LECTURE 20

3. Measurable spaces and measurable maps

In this section we discuss a certain type of maps related to σ-algebras.

DEFINITIONS. A *measurable space* is a pair $(X, \mathcal{A})$ consisting of a (non-empty) set X and a σ-algebra $\mathcal{A}$ on X.

Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a *measurable map* $T : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is simply a map $T : X \to Y$, with the property

$$T^{-1}(B) \in \mathcal{A}, \ \forall B \in \mathcal{B}. \tag{1}$$

REMARK 3.1. In terms of the constructions outlined in Section 2, measurability for maps can be characterized as follows. *Given measurable spaces* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$*, and a map* $T : X \to Y$*, the following are equivalent:*

- (i) $T : (X, \mathcal{A}) \to (Y, \mathcal{B})$ *is measurable;*
- (ii) $T^*\mathcal{B} \subset \mathcal{A}$*;*
- (iii) $T_*\mathcal{A} \supset \mathcal{B}$.

Recall

$$T^*\mathcal{B} = \{T^{-1}(B) \, : \, B \in \mathcal{B}\};$$
$$T_*\mathcal{A} = \{B \subset Y \, : \, T^{-1}(B) \in \mathcal{A}\}.$$

With these equalities, everything is immediate.

The following summarizes some useful properties of measurable maps.

PROPOSITION 3.1. *Let* $(X, \mathcal{A})$ *be a measurable space.*

- (i) *If* $\mathcal{A}'$ *is any* σ*-algebra, with* $\mathcal{A}' \subset \mathcal{A}$*, then the identity map* $\mathrm{Id}_X : (X, \mathcal{A}) \to (X, \mathcal{A}')$ *is measurable.*
- (ii) *For any subset* $M \subset X$*, the inclusion map* $\iota : (M, \mathcal{A}\big|_M) \hookrightarrow (X, \mathcal{A})$ *is measurable.*
- (iii) *If* $(Y, \mathcal{B})$ *and* $(Z, \mathcal{C})$ *are measurable spaces, and if* $(X, \mathcal{A}) \xrightarrow{T} (Y, \mathcal{B}) \xrightarrow{S} (Z, \mathcal{C})$ *are measurable maps, then the composition* $S \circ T : (X, \mathcal{A}) \to (Z, \mathcal{C})$ *is again a measurable map.*

PROOF. (i). This is trivial, since $(\mathrm{Id}_X)^*\mathcal{A}' = \mathcal{A}' \subset \mathcal{A}$.

(ii). This is again trivial, since $\iota^*\mathcal{A} = \mathcal{A}\big|_M$.

(iii). Start with some set $C \in \mathcal{C}$, and let us prove that $(S \circ T)^{-1}(C) \in \mathcal{A}$. We know that $(S \circ T)^{-1} = T^{-1}\big(S^{-1}(C)\big)$. Since S is measurable, we have $S^{-1}(C) \in \mathcal{B}$, and since T is measurable, we have $T^{-1}\big(S^{-1}(C)\big) \in \mathcal{A}$. □

Often, one would like to check the measurability condition (1) on a small collection of B's. Such a criterion is the following.

Lemma 3.1. *Let $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ be masurable spaces. Assume $\mathcal{B}=\mathbf{\Sigma}(\mathcal{E})$, for some collection of sets $\mathcal{E}\subset\mathcal{P}(Y)$. For a map $T:X\to Y$, the following are equivalent:*

(i) $T:(X,\mathcal{A})\to(Y,\mathcal{B})$ *is measurable;*
(ii) $T^{-1}(E)\in\mathcal{A}$, $\forall\, E\in\mathcal{E}$.

Proof. The implication $(i)\Rightarrow(ii)$ is trivial.

To prove the implication $(ii)\Rightarrow(i)$, assume (ii) holds. We first observe that condition (ii) reads $f^*\mathcal{E}\subset\mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra, we get the inclusion

$$\mathbf{\Sigma}(f^*\mathcal{E})\subset\mathcal{A}.$$

Using the Generating Theorem 2.2, we have

$$f^*\mathcal{B}=f^*\mathbf{\Sigma}(\mathcal{E})=\mathbf{\Sigma}(f^*\mathcal{E})\subset\mathcal{A},$$

and, by the preceding remark, we are done. □

Corollary 3.1. *Let $(X,\mathcal{A})$ be a measurable space, let Y be a topological Hausdorff space which is second countable, and let $\mathcal{S}$ be a sub-base for the topology of Y. For a map $T:X\to Y$, the following are equivalent:*

(i) $T:(X,\mathcal{A})\to\big(Y,Bor(Y)\big)$ *is a measurable map;*
(ii) $T^{-1}(S)\in\mathcal{A}$, $\forall\, S\in\mathcal{S}$.

Proof. Immediate from the above Lemma, and Proposition 2.2, which states that $Bor(Y)=\mathbf{\Sigma}(\mathcal{S})$. □

We know (see Section 19) that the type $\mathbf{\Sigma}$ is consistent and natural. In particular, measurability behaves nicely with respect to products and disjoint unions. More explicitly one has the following.

Proposition 3.2. *Let $(X_i,\mathcal{A}_i)_{i\in I}$ be a collection of measurable spaces. Consider the sets $X=\prod_{i\in I}X_i$ and $Y=\bigsqcup_{i\in I}X_i$, and the σ-algebras*

$$\mathcal{A}=\mathbf{\Sigma}\text{-}\bigtimes_{i\in I}\mathcal{A}_i \text{ and } \mathcal{B}=\bigvee_{i\in I}\mathcal{A}_i.$$

Let $(Z,\mathcal{G})$ be a measurable space.

(i) *If we denote by $\pi_i:X\to X_i$, $i\in I$, the projection maps, then a map $f:(Z,\mathcal{G})\to(X,\mathcal{A})$ is measurable, if and only if, all the maps $\pi_i\circ f:(Z,\mathcal{G})\to(X_i,\mathcal{A}_i)$, $i\in I$, are measurable.*
(ii) *If we denote by $\epsilon_i:X_i\to Y$, $i\in I$, the inclusion maps, then a map $g:(Y,\mathcal{B})\to(Z,\mathcal{G})$ is measurable, if and only if, all the maps $g\circ\epsilon_i\circ f:(X_i,\mathcal{A}_i)\to(Z,\mathcal{G})$, $i\in I$, are measurable.*

Proof. (i). By the definition of the product σ-algebra, we know that

$$\mathcal{A}=\mathbf{\Sigma}\big(\bigcup_{i\in I}\pi_i^*\mathcal{A}_i\big). \tag{2}$$

If we fix some index $i\in I$, then the obvious inclusion $\pi_i^*\mathcal{A}_i\subset\mathcal{A}$ immediately shows that $\pi_i:(X,\mathcal{A})\to(X_i,\mathcal{A}_i)$ is measurable. Therefore, if $f:(Z,\mathcal{G})\to(X,\mathcal{A})$ is measurable, then by Proposition 3.1 it follows that all compositions $\pi_i\circ f:(Z,\mathcal{G})\to(X_i,\mathcal{A}_i)$, $i\in I$, are measurable.

Conversely, assume all the compositions $\pi_i \circ f$ are measurable, and let us show that $f : (Z, \mathcal{G}) \to (X, \mathcal{A})$ is measurable. By Lemma 3.1 and (2), all we need to prove is the fact that

$$f^*\big(\bigcup_{i\in I} \pi_i^* \mathcal{A}_i\big) \subset \mathcal{G},$$

which is equivalent to

$$f^*\big(\pi_i^* \mathcal{A}_i\big) \subset \mathcal{G}, \ \ \forall\, i \in I.$$

But this is obvious, because $f^*\big(\pi_i^* \mathcal{A}_i\big) = (\pi_i \circ f)^* \mathcal{A}_i$, and $\pi_i \circ f$ is measurable, for all $i \in I$.

(ii). By the definition of the σ-algebra sum, we know that

$$\mathcal{B} = \bigcap_{i\in I} \epsilon_{i*} \mathcal{A}_i. \tag{3}$$

If we fix some index $i \in I$, then the obvious inclusion $\epsilon_{i*}\mathcal{A}_i \supset \mathcal{B}$ immediately shows that $\epsilon_i : (X_i, \mathcal{A}_i) \to (Y, \mathcal{B})$ is measurable. Therefore, if $g : (Y, \mathcal{B}) \to (Z, \mathcal{G})$ is measurable, then by Proposition 3.1 it follows that all compositions $g \circ \epsilon_i : (X_i, \mathcal{A}_i) \to (Z, \mathcal{G})$, $i \in I$, are measurable.

