{vol}_n(B_\delta). \tag{12}$$

To prove (10), we fix some $\varepsilon > 0$, and we choose positive numbers δ and $(\delta_k)_{k=1}^\infty$, such that

$$\operatorname{vol}_n(A_\delta) > \operatorname{vol}_n(A) - \varepsilon, \text{ and } \operatorname{vol}_n\big((A_k)^{\delta_n}\big) < \frac{\varepsilon}{2^k} + \operatorname{vol}_n(A_k), \ \forall\, k \in \mathbb{N}. \tag{13}$$

Notice now that, using (11), we have the inclusions

$$\overline{A_\delta} \subset A \subset \bigcup_{k=1}^\infty A_k \subset \operatorname{Int}\big((A_k)^{\delta_n}\big),$$

and using the compactness of $\overline{A_\delta}$, there exists some $N \geq 1$, such that

$$\overline{A_\delta} \subset \bigcup_{k=1}^N \operatorname{Int}\big((A_k)^{\delta_n}\big).$$

This immediately gives the inclusion

$$A_\delta \subset \bigcup_{k=1}^N (A_k)^{\delta_n}.$$

Using sub-additivity (see Corollary 4.1) we now get

$$\operatorname{vol}_n(A_\delta) \leq \sum_{k=1}^N \operatorname{vol}_n\big((A_k)^{\delta_n}\big),$$

and using (13) we have

$$\operatorname{vol}_n(A) - \varepsilon \leq \sum_{k=1}^N \big[\frac{\varepsilon}{2^k} + \operatorname{vol}_n(A_k)\big] \leq \varepsilon + \sum_{k=1}^N \operatorname{vol}_n(A_k) \leq \varepsilon + \sum_{k=1}^\infty \operatorname{vol}_n(A_k).$$

This gives

$$\operatorname{vol}_n(A) - 2\varepsilon \leq \sum_{k=1}^\infty \operatorname{vol}_n(A_k).$$

But since this inequality holds for all $\varepsilon > 0$, the inequality (10) immediately follows. □