Conversely, assume all the compositions $g \circ \epsilon_i$ are measurable, and let us show that $g : (Y, \mathcal{B}) \to (Z, \mathcal{G})$ is measurable. This is equivalent to the inclusion $g_*\mathcal{B} \supset \mathcal{G}$. By (3) we immediately have

$$g_*\mathcal{B} = g_*\big(\bigcap_{i\in I} \epsilon_{i*} \mathcal{A}_i\big) = \bigcap_{i\in I} g_*\big(\epsilon_{i*} \mathcal{A}_i\big). \tag{4}$$

We know however that, since $g \circ \epsilon_i$ are all measurable, we have

$$g_*\big(\epsilon_{i*} \mathcal{A}_i\big) = (g \circ \epsilon_i)_* \mathcal{A}_i \supset \mathcal{G}, \ \ \forall\, i \in I,$$

so the desired inclusion is an immediate consequence of (4). $\square$

Conventions. Let $(X, \mathcal{A})$ be a measurable space. An extended real-valued function $f : (X, \mathcal{A}) \to [-\infty, \infty]$ is said to be a *measurable function*, if it is measurable in the above sense as a map $f : (X, \mathcal{A}) \to \big([-\infty, \infty], Bor([-\infty, \infty])\big)$. If f has values in $\mathbb{R}$, this is equivalent to the fact that f is a measurable map $f : (X, \mathcal{A}) \to \big(\mathbb{R}, Bor(\mathbb{R})\big)$ is measurable. Likewise, a complex valued function $f : (X, \mathcal{A}) \to \mathbb{C}$ is measurable, if it is measurable as a map $f : (X, \mathcal{A}) \to \big(\mathbb{C}, Bor(\mathbb{C})\big)$. If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, we define the set

$$\mathbf{B}_{\mathbb{K}}(X, \mathcal{A}) = \big\{ f : (X, \mathcal{A}) \to \mathbb{K} \,:\, f \text{ measurable function} \big\}.$$

Remark 3.2. Let $(X, \mathcal{A})$ be a measurable space. If $A \subset \mathbb{R}$ is a dense subset, then the results from Section 2, combined with Lemma 2.1, show that the measurability of a function $f : (X, \mathcal{A}) \to [-\infty, \infty]$ is equivalent to any of the following conditions:

- $f^{-1}\big((a, \infty]\big) \in \mathcal{A}$, $\forall\, a \in A$;
- $f^{-1}\big([a, \infty]\big) \in \mathcal{A}$, $\forall\, a \in A$;
- $f^{-1}\big([-\infty, a)\big) \in \mathcal{A}$, $\forall\, a \in A$;
- $f^{-1}\big([-\infty, a]\big) \in \mathcal{A}$, $\forall\, a \in A$.

Definition. If X and Y are topological Hausdorff spaces, a map $T : X \to Y$ is said to be *Borel measurable*, if T is measurable as a map

$$T : \big(X, Bor(X)\big) \to \big(Y, Bor(Y)\big).$$

In the cases when $Y = \mathbb{R}$, $\mathbb{C}$, $[-\infty, \infty]$, a Borel measurable map will be simply called a *Borel measurable function.*

For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, we define

$$\mathbf{B}_{\mathbb{K}}(X) = \big\{ f : X \to \mathbb{K} \,:\, f \text{ Borel measurable function} \big\}.$$

REMARK 3.3. If X and Y are topological Hausdorff spaces, then any continuous map $T : X \to Y$ is Borel measurable. This follows from Lemma 3.1, from the fact that

$$Bor(Y) = \mathbf{\Sigma}\big(\{ D \subset Y \,:\, D \text{ open} \}\big),$$

and the fact that $T^{-1}(D)$ is open, hence in $Bor(X)$, for every open set $D \subset Y$.

Measurable maps behave nicely with respect to "measurable countable operations," as suggested by the following result.

PROPOSITION 3.3. *Let $(X, \mathcal{A})$ and $(Z, \mathcal{B})$ be a measurable spaces, let I be a set which is at most countable, and let $(Y_i)_{i \in I}$ be a family of topological Hausdorff spaces, each of which is second countable. Suppose a measurable map $T_i : (X, \mathcal{A}) \to \big(Y_i, Bor(Y_i)\big)$ is given, for each $i \in I$. Define the map $T : X \to \prod_{i \in I} Y_i$ by*

$$T(x) = \big(T_i(x)\big)_{i \in I}, \ \ \forall\, x \in X.$$

Equip the product space $Y = \prod_{i \in I} Y_i$ with the product topology.

For any measurable map $g : \big(Y, Bor(Y)\big) \to (Z, \mathcal{B})$, the composition $g \circ T : (X, \mathcal{A}) \to (Z, \mathcal{B})$ is measurable.

PROOF. We know (see Corollary 2.3) that we have the equality

$$Bor(Y) = \mathbf{\Sigma}\text{-}\bigtimes_{i \in I} Bor(Y_i).$$

By Proposition 3.2, the map $T : (X, \mathcal{A}) \to \big(Y, Bor(Y)\big)$ is measurable, so by Proposition 3.1, the composition $g \circ T : (X, \mathcal{A}) \to (Z, \mathcal{B})$ is also measurable. ☐

The above result has many useful applications.

COROLLARY 3.2. *Suppose $(X, \mathcal{A})$ is a measurable space, and $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then, when equipped with point-wise addition and multiplication, the set $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ is a unital $\mathbb{K}$-algebra.*

PROOF. Clearly the constant function 1 is measurable.

Also, if $f \in \mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ and $\lambda \in \mathbb{K}$, then the function λf is again measurable, since it can be written as the composition $M_\lambda \circ f$, where $M_\lambda : \mathbb{K} \ni \alpha \longmapsto \lambda\alpha \in \mathbb{K}$ is obviously continuous.

Finally, let us show that if $f_1, f_2 \in \mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$, then $f_1 + f_2$ and $f_1 \cdot f_2$ again belong to $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$. This is however immediate from Proposition 3.3, applied to the index set $I = \{1, 2\}$, the spaces $Y_1 = Y_2 = \mathbb{K}$, and the continuous maps

$$\begin{aligned} &g_1 : \mathbb{K}^2 \ni (\lambda_1, \lambda_2) \longmapsto \lambda_1 + \lambda_2 \in \mathbb{K}, \\ &g_2 : \mathbb{K}^2 \ni (\lambda, \lambda_2) \longmapsto \lambda_1 \cdot \lambda_2 \in \mathbb{K}. \end{aligned}$$

☐

COROLLARY 3.3. *If $(X, \mathcal{A})$ is a measurable space, then a complex valued function $f : X \to \mathbb{C}$ is measurable, if and only if the real valued functions* $\operatorname{Re} f$, $\operatorname{Im} f$: *$X \to \mathbb{R}$ are measurable.*

Proof. If f is measurable, the composing f with the continuous maps

$$\rho : \mathbb{C} \ni z \longmapsto \operatorname{Re} z \in \mathbb{R} \text{ and } \gamma : \mathbb{C} \ni z \longmapsto \operatorname{Im} z \in \mathbb{R},$$

immediately gives the measurability of $\operatorname{Re} f = \rho \circ f$ and $\operatorname{Im} f = \gamma \circ f$.

Conversely, if both $\operatorname{Re} f,\ \operatorname{Im} f : X \to \mathbb{R}$ then the measurability of f follows from Proposition 3.3, applied to $Y_1 = Y_2 = \mathbb{R}$, the functions $f_1 = \operatorname{Re} f$ and $f_2 = \operatorname{Im} f$, and to the continuous function

$$g : \mathbb{R}^2 \ni (a,b) \longmapsto a + bi \in \mathbb{C}. \quad \square$$

Corollary 3.4. *Let $(X, \mathcal{A})$ be a measurable space, let I be a set which is at most countable, and let $f_i : (X, \mathcal{A}) \to [-\infty, \infty]$, $i \in I$ be collection of measurable functions. Then the functions $g, h : X \to [-\infty, \infty]$, defined by*

$$g(x) = \inf \big\{ f_i(x) \,:\, i \in I \big\} \text{ and } h(x) = \sup \big\{ f_i(x) \,:\, i \in I \big\},\ \ \forall\, x \in X,$$

are both measurable.

Proof. Define the maps $m, M : \prod_{i\in I}[-\infty, \infty] \to [-\infty, \infty]$ by

$$m(x) = \inf\{x_i \,:\, i \in I\} \text{ and } M(x) = \sup\{x_i \,:\, i \in I\},\ \ \forall\, x = (x_i)_{i\in I} \in \prod_{i\in I}[-\infty, \infty].$$

By Proposition 3.3, it suffices to prove the (Borel) measurability of the maps m and M.

To prove the measurability of m, we are going to show that

$$m^{-1}\big([-\infty, a)\big) \in Bor\Big(\prod_{i\in I}[-\infty, \infty]\Big),\ \ \forall\, a \in \mathbb{R}.$$

But this is quite obvious, since a point $x = (x_i)_{i\in I}$ belongs to $m^{-1}\big([-\infty, a)\big)$, if and only if there exists some $j \in I$ with $x_i < a$. In other words, if we define the projections $\pi_j : \prod_{i\in I}[-\infty, \infty] \to [-\infty, \infty]$, then we have

$$m^{-1}\big([-\infty, a)\big) = \bigcup_{j\in I} \pi_j\big([-\infty, a)\big).$$

This shows that in fact $m^{-1}\big([-\infty, a)\big)$ is open, hence clearly Borel.

To prove the measurability of M, we are going to show that

$$M^{-1}\big((a, \infty]\big) \in Bor\Big(\prod_{i\in I}[-\infty, \infty]\Big),\ \ \forall\, a \in \mathbb{R}.$$

But this is again clear, since, as before, we have the equality

$$M^{-1}\big((a, \infty]\big) = \bigcup_{j\in I} \pi_j\big((a, \infty]\big),$$

which shows that in fact $M^{-1}\big((a, \infty]\big)$ is open, hence Borel. $\square$

Corollary 3.5. *Let $(X, \mathcal{A})$ be a measurable space, and let $f_n : (X, \mathcal{A}) \to [-\infty, \infty]$, $n \in \mathbb{N}$ be sequence of measurable functions. Then the functions $g, h : X \to [-\infty, \infty]$, defined by*

$$g(x) = \liminf_{n\to\infty} f_n(x) \text{ and } h(x) = \limsup_{n\to\infty} f_n(x),\ \ \forall\, x \in X,$$

are both measurable.

Proof. For every $n \in \mathbb{N}$, define the functions $g_n, h_n : X \to [-\infty, \infty]$ by

$$g_n(x) = \inf \big\{ f_k(x) \, : \, k \geq n \big\} \text{ and } h_n(x) = \sup \big\{ f_k(x) \, : \, k \geq n \big\}, \ \ \forall \, x \in X.$$

By Corollary 3.5, we know that g_n and h_n are measurable for all $n \in \mathbb{N}$. Since

$$g(x) = \sup \big\{ g_n(x) \, : \, n \in \mathbb{N} \big\} \text{ and } h(x) = \inf \big\{ h_n(x) \, : \, n \in \mathbb{N} \big\}, \ \ \forall \, x \in X,$$

the fact that both g and h are measurable follows again from Corollary 3.5. □

Corollary 3.6. *Let $(X, \mathcal{A})$ be a measurable space, and let*

$$f_n : (X, \mathcal{A}) \to [-\infty, \infty], \ \ n \in \mathbb{N}$$

be sequence of measurable functions, with the property that, for each $x \in X$, the sequence $\big(f_n(x)\big)_{n=1}^{\infty} \subset [-\infty, \infty]$ has a limit. Then the function $f : X \to [-\infty, \infty]$, defined by

$$f(x) = \lim_{n \to \infty} f_n(x), \ \ \forall \, x \in X,$$

is again measurable.

Proof. Immediate from the above result. □

Exercise 1. If $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are continuous functions, and if $f(x) = \lim_{n \to \infty} f_n(x)$ exists, for every $x \in \mathbb{R}$, then by the above Corollary we know that $f : \mathbb{R} \to [-\infty, \infty]$ is Borel measurable. Prove that the converse is not true. More explicitly, prove that there is no sequence $(f_n)_{n=1}^{\infty}$ of continuous functions, with

$$\lim_{n \to \infty} f_n(x) = \varkappa_{\mathbb{Q}}(x), \ \ \forall \, x \in \mathbb{R}.$$

Hint: Use Baire's Theorem.

Exercise 2. Prove that a function $f : \mathbb{R} \to \mathbb{R}$, which is continuous everywhere, except for a countable set of points, is Borel measurable. As an application, prove that any monotone function is Borel measurable.

Corollary 3.6 can be generalized, as follows.

Theorem 3.1. *Let $(X, \mathcal{A})$ be a measurable space, let Y be a separable metric space, and let*

$$T_n : (X, \mathcal{A}) \to \big(Y, Bor(Y)\big), \ \ n \in \mathbb{N}$$

be a sequence of measurable maps. Assume that, for every $x \in X$, the sequence $\big(T_n(x)\big)_{n=1}^{\infty} \subset Y$ is convergent. Define the map $T : X \to Y$ by

$$T(x) = \lim_{n \to \infty} T_n(x), \ \ \forall \, x \in X.$$

Then $T : (X, \mathcal{A}) \to \big(Y, Bor(Y)\big)$ is a measurable map.

Proof. Denote by d the metric on Y. The collection

$$\mathcal{V} = \big\{ \mathcal{B}_r(y) \, : \, y \in Y, \, r > 0 \big\}$$

is a base for the topology of Y. Since Y is second countable, it suffices then to show that

$$T^{-1}\big(\mathcal{B}_r(y)\big) \in \mathcal{A}, \ \ \forall \, y \in Y, \, r > 0. \tag{5}$$

Claim: *For every $y \in Y$ and $r > 0$ one has the equality*

$$T^{-1}\big(\mathcal{B}_r(y)\big) = \bigcup_{m,n=1}^{\infty} \Big[\bigcap_{k=m}^{\infty} T_k^{-1}\big(\mathcal{B}_{r - \frac{1}{n}}(y)\big) \Big]. \tag{6}$$

Denote the set in the right hand side simply by A. Start first with some $x \in A$. There exist some $m, n \in \mathbb{N}$ such that

$$x \in \bigcap_{k=m}^{\infty} T_k^{-1}\big(\mathcal{B}_{r-\frac{1}{n}}(y)\big),$$

which means that

$$T_k(x) \in \mathcal{B}_{r-\frac{1}{n}}(y), \ \forall\, k \geq m,$$

that is,

$$d\big(T_k(x), y\big) < r - \frac{1}{n}, \ \forall\, k \geq m.$$

Pasing to the limit ($k \to \infty$) then yields

$$d\big(T(x), y\big) \leq r - \frac{1}{n} < r,$$

which means that $T(x) \in \mathcal{B}_r(y)$, i.e. $x = T^{-1}\big(\mathcal{B}_r(y)\big)$, thus proving the inclusion $A \subset T^{-1}\big(\big(\mathcal{B}_r(y)\big)$.

Conversely, if $x \in T^{-1}\big(\mathcal{B}_r(y)\big)$, we get $T(x) \in (\mathcal{B}_r(y)$, i.e. $d\big(T(x), y\big) < r$. Choose an integer n such that

$$d\big(T(x), y\big) < r - \frac{2}{n}. \tag{7}$$

Since $\lim_{k\to\infty} T_k(x) = T(x)$, there exists some $m \in \mathbb{N}$ such that

$$d\big(T_k(x), T(x)\big) < \frac{2}{n}, \ \forall\, k \geq m.$$

Combining this with (7) then gives

$$d\big(T_k(x), y\big) \leq d\big(T(x), y\big) + d\big(T_k(x), T(x)\big) < r - \frac{2}{n} + \frac{1}{n} = r - \frac{1}{n}, \ \forall\, k \geq m,$$

which means that

$$x \in \bigcap_{k=m}^{\infty} T_k^{-1}\big(\mathcal{B}_{r-\frac{1}{n}}(y)\big),$$

hence x indeed belongs to A.

Having proven (6) we now observe that, since the T_k's are measurable, it follows that

$$T_k^{-1}\big(\mathcal{B}_{r-\frac{1}{n}}(y)\big) \in \mathcal{A}, \ \forall\, k, n \in \mathbb{N}, \ r > 0.$$

Using the fact that $\mathcal{A}$ is closed under countable intersections, it follows that

$$\bigcap_{k=m}^{\infty} T_k^{-1}\big(\mathcal{B}_{r-\frac{1}{n}}(y)\big) \in \mathcal{A}, \ \forall\, m, n \in \mathbb{N}, \ r > 0.$$

Finally, using the fact that $\mathcal{A}$ is closed under countable unions, the desired property (5) follows. □

Exercise 3. Let $(X, \mathcal{A})$ be a measurable space, and let $(X_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{A}$, with $X = \bigcup_{n=1}^{\infty} X_n$. Suppose $(Y, \mathcal{B})$ is a measurable space, and $F : X \to Y$ is a map, such that

$$F\big|_{X_n} : \big(X_n, \mathcal{A}\big|_{X_n}\big) \to (Y, \mathcal{B})$$

is measurable, for all $n \in \mathbb{N}$. Prove that $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is measurable.

*Exercise 4**. Let $\Omega_1 \subset \mathbb{R}^n$ be an open set, and let $f_1, \dots, f_n : \Omega_1 \to \mathbb{R}$ be C^1 functions, with the property that the matrix

$$A(p) = \left[\frac{\partial f_j}{\partial x_k}(p)\right]_{j,k=1}^{n}$$

is invertible, for every point $p \in \Omega_1$. Define the map

$$F : \Omega_1 \ni p \longmapsto \big(f_1(p), \dots, f_n(p)\big) \in \mathbb{R}^n.$$

(i) Prove that the set $\Omega_2 = F(\Omega_1)$ is open in $\mathbb{R}^n$.
(ii) Although $F : \Omega_1 \to \Omega_2$ may fail to be injective, prove that there exists a Borel measurable map $\phi : \Omega_2 \to \Omega_1$, with $F \circ \phi = \mathrm{Id}_{\Omega_2}$.

Hint: Use the Inverse Function Theorem, combined with Exercises 2 and 3. exercise.

*Exercise 5**. Let $P(z)$ be a non-constant polynomial with complex coefficients. Prove that there exists a Borel measurable function $f : \mathbb{C} \to \mathbb{C}$, such that

$$P\big(f(z)\big) = z, \ \ \forall\, z \in \mathbb{C}.$$

Hint: Use the preceding exercise, applied to the set $\Omega_1 = \{z \in \mathbb{C} : P'(z) \neq 0\}$.

The preceding exercise can be generalized:

*Exercise 6**. Let $\Omega_1 \subset \mathbb{C}$ be a connected open set, and let $f : \Omega_1 \to \mathbb{C}$ be a non-constant holomorphic function. By the Open Mapping Theorem we know that the set $\Omega_2 = f(\Omega_1)$ is open. Prove that there exists a Borel measurable function $\phi : \Omega_2 \to \Omega_1$, such that $f \circ \phi = \mathrm{Id}\big|_{\Omega_2}$.

Hint: Use Exercise 4, applied to the set $\Omega_0 = \{z \in \Omega_1 : f'(z) \neq 0\}$. Since f is non-constant, the set $\Omega_1 \smallsetminus \Omega_0$ is countable.

We continue with a discussion on the role of elementary functions.

Proposition 3.4. *Let $(X, \mathcal{A})$ be a measurable space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For an elementary function $f \in \mathrm{Elem}_{\mathbb{K}}(X)$, the following are equivalent:*

(i) *$f \in \mathcal{A}$-$\mathrm{Elem}_{\mathbb{K}}(X)$;*
(ii) *$f : (X, \mathcal{A}) \to \mathbb{K}$ is measurable.*

Proof. *(i)* ⇒ *(ii)*. We know that $\mathcal{A}$-$\mathrm{Elem}_{\mathbb{K}} = \mathrm{Span}_{\mathbb{K}}\{\varkappa_A : A \in \mathcal{A}\}$. Since $\mathbf{B}_{\mathbb{K}}(X, \mathcal{A})$ is a vector space, it suffices to show only that $\varkappa_A : (X, \mathcal{A}) \to \mathbb{K}$ is measurable, for all $A \in \mathcal{A}$. But this is trivial, since for every Borel set $B \subset \mathbb{R}$ one has either $\varkappa_A^{-1}(B) = \varnothing$, or $\varkappa_A^{-1}(B) = A$, or $\varkappa_A^{-1}(B) = X$.

(ii) ⇒ *(i)*. Assume now f is measurable. List the range of f as

$$f(X) = \{\lambda_1, \dots, \lambda_n\},$$

with $\lambda_j \neq \lambda_k$, for all $j, k \in \{1, \dots, n\}$ with $j \neq k$. Since f is measurable, and the singleton sets $\{\lambda_1\}, \dots, \{\lambda_n\}$ are in $Bor(\mathbb{K})$, it follows that the sets $A_j = f^{-1}\big(\{\lambda_j\}\big)$, $j = 1, \dots, n$ are all in $\mathcal{A}$. Since we clearly have

$$f = \lambda_1 \varkappa_{A_1} + \dots + \lambda_n \varkappa_{A_n},$$

it follows that f indeed belongs to $\mathcal{A}$-$\mathrm{Elem}_{\mathbb{K}}(X)$. □

REMARKS 3.4. A. If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are measurable spaces, if $T : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a measurable map, and if $f \in \mathcal{B}\text{-Elem}_{\mathbb{K}}(Y)$, then $f \circ T \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$. This follows from the fact that the composition $f \circ T : (X, \mathcal{A}) \to \mathbb{K}$ is measurable, and elementary.

B. If $(X, \mathcal{A})$ is a measurable space, if $f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$, and if $g : f(X) \to \mathbb{K}$ is an arbitrary function, then $g \circ f \in \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$. This follows from the fact that, if one considers the finite set $Y = f(X)$, and the σ-algebra $\mathcal{P}(Y)$ on it, then

$$(X, \mathcal{A}) \xrightarrow{f} \big(Y, \mathcal{P}(Y)\big) \xrightarrow{g} \mathbb{K}$$

are measurable. So $g \circ f$ is also measurable, and obviously elementary.

The following is an interesting converse of Corollary 3.6.

THEOREM 3.2. *Let $(X, \mathcal{A})$ be a measurable space, and let $f : (X, \mathcal{A}) \to [-\infty, \infty]$ be a measurable function. Then there exists a sequence $(f_n)_{n=1}^{\infty} \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that*

- $\inf \big\{ f(y) \,:\, y \in X \big\} \leq f_n(x) \leq \sup \big\{ f(z) \,:\, z \in X \big\}$, $\forall\, x \in X$, $n \geq 1$;
- $\lim_{n \to \infty} f_n(x) = f(x)$, $\forall\, x \in X$.

Moreover,

(i) *if* $\inf \big\{ f(x) \,:\, x \in X \big\} > -\infty$, *then the sequence* $(f_n)_{n=1}^{\infty}$ *can be chosen to be non-decreasing, i.e.* $f_n \leq f_{n+1}$, $\forall\, n \in \mathbb{N}$;

(ii) *if* $\sup \big\{ f(x) \,:\, x \in X \big\} < \infty$, *then the sequence* $(f_n)_{n=1}^{\infty}$ *can be chosen to be non-increasing, i.e.* $f_n \geq f_{n+1}$, $\forall\, n \in \mathbb{N}$;

(iii) *if* $\inf \big\{ f(x) \,:\, x \in X \big\} > -\infty$ *and* $\sup \big\{ f(x) \,:\, x \in X \big\} < \infty$, *then the sequence* $(f_n)_{n=1}^{\infty}$ *can be chosen eiher non-decreasing, or non-increasing, and such that it converges uniformly to* f, *i.e.*

$$\lim_{n \to \infty} \left[\sup_{x \in X} \big| f_n(x) - f(x) \big| \right] = 0.$$

PROOF. We begin with a special case of (iii). Assume $X = [0, 1]$, $\mathcal{A} = Bor([0, 1])$, and consider the inclusion $F : [0, 1] \hookrightarrow [-\infty, \infty]$. For each $n \in \mathbb{N}$, define the intervals I_k^n, J_k^n, $0 \leq k \leq 2^n - 1$ by

$$I_k^n = \big[k/2^n, (k+1)/2^n\big), \text{ if } 0 \leq k \leq 2^n - 2; \qquad I_{2^n-1}^n = \big[(2^n - 1)/2^n, 1\big],$$
$$J_k^n = \big(k/2^n, (k+1)2^n\big], \text{ if } 1 \leq k \leq 2^n - 1; \qquad J_0^n = \big[0, 1/2^n\big].$$

We then define, for each $n \in \mathbb{N}$, the functions $g_n, h_n : [0, 1] \to \mathbb{R}$ by

$$g_n = 2^{-n} \sum_{k=0}^{2^n - 1} k \varkappa_{I_k^n} \text{ and } h_n = 2^{-n} \sum_{k=0}^{2^n - 1} (k+1) \varkappa_{J_k^n}.$$

Remark that

$$0 \leq g_n(s) < 1 \text{ and } 0 < h_n(s) \leq 1, \ \ \forall\, s \in [0, 1]. \tag{8}$$

Note that, for every $n \in \mathbb{N}$, we have

$$g_n(0) = 0; \qquad g_n(1) = (2^n - 1)/2^n; \tag{9}$$

$$h_n(0) = 1/2^n; \qquad h_n(1) = 1. \tag{10}$$

Claim 1**:** *The sequence* $(g_n)_{n=1}^{\infty}$ *is non-decreasing, and the sequence* $(h_n)_{n=1}^{\infty}$ *is non-increasing.*

Using (9) and (10), we only need to examine the restrictions to the open interval $(0,1)$. Fix some point $s \in (0,1)$. For every integer $n \geq 1$, define

$$p_n^s = \max\big\{k \in \mathbb{Z} \,:\, 0 \leq \frac{k}{2^n} < s\big\}.$$

We clearly have $p_n^s < 2^n$ and

$$\frac{p_n^s}{2^n} < s \leq \frac{p_n^s + 1}{2^n}. \tag{11}$$

We then have

$$g_n(s) = \begin{cases} p_n^s/2^n & \text{if } s \neq (p_n^s+1)/2^n \\ (p_n^s+1)/2^n & \text{if } s = (p_n^s+1)/2^n \end{cases} \quad \text{and } h_n(s) = \frac{p_n^s+1}{2^n} \tag{12}$$

We now estimate $g_{n+1}(s)$ and $h_{n+1}(s)$. First of all, using (11), we have

$$\frac{2p_n^s}{2^{n+1}} < x \leq \frac{2p_n^s + 2}{2^{n+1}},$$

which means that either $p_{n+1}^s = 2p_n^s$, or $p_{n+1}^s = 2p_n^s + 1$. This immediately gives

$$h_{n+1}(s) = \frac{p_{n+1}^s + 1}{2^{n+1}} \leq \frac{2p_n^s + 2}{2^{n+1}} = \frac{p_n^s + 1}{2^n} = h_n(s).$$

Note that, if $s = (p_n^s + 1)/2^n$, we will have $p_{n+1}^s = 2p^s + 1$ and $s = (p_{n+1}^s + 1)/2^{n+1}$, so we get

$$g_{n+1}(s) = (p_{n+1}^s + 1)/2^{n+1} = (2p_n^s + 2)/2^{n+1} = (p_n^s + 1)/2^n = g_n(s).$$

If $s \neq (p_n^s + 1)/2^n$, then

$$g_n(s) = \frac{p_n^s}{2^n} = \frac{2p_n^s}{2^n} \leq \frac{p_{n+1}^s}{2^{n+1}} \leq g_{n+1}(s).$$

Claim 2: *For every $s \in [0,1]$ one has*

$$\lim_{n\to\infty}\Big[\sup_{s\in[0,1]} \big|g_n(s) - s\big|\Big] = \lim_{n\to\infty}\Big[\sup_{s\in[0,1]} \big|h_n(s) - s\big|\Big] = 0.$$

To prove this fact we are going to estimate the differences $|g_n(s)-s|$ and $|h_n(s)-s|$.

If $s = 0$ or $s = 1$, then the equalities (9) and (10) immediately show that

$$|g_n(s) - s| \leq \frac{1}{2^n} \text{ and } |h_n(s) - s| \leq \frac{1}{2^n}, \ \ \forall\, n \in \mathbb{N}. \tag{13}$$

If $s \in (0,1)$, then the definitions of $g_n(s)$ and $h_n(s)$ clearly show that

$$s,\ g_n(s),\ h_n(s) \in \big[p_n^s/2^n, (p_n^s + 1)/2^n\big],$$

and then we see that we again have the inequalities (13). Since (13) now holds for all $s \in [0,1]$, the Claim immediately follows.

We proceed now with the proof of the theorem. Define

$$\alpha = \inf\big\{f(x) \,:\, x \in X\big\} \text{ and } \beta = \sup\big\{f(x) \,:\, x \in X\big\}.$$

If $\alpha = \beta$, there is nothing to prove. Assume $\alpha < \beta$. Depending on the finitude of α and β, we define a homeomorphism $\Phi : [\alpha,\beta] \to [0,1]$, as follows.

(*a*) If $\alpha > -\infty$ and $\beta < \infty$, we define

$$\Phi(s) = \frac{s-\alpha}{\beta-\alpha}, \ \ \forall\, s \in [\alpha,\beta].$$

(*b*) If $\alpha > -\infty$ and $\beta = \infty$, we define

$$\Phi(s) = \begin{cases} \frac{2}{\pi}\arctan(s-\alpha) & \text{if } s \neq \beta \\ 1 & \text{if } s = \beta \end{cases}$$

(*c*) If $\alpha = -\infty$ and $\beta < \infty$, we define

$$\Phi(s) = \begin{cases} 1 + \frac{2}{\pi}\arctan(s-\beta) & \text{if } s \neq \alpha \\ 0 & \text{if } s = \alpha \end{cases}$$

(*d*) If $\alpha = -\infty$ and $\beta = \infty$, we define

$$\Phi(s) = \begin{cases} 0 & \text{if } s = \alpha \\ \frac{1}{2} + \frac{1}{\pi}\arctan(s-\beta) & \text{if } \alpha < s\beta \\ 1 & \text{if } s = \beta \end{cases}$$

Notice that $\Phi(\alpha) = 0$, $\Phi(\beta) = 1$, and

$$\alpha \leq s < t \leq \beta \Rightarrow \Phi(s) < \Phi(t).$$

After these preparations, we proceed with the proof. We begin with the special cases (i) (ii) and (iii).

If $\alpha > -\infty$, we define the functions $f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f$. Since Φ and Φ^{-1} are increasing, and $(g_n)_{n=1}^\infty$ is non-decreasing, it follows that $(f_n)_{n=1}^\infty$ is non-decreasing. Since $0 \leq g_n(s) < 1$, $\forall\, s \in [0,1]$, we see that $\alpha \leq f_n(x) < \beta$, $\forall\, x \in X$. In particular, we have $-\infty < f_n(x) < \infty$, for all n and x. It it obvious that f_n is elementary, measurable, and since $\lim_{n\to\infty} g_n(s) = s$, $\forall\, s \in [0,1]$ (by Claim 2), we immediately get $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall\, x \in X$.

If $\beta < \infty$, we define the functions $f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f$. Since Φ and Φ^{-1} are increasing, and $(h_n)_{n=1}^\infty$ is non-increasing, it follows that $(f_n)_{n=1}^\infty$ is non-increasing. Since $0 < h_n(s) \leq 1$, $\forall\, s \in [0,1]$, we see that $\alpha < f_n(x) \leq \beta$, $\forall\, x \in X$. In particular, we have $-\infty < f_n(x) < \infty$, for all n and x. It it obvious that f_n is elementary, measurable, and since $\lim_{n\to\infty} h_n(s) = s$, $\forall\, s \in [0,1]$ (by Claim 2), we immediately get $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall\, x \in X$.

If $\alpha > -\infty$ and $\beta < \infty$, then we can take $f_n = \Phi^{-1} \circ g_n \circ \Phi \circ f$, $\forall\, n$, or we can take $f_n = \Phi^{-1} \circ h_n \circ \Phi \circ f$, $\forall\, n$. The inequalities (13), combined with the definition (*c*) of Φ, show that

$$|f_n(x) - f| \leq \frac{\beta - \alpha}{2^n},\ \ \forall\, x \in X,\ \ n \in \mathbb{N},$$

with any of the above choices for $(f_n)_{n=1}^\infty$.

Having proven the cases (i), (ii) and (iii), we now examine the general situation, when $\alpha = -\infty$ and $\beta = \infty$. Consider the functions $f', f'' : X \to [-\infty, \infty]$ defined by

$$f'(x) = \max\{f(x), 0\} \text{ and } f''(x) = \min\{f(x), 0\},\ \ \forall\, x \in X.$$

By Corollary 3.4, both f' and f'' are measurable. Since $\inf_{x\in X} f'(x) \geq 0$, by part (i), there exists a sequence $(f'_n)_{n=1}^\infty \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that $\lim_{n\to\infty} f'_n(x) = f'(x)$, $\forall\, x \in X$. Since $\sup_{x\in X} f''(x) \leq 0$, by part (ii), there exists a sequence $(f''_n)_{n=1}^\infty \in \mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$, such that $\lim_{n\to\infty} f''_n(x) = f''(x)$, $\forall\, x \in X$. Define the elementary functions $f_n = f'_n + f''_n$, $n \in \mathbb{N}$. Clearly the f_n's are all in $\mathcal{A}\text{-Elem}_{\mathbb{R}}(X)$.

We now check that

$$\lim_{n\to\infty} f_n(x) = f(x),\ \ \forall\, x \in X. \tag{14}$$

There are two cases to examine: (*a*) $f(x) \geq 0$; (*b*) $f(x) \leq 0$.

In case (a), we have $f'(x) = f(x)$ and $f''(x) = 0$, so $\lim_{n\to\infty} f_n'(x) = f(x)$ and $\lim_{n\to\infty} f_n''(x) = 0$.

In case (b), we have $f'(x) = 0$ and $f''(x) = f(x)$, so $\lim_{n\to\infty} f_n'(x) = 0$ and $\lim_{n\to\infty} f_n''(x) = f(x)$.

In either case, the equality (14) follows. □

We conclude this section with a discussion on an interesting measurable space, that appears often in connection with probability theory.

EXAMPLE 3.1. Consider the space $T = \{0,1\}^{\aleph_0}$, i.e.

$$T = \big\{a = (\alpha_n)_{n=1}^\infty \,:\, \alpha_n \in \{0,1\},\ \forall\, n \in \mathbb{N}\big\}.$$

We call T the *space of infinite coin flippings*, having in mind that an element of T is the same as the outcome of an infinite sequence of coin flips (think 0 as corresponding to tails, and 1 as corresponding to heads). Equipp T with the product topology. By Tihonov's Theorem, T is compact. The product topology on T is in fact given by a metric d defined by

$$d(a,b) = \sum_{n=1}^\infty \frac{|\alpha_n - \beta_n|}{2^n},\quad \forall\, a = (\alpha_n)_{n=1}^\infty,\, b = (\beta_n)_{n=1}^\infty \in T.$$

For every number $r \geq 2$ we define a map $\phi_r : T \to [0,1]$ by

$$\phi_r(a) = (r-1)\sum_{n=1}^\infty \frac{\alpha_n}{r^n},\ \ \forall\, a = (\alpha_n)_{n=1}^\infty \in T.$$

It is pretty clear that

$$\big|\phi_r(a) - \phi_r(b)\big| \leq (r-1)d(a,b),\ \ \forall\, a,b \in T,$$

so the maps $\phi_r : T \to [0,1]$, $r \geq 2$ are continuous. In particular, the set $K_r = \phi_r(T)$ is a compact subset of $[0,1]$.

Define

$$T_0 = \big\{a = (\alpha_n)_{n\in\mathbb{N}} \in T \,:\, \text{ the set } \{n \in \mathbb{N} \,:\, \alpha_n = 0\} \text{ is infinite}\big\}.$$

The set $T \smallsetminus T_0$ can be described as:

$$T \smallsetminus T_0 = \big\{(\alpha_n)_{n\in\mathbb{N}} \in T \,:\, \text{ there exists } N \in \mathbb{N}, \text{ such that } \alpha_n = 1,\ \forall\, n \geq N\big\}.$$

The following are well known (see Appendix B, the proof of Proposition B.2).

Facts: *1. The set $T \smallsetminus T_0$ is countable*

2. For any $r \geq 2$, and elements $a = (\alpha_n)_{n=1}^\infty$, $b = (\beta_n)_{n=1}^\infty \in T_0$, the following are equivalent:

- *there exists $N \in \mathbb{N}$ such that $\alpha_N = 1$, $\beta_N = 0$, and $\alpha_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N$;*
- *$\phi_r(a) > \phi(b)$.*

In particular, the map $\phi_r\big|_{T_0} : T_0 \to [0,1]$ is injective.

The above constructions have a remarkable feature.

THEOREM 3.3. *Use the notations above. For a number $r \geq 2$ and subset $A \subset T$, the following are equivalent:*

(i) *$A \in Bor(T)$;*
(ii) *$\phi_r(A) \in Bor(K_r)$.*

Proof. Throughout the proof the number r will be fixed. The map ϕ_r will be denoted by ϕ, and the compact set K_r will be denoted by K.

Since $\phi : T \to K$ is continuous, it is measurable, i.e. we have the implication

$$B \in Bor(K) \Rightarrow \phi^{-1}(B) \in Bor(T). \tag{15}$$

Before we proceed with the actual proof, we need some preparations. Remark that, since $\phi : T \to K$ is surjective, we have the equality

$$\phi\big(\phi^{-1}(C)\big) = C, \ \forall\, C \subset K. \tag{16}$$

Claim 1: *If a subset $C \subset K$ is at most countable, if and only if the set $\phi^{-1}(C) \subset T$ is at most countable.*

Suppose C is at most countable countable. If we take $A_0 = \phi^{-1}(C) \cap T_0$, and $A_1 = \phi^{-1}(C) \smallsetminus T_0$, then obviously $\phi^{-1}(C) = A_0 \cup A_1$. Since $A_1 \subset T \smallsetminus T_0$, and $T \smallsetminus T_0$ is countable, it follows that A_1 is at most countable, so we only need to prove that A_0 is at most countable. But since $\phi\big|_{T_0}$ is injective, and $A_0 \subset T_0$, it follows that $\phi\big|_{A_0} : A_0 \to C$ is injective, and then the fact that C is at most countable, forces A_0 to be at most countable.

Conversely, if $\phi^{-1}(C)$ is at most countable, then so is $\phi\big(\phi^{-1}(C)\big)$. By (16) we are done.

For each subset $A \subset T$, we define

$$\langle A \rangle = \phi^{-1}\big(\phi(A)\big).$$

Remark that $A \subset \langle A \rangle$, $\forall\, A \subset T$. Note also that, for any family $(A_i)_{i \in I}$ of subsets of T, one has the equality

$$\langle \bigcup_{i\in I} A_i \rangle = \phi^{-1}\Big(\phi\big(\bigcup_{i\in I} A_i\big)\Big) = \phi^{-1}\Big(\bigcup_{i\in I} \phi(A_i)\Big) = \bigcup_{i\in I} \phi^{-1}\big(\phi(A_i)\big) = \bigcup_{i\in I} \langle A_i \rangle. \tag{17}$$

As an application of Claim 1, to the set $C = \phi(T \smallsetminus T_0)$, we see that

($*$) *the set $\langle T \smallsetminus T_0 \rangle$ is at most countable.*

Claim 2: *For any subset $A \subset T_0$, one has the inclusion*

$$\langle A \rangle \smallsetminus A \subset \langle T \smallsetminus T_0 \rangle.$$

In particular, the difference $\langle A \rangle \smallsetminus A$ is at most countable.

Start with an arbitrary element $x \in \langle A \rangle \smallsetminus A$. This means that $x \notin A$, but $\phi(x) \in \phi(A)$, which means that there exists some $a \in A$, with $\phi(x) = \phi(a)$. Assume now $x \notin T \smallsetminus T_0$, which means that $x \in T_0$. But then, the fact that $x, a \in T_0$, combined with the injectivity of $\phi\big|_{T_0}$ will force $x = a$, which is impossible since $a \in A$.

Claim 3: *For any set $A \subset T$, the difference $\langle A \rangle \smallsetminus A$ is at most countable.*

Take $A_0 = A \cap T_0$ and $A_1 = A \smallsetminus A_0$. Notice that, since $A_1 \subset T \smallsetminus T_0$, we have

$$\langle A_1 \rangle = \phi^{-1}\big(\phi(A_1)\big) \subset \phi^{-1}\big(\phi(T \smallsetminus T_0)\big) = \langle T \smallsetminus T_0 \rangle,$$

so it follows that $\langle A_1 \rangle$ is at most countable. We obviously have $A = A_0 \cup A_1$, so by (17)

$$\langle A \rangle = \langle A_0 \rangle \cup \langle A_1 \rangle.$$

But now we are done, since

$$\langle A \rangle \smallsetminus A = \big(\langle A_0 \rangle \cup \langle A_1 \rangle\big) \smallsetminus \big(A_0 \cup A_1\big) \subset \big(\langle A_0 \rangle \smallsetminus A_0\big) \cup \langle A_1 \rangle,$$

and both $\langle A_0 \rangle \smallsetminus A_0$ (by Claim 2) and $\langle A_1 \rangle$ are at most countable.

*Claim 4***:** *For any subset $A \subset T$, one has the inclusion*

$$\phi(T \smallsetminus A) \supset K \smallsetminus \phi(A), \tag{18}$$

and the difference $\phi(T \smallsetminus A) \smallsetminus \big(K \smallsetminus \phi(A)\big)$ is at most countable.

The inclusion (18) is pretty obvious, from the surjectivity of ϕ. In order to prove that the difference

$$C = \phi(T \smallsetminus A) \smallsetminus \big(K \smallsetminus \phi(A)\big) = \phi(T \smallsetminus A) \cap \phi(A)$$

is countable, by Claim 1, it suffices to prove that $\phi^{-1}(C)$ is countable. We have

$$\phi^{-1}(C) = \phi^{-1}\big(\phi(T \smallsetminus A) \cap \phi(A)\big) = \phi^{-1}\big(\phi(T \smallsetminus A)\big) \cap \phi^{-1}\big(\phi(A)\big) = \langle T \smallsetminus A\rangle \cap \langle A\rangle.$$

We can write $\phi^{-1}(C) = A_1 \cup A_2$, where

$$A_1 = (T \smallsetminus A) \cap \langle A\rangle \text{ and } A_2 = \big[\langle T \smallsetminus A\rangle \smallsetminus (T \smallsetminus A)\big] \cap \langle A\rangle,$$

so it suffices to prove that both A_1 and A_2 are at most countable. But these facts are immediate from Claim 3, since $A_1 = \langle A\rangle \smallsetminus A$, and $A_2 \subset \langle T \smallsetminus A\rangle \smallsetminus (T \smallsetminus A)$.

We can now proceed with the proof of the theorem. Define

$$\mathcal{A} = \big\{A \subset T \,:\, \phi(A) \in Bor(K)\big\},$$

so that what we need to prove is the equality $\mathcal{A} = Bor(T)$.

First, remark that, if $A \in \mathcal{A}$, then $\phi(A) \in Bor(K)$, and the fact that ϕ is Borel measurable will force $\langle A\rangle = \phi^{-1}\big(\phi(A)\big)$ to be a Borel set in T. But since $\langle A\rangle \smallsetminus A$ is countable, hence Borel, it follows that

$$A = \langle A\rangle \smallsetminus \big(\langle A\rangle \smallsetminus A\big)$$

is again Borel. Therefore, we have the inclusion $\mathcal{A} \subset Bor(T)$.

Second, remark that if $F \subset T$ is a compact subset, then the continuity of ϕ gives the fact that $\phi(F)$ is compact, hence Borel. This then forces $F \in \mathcal{A}$. Therefore $\mathcal{A}$ contains the collection $\mathcal{C}_T$ of all compact subsets of T.

Now we have

$$\mathcal{C}_T \subset \mathcal{A} \subset Bor(T) = \mathbf{\Sigma}(\mathcal{C}_T),$$

so all we need to prove is the fact that $\mathcal{A}$ is a σ-algebra, i.e. we have the properties

- (*a*) $A \in \mathcal{A} \Rightarrow T \smallsetminus A \in \mathcal{A}$;
- (*b*) for any sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$, the union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{A}$..

To check (*a*) start with some set $A \in \mathcal{A}$. We know that $\phi(A) \in Bor(K)$, and we want to show that $\phi(T \smallsetminus A)$ is again Borel. By Claim 4, we know we can write

$$\phi(T \smallsetminus A) = \big[K \smallsetminus \phi(A)\big] \cup C,$$

for some set $C \subset K$ which is at most countable. Since C and $K \smallsetminus \phi(A)$ are Borel, this shows that $\phi(T \smallsetminus A)$ is also Borel.

Property (*b*) is obvious, since $\phi(A_n)$, $n \geq 1$ are all Borel, and

$$\phi\bigg(\bigcup_{n=1}^{\infty} A_n\bigg) = \bigcup_{n=1}^{\infty} \phi(A_n). \quad \square$$

Corollary 3.7. *Use the above notations. For a number $r \geq 2$ and a subset $B \subset K_r$, the following are equivalent:*

- (i) *$B \in Bor(K_r)$;*
- (ii) *$\phi_r^{-1}(B) \in Bor(T)$.*

PROOF. The implication $(i) \Rightarrow (ii)$ is trivial, since ϕ_r is continuous, hence measurable.

Conversely, if the set $A = \phi_r^{-1}(B)$ is Borel, then by the Theorem, $\phi_r(A)$ is Borel. But since ϕ_r is surjective, we have $B = \phi_r(A)$. □

COMMENTS. From the above results, we see that $\phi_r : T \to K_r$ "almost preserves Borel structures." More explicitly, if one considers the maps

$$\Phi_r : \mathcal{P}(T) \ni A \longmapsto \phi_r(A) \in \mathcal{P}(K_r),$$
$$\Psi_r : \mathcal{P}(K_r) \ni B \longmapsto \phi_r^{-1}(B) \in \mathcal{P}(T),$$

then

- $(\Phi_r \circ \Psi_r)(B) = B$, for all $B \subset K_r$;
- $(\Psi_r \circ \Phi_r)(A) \supset A$, and $(\Phi_r \circ \Psi_r)(A) \smallsetminus A$ is at most countable, for all $A \subset T$;
- $B \in Bor(K_r) \Leftrightarrow \Psi_r(B) \in Bor(T)$;
- $A \in Bor(T) \Leftrightarrow \Phi_r(A) \in Bor(K_r)$.

In the particular case $r = 2$, we know that $K_2 = [0, 1]$, so we can think the measurable space $\big([0, 1], Bor([0, 1])\big)$ as "approximatively the same" as the measurable space $\big(T, Bor(T)\big)$.

The case $r = 3$ will be an interesting one, especially for constructing various counter-examples. The compact set $K_3 \subset [0, 1]$ is called the *ternary Cantor set*.

It turns out that there exists another useful description of the ternary Cantor set K_3, which yields some interesting properties.

NOTATIONS. We keep the notations above. An element $a = (\alpha)_{n=1}^{\infty} \in T$ will be called *finite*, if there exists some $N \in \mathbb{N}$, such that $\alpha_n = 0$, $\forall\, n \geq 0$. We define

$$T_{\mathit{fin}} = \big\{ a \in T \,:\, a \text{ finite} \big\}.$$

Remark that $T_{\mathit{fin}} \subset T_0$. In particular the map $\phi_3\big|_{T_{\mathit{fin}}} : T_{\mathit{fin}} \to K_3$ is injective.

For $a \in T_{\mathit{fin}}$ we define its length as

$$\ell(a) = \min\{N \in \mathbb{N} \,:\, \alpha_n = 0,\ \forall\, n \geq N\} - 1.$$

With this definition, for every $a = (\alpha_n)_{n=1}^{\infty} \in T_{\mathit{fin}}$, we have

$$\alpha_{\ell(a)} = 1 \text{ and } \alpha_n = 0,\ \ \forall\, n > \ell(a). \tag{19}$$

We define

$$\Lambda = \big\{ (k, a) \in \mathbb{Z} \times T_{\mathit{fin}} \,:\, k \geq \ell(a) \big\}.$$

Finally, for every pair $\lambda = (k, a) \in \Lambda$, we define the open interval

$$I_\lambda = \big(\phi_3(a) + \frac{1}{3^{k+1}}, \phi_3(a) + \frac{2}{3^{k+1}} \big).$$

Remark that, using (19) we have

$$\phi_3(a) \leq 2 \sum_{n=1}^{\ell a} \frac{2}{3^n} = 1 - \frac{1}{3^{\ell(a)}},$$

with the convention that the sum is 0, if $\ell(a) = 0$. We then get

$$\phi_3(a) + \frac{2}{3^{k+1}} \leq 1 - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{k+1}} < 1 - \frac{1}{3^{\ell(a)}} + \frac{1}{3^k} \leq 1,$$

which gives the inclusion $I_\lambda \subset (0,1)$.

The following result is describes an alternative construction of K_3.

THEOREM 3.4. *Use the notations above.*

(i) *The set T_{fin} is dense in T;*
(ii) *The system $(I_\lambda)_{\lambda\in\Lambda}$ is pair-wise disjoint.*
(iii) $\bigcup_{\lambda\in\Lambda} = [0,1] \smallsetminus K_3$.

PROOF. The map ϕ_3 will be simply denoted by ϕ, and the Cantor set K_3 will be denoted simply by K.

(i). Fix some element $a = (\alpha_n)_{n=1}^\infty \in T$. For every integer $k \geq 1$ define the element $a_k = (\alpha_n^k)_{n=1}^\infty \in T$, by

$$\alpha_n^k = \begin{cases} \alpha_n & \text{if } n \leq k \\ 0 & \text{if } n > k \end{cases}$$

It is obvious that $a_k \in T_{\mathit{fin}}$, $\forall\, k \in \mathbb{N}$. The inequality

$$d(a, a_k) = \sum_{n=k+1}^{\infty} \frac{\alpha_n}{2^n} \leq \sum_{n=k+1} \frac{1}{2^n} = \frac{1}{2^k},\ \ \forall\, k \in \mathbb{N}$$

then immediately shows that $\lim_{k\to\infty} a_k = a$.

(ii). Assume $\lambda, \mu \in \Lambda$ are such that $\lambda \neq \mu$, and let us prove that $I_\lambda \cap I_\mu = \varnothing$. Let $\lambda = (j, a)$ and $\mu = (k, b)$, where $a = (\alpha_n)_{n=1}^\infty$ and $b = (\beta_n)_{n=1}^\infty$ are elements int T_{fin} with $\ell(a) \leq j$ and $\ell(b) \leq k$. Since $\lambda \neq \mu$, we have one (or both) of the following cases: (A) $a \neq b$, or (B) $j \neq k$.

In case (A) we take

$$m = \min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\}.$$

Without any loss of generality, we can assume that $\alpha_m = 0$ and $\beta_m = 1$. Note that $k \geq \ell(b) \geq m \geq 1$. We are going to prove that $I_\lambda \cap I_\mu = \varnothing$, by showing that the right end-point of I_λ is not greater than the left end-point of I_μ, that is,

$$\phi(a) + \frac{2}{3^{k+1}} \leq \phi(b) + \frac{1}{3^{k+1}}. \tag{20}$$

Define the number

$$M = \sum_{n=1}^{m-1} \frac{\alpha_n}{3^n} = \sum_{n=1}^{m-1} \frac{\beta_n}{3^n},$$

with the convention that $M = 0$, if $m = 1$. We have:

$$\phi(a) = 2M + 2\sum_{n=m+1}^{\ell(a)} \frac{\alpha_n}{3^n} \leq 2M + 2\sum_{m+1}^{\ell(a)} \frac{1}{3^n} = 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}};$$

$$\phi(b) = 2M + \frac{2}{3^m} + 2\sum_{n=m+1}^{\ell(b)} \frac{\beta_n}{3^n} \geq 2M + \frac{2}{3^m}.$$

The inequality (20) then follows immediately from:

$$\phi(a) + \frac{2}{3^{j+1}} \leq 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}} + \frac{2}{3^{j+1}} << 2M + \frac{1}{3^m} - \frac{1}{3^{\ell(a)}} + \frac{1}{3^j} \leq$$
$$\leq 2M + \frac{1}{3^m} < 2M + \frac{2}{3^m} \leq \phi(b) < \phi(b) + \frac{1}{3^{k+1}}.$$

In case (B), based on the fact that we have proven case (A), we can assume, without any loss of generality, that $a = b$ and $j < k$. In this case we have

$$\phi(b) + \frac{2}{3^{k+1}} = \phi(a) + \frac{2}{3^{k+1}} < \phi(a) + \frac{1}{3^k} \leq \phi(a) + \frac{1}{3^{j+1}},$$

which means that the right end-point of I_μ is not greater than the left end-point of I_λ, so again we get $I_\lambda \cap I_\mu = \varnothing$.

For the proof of (iii) we are going to use the space

$$P = \{0,1,2\}^{\aleph_0} = \big\{(\alpha_n)_{n=1}^\infty : \alpha_n \in \{0,1,2\},\ \forall\, n \in \mathbb{N}\big\}.$$

Exactly as is the case with T, the product space P is compact with respect to the product topology, which is given by the metric

$$d(a,b) = \sum_{n=1}^\infty \frac{|\alpha_n - \beta_n|}{2^n},\ \ \forall\, a = (\alpha_n)_{n=1}^\infty,\ b = (\beta_n)_{n=1}^\infty \in P.$$

Then map $\psi : P \to [0,1]$, defined by

$$\psi(a) = \sum_{n=1}^\infty \frac{\alpha_n}{3^n},\ \ \forall\, a = (\alpha_n)_{n=1}^\infty \in P,$$

satisfies

$$\big|\psi(a) - \psi(b)\big| \leq d(a,b),\ \ \forall\, a,b \in P,$$

hence it is continuous. Note also that ψ is surjective. We can write $\phi = \psi \circ \rho$, where

$$\rho : \{0,1\}^{\aleph_0} \ni (\alpha_n)_{n=1}^\infty \longmapsto (2\alpha_n)_{n=1}^\infty \in \{0,1,2\}^{\aleph_0}.$$

Note also that $\rho : T \to P$ is continuous, since we clearly have

$$d\big(\rho(a), \rho(b)\big) \leq 2d(a,b),\ \ \forall\, a,b \in T.$$

We now proceed with the proof of (iii). Denote the open set $\bigcup_{\lambda\in\Lambda} I_\lambda$ simply by D. Since T_{fin} is dense in T, it follows that $\phi(T_{\mathit{fin}})$ is dense in $K = \phi(T)$. Therefore, in order to prove the inclusion $K \subset [0,1] \smallsetminus D$, using the surjectivity of ψ, it suffices to prove the inclusion

$$\phi(T_{\mathit{fin}}) \subset [0,1] \smallsetminus D.$$

Using the map $\psi : P \to [0,1]$, the above inclusion is equivalent to

$$P \smallsetminus \rho(T_{\mathit{fin}}) \supset \psi^{-1}(D). \tag{21}$$

In order to prove the inclusion $[0,1] \smallsetminus D \subset K$, again using the surjectivity of ψ, it suffices to prove the inclusion

$$\psi^{-1}(D) \supset P \smallsetminus \psi^{-1}(K). \tag{22}$$

To prove (21) start with some element $a = (\alpha_n)_{n=1}^\infty \in \psi^{-1}(D)$, which means that there exists some $b \in T_{\mathit{fin}}$, and an integer $k \geq \ell(b)$, such that $\psi(a) \in I_{(k,b)}$, i.e.

$$\frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{1}{3^{k+1}} < \sum_{n=1}^\infty \frac{\alpha_n}{3^n} < \frac{2\beta_1}{3} + \cdots + \frac{2\beta_k}{3^k} + \frac{2}{3^{k+1}}. \tag{23}$$

We prove that $a \notin \rho(T_{\mathit{fin}})$ by contradiction. Assume $a \in \rho(T_{\mathit{fin}})$, which means that there exists $c = (\gamma_n)_{n=1}^{\infty} \in T_{\mathit{fin}}$, such that $\alpha_n = 2\gamma_n$, $\forall\, n \in \mathbb{N}$. Define the element $\tilde{b} = (\tilde{\beta}_n)_{n=1}^{\infty} \in T_{\mathit{fin}}$ by

$$\tilde{\beta}_n = \begin{cases} \beta_n & \text{if } n \leq k \\ 1 & \text{if } n = k+1 \\ 0 & \text{if } n > k+1 \end{cases}$$

With this definition, the inequalities (23) give

$$\phi(b) < \phi(b) + \frac{1}{3^{k+1}} < \phi(c) < \phi(\tilde{b}). \tag{24}$$

By Fact 2 above, there exist $N, N' \in \mathbb{N}$ such that

- $\gamma_N = 1$, $\beta_N = 0$, and $\gamma_n = \beta_n$, for all $n \in \mathbb{N}$ with $n < N$;
- $\gamma_{N'} = 0$, $\tilde{\beta}_{N'} = 1$, and $\gamma_n = \tilde{\beta}_n$, for all $n \in \mathbb{N}$ with $n < N'$.

We will examine three cases: (A) $N < N'$, (B) $N = N'$, or (C) $N > N''$.

Case (B) is clearly impossible. In case (A), the inequality $N < N'$ forces $\beta_N = 0$, $\gamma_N = 1$ and $\tilde{\beta}_N = \gamma_N$, which means that $\tilde{\beta}_N = 1 \neq \beta_N = 0$. This clearly forces $N = k+1 > \ell(b)$, which in particular gives $\beta_n = \tilde{\beta}_n = 0$, $\forall\, n > N$, so we clearly have $\gamma_n \geq \tilde{\beta}_n$, $\forall\, n \in \mathbb{N}$, so we get $\phi(c) \geq \phi(\tilde{b})$, thus contradicting (24). In case (C), we have $\gamma_{N'} = 0$, $\tilde{\beta}_{N'} = 1$, and since $N' < N$, we also have $\beta_{N'} = \gamma_{N'} = 0$. As before this would force $N' = k+1$. We then have

$$\begin{aligned} \phi(c) = 2\sum_{n=1}^{\infty} \frac{\gamma_n}{3^n} = 2\sum_{n=1}^{N'-1} \frac{\gamma_n}{3^n} + \frac{2\gamma_{N'}}{3^{N'}} + 2\sum_{n=N'+1}^{\infty} \frac{\gamma_n}{3^n} = 2\sum_{n=1}^{k} \frac{\beta_n}{3^n} + 0 + 2\sum_{n=k+2}^{\infty} \frac{\gamma_n}{3^n} = \\ = \phi(b) + 2\sum_{n=k+2}^{\infty} \frac{\gamma_n}{3^n} \leq \phi(b) + 2\sum_{n=k+1}^{\infty} \frac{1}{3^n} = \phi(b) + \frac{1}{3^{k+1}}, \end{aligned}$$

again contradicting (24).

To prove (22), we start with some element $a \in P \smallsetminus \psi^{-1}(K)$, and we show that $\psi(a) \in D$. The fact that $a \notin \psi^{-1}(K)$ forces the fact that $a \notin \rho(T)$. In particular, this gives the fact that $a = (\alpha_n)_{n=1}^{\infty} \in \{0,1,2\}^{\aleph_0}$ and there exists some $n \in \mathbb{N}$ such that $\alpha_n = 1$. Put

$$N = \min\{n \in \mathbb{N} : \alpha_n = 1\}.$$

Define the elements $b = (\beta_n)_{n=1}^{\infty} \in \{0,1\}^{\aleph_0}$, by

$$\beta_n = \begin{cases} \alpha_n/2 & \text{if } n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Notice that $b \in T_{\mathit{fin}}$, and $\ell(b) \leq N-1$. Notice also that $2\beta_n = \alpha_n$, for all $n \in \mathbb{N}$ with $n < N-1$. In particular, using the equality $\alpha_N = 1$, this gives

(25)

$$\phi(b) + \frac{1}{3^N} = 2\sum_{n=1}^{N-1} \frac{\beta_n}{3^n} + \frac{\alpha_N}{3^N} = \sum_{n=1}^{N} \frac{\alpha_n}{3^n} \leq \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \psi(a);$$

(26)

$$\phi(b) + \frac{2}{3^N} = 2\sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{\alpha_N}{3^N} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{N} \frac{\alpha_n}{3^n} + \sum_{n=N+1}^{\infty} \frac{2}{3^n} \geq \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n} = \psi(a).$$

Consider the pair $\lambda = (N-1, \beta) \in \Lambda$. We are going to show that $\psi(a) \in I_\lambda$, i.e. we have the inequalities

$$\phi(b) + \frac{1}{3^N} < \psi(a) < \phi(b) + \frac{2}{3^N}. \tag{27}$$

By (25) and (26) it suffices to prove only that

$$\psi(a) \neq \phi(b) + \frac{1}{3^N} \text{ and } \psi(a) \neq \phi(b) + \frac{2}{3^N}.$$

If $\psi(a) = \phi(b) + \frac{1}{3^N}$, then by the inequalities (25), we are forced to have

$$\alpha_n = 0, \ \forall\, n > N.. \tag{28}$$

If $\psi(a) = \phi(b) + \frac{2}{3^N}$, then by the inequalities (26), we are forced to have

$$\alpha_n = 2, \ \forall\, n > N.. \tag{29}$$

If (28) holds, we define $c = (\gamma_n)_{n=1}^\infty \in T$, by

$$\gamma_n = \begin{cases} \alpha_n/2 & \text{if } n < N \\ 0 & \text{if } n = N \\ 1 & \text{if } n > N \end{cases}$$

and we will have

$$\phi(c) = 2\sum_{n=1}^\infty \frac{\gamma_n}{3^n} = \sum_{n=1}^{N-1} \frac{2\gamma_n}{3^n} + 2\sum_{n=N+1}^\infty \frac{1}{3^n} = \sum_{n=1}^{N-1} \frac{\alpha_n}{3^n} + \frac{1}{3^N} = \psi(a),$$

thus forcing $\psi(a) \in K$, which is impossible.

If (29) holds, we define $c = (\gamma_n)_{n=1}^\infty \in T$, by

$$\gamma_n = \begin{cases} \alpha_n/2 & \text{if } n \neq N \\ 1 & \text{if } n = N \\ 0 & \text{if } n > N \end{cases}$$

and we will have

$$\phi(c) = 2\sum_{n=1}^{N-1} \frac{\gamma_n}{3^n} + \frac{2}{3^N} = \sum_{n=1}^{N-1} \frac{2\gamma_n}{3^n} + \frac{1}{3^N} + \sum_{n=N+1}^\infty \frac{2}{3^n} = \sum_{n=1}^\infty \frac{\alpha_n}{3^n} = \psi(a),$$

thus forcing again $\psi(a) \in K$, which is impossible. □

Exercise 7. Using the notations above, prove that the set

$$[0,1] \smallsetminus K_3 = \bigcup_{\lambda \in \Lambda} I_\lambda$$

is dense in $[0,1]$.

Hints: Define the set

$$P_0 = \big\{ (\alpha_n)_{n=1}^\infty \in \{0,1,2\}^{\aleph_0} \ : \ \text{the set } \{n \in \mathbb{N} \,:\, \alpha_n = 1\} \text{ is infinite} \big\}.$$

Prove that P_0 is dense in P, and prove that $\psi(P) \subset [0,1] \smallsetminus K$. (Use the arguments employed in the proof of part (iii).)

Remarks 3.5. If we set $\Lambda_n = \Lambda \cap \big(\{n\} \times P\big)$, then we can write the complement of the ternary Cantor set as

$$[0,1] \smallsetminus K_3 = \bigcup_{n=0}^\infty D_n,$$

where

$$D_n = \bigcup_{\lambda \in \Lambda_n} I_\lambda.$$

Then the system of open sets $(D_n)_{n \geq 0}$ is pair-wise disjoint. Morever, each D_n is a union of 2^n disjoint intervals of length $1/3^{n+1}$.

Since $\operatorname{card} T_0 = \mathfrak{c}$, and the map $\phi_3\big|_{T_0} : T_0 \to K_3$ is injective, we get $\operatorname{card} K_3 \geq \mathfrak{c}$. Since we also have $\operatorname{card} K_3 \leq \operatorname{card} \mathbb{R} = \mathfrak{c}$, we get in fact the equality

$$\operatorname{card} K_3 = \mathfrak{c}.$$