

LECTURE 19

2. Constructing (σ -)rings and (σ -)algebras

In this section we outline three methods of constructing (σ -)rings and (σ -)algebras. It turns out that one can devise some general procedures, which work for all the types of set collections considered, so it will be natural to begin with some very general considerations.

DEFINITION. Suppose one has a *type* Θ of set collections. In other words, for any set X , one defines what it means for a collection $\mathcal{C} \subset \mathcal{P}(X)$ to be *of type* Θ . The type Θ is said to be *consistent*, if for every set X , one has the following conditions:

- the collection $\mathcal{P}(X)$, of all subsets of X , is of type Θ ;
- if $\mathcal{C}_i, i \in I$ are collections of type Θ , then the intersection $\bigcap_{i \in I} \mathcal{C}_i$ is again of type Θ .

EXAMPLES 2.1. The following types are consistent:

- The type \mathbf{R} of *rings*;
- The type \mathbf{A} of *algebras*;
- The type \mathbf{S} of *σ -rings*;
- The type $\mathbf{\Sigma}$ of *σ -algebras*;
- The type \mathbf{M} of *monotone classes*.

The reason for the consistency is simply the fact that each of these types is defined by means of set operations.

DEFINITION. Let Θ be a consistent type, let X be a set, and let $\mathcal{E} \subset \mathcal{P}(X)$ be an arbitrary collection of sets. Define

$$\mathcal{F}_\Theta(\mathcal{E}, X) = \{\mathcal{C} \subset \mathcal{P}(X) : \mathcal{E} \subset \mathcal{C}, \text{ and } \mathcal{C} \text{ is of type } \Theta \text{ on } X\}.$$

Notice that the family $\mathcal{F}_\Theta(\mathcal{E}, X)$ is non-empty, since it contains at least the collection $\mathcal{P}(X)$. The collection

$$\Theta_X(\mathcal{E}) = \bigcap_{\mathcal{C} \in \mathcal{F}_\Theta(\mathcal{E}, X)} \mathcal{C}$$

is of type Θ on X , and is called the *type Θ class generated by \mathcal{E}* . When there is no danger of confusion, the ambient set X will be omitted.

COMMENT. In the above setting, the class $\Theta(\mathcal{E})$ is *the smallest collection of type Θ on X , which contains \mathcal{E}* . In other words, if \mathcal{C} is a collection of type Θ on X , with $\mathcal{E} \subset \mathcal{C}$, then $\mathcal{E} \subset \Theta(\mathcal{E})$. This follows immediately from the fact that \mathcal{C} belongs to $\mathcal{F}_\Theta(\mathcal{E}, X)$.

EXAMPLES 2.2. Let X be a (non-empty) set, and let \mathcal{E} be an arbitrary collection of subsets of X . According to the previous list of consistent types \mathbf{R} , \mathbf{A} , \mathbf{S} , $\mathbf{\Sigma}$, and \mathbf{M} , one can construct the following collections.

- (i) $\mathbf{R}(\mathcal{E})$, *the ring generated by \mathcal{E}* ; this is the smallest ring that contains \mathcal{E} .

- (ii) $\mathbf{A}(\mathcal{E})$, the algebra generated by \mathcal{E} ; this is the smallest algebra that contains \mathcal{E} .
- (iii) $\mathbf{S}(\mathcal{E})$, the σ -ring generated by \mathcal{E} ; this is the smallest σ -ring that contains \mathcal{E} .
- (iv) $\mathbf{\Sigma}(\mathcal{E})$, the σ -algebra generated by \mathcal{E} ; this is the smallest σ -algebra that contains \mathcal{E} .
- (v) $\mathbf{M}(\mathcal{E})$, the monotone class generated by \mathcal{E} ; this is the smallest monotone class that contains \mathcal{E} .

COMMENT. Assume Θ is a consistent type. Suppose \mathcal{E} is an arbitrary collection of subsets of some fixed non-empty set X . There are instances when we would like to decide whether a class $\mathcal{C} \supset \mathcal{E}$ coincides with $\Theta(\mathcal{E})$. The following is a useful test:

- (i) check that \mathcal{C} is of type Θ ;
- (ii) check the inclusion $\mathcal{C} \subset \Theta(\mathcal{E})$.

By (i) we must have $\mathcal{C} \supset \Theta(\mathcal{E})$, so by (ii) we will indeed have equality.

A simple illustration of the above technique allows one to describe the ring and the algebra generated by a collection of sets.

PROPOSITION 2.1. *Let X be a non-empty set, and let \mathcal{E} be an arbitrary collection of subsets of X .*

A. *For a set $A \subset X$, the following are equivalent:*

- (i) $A \in \mathbf{R}(\mathcal{E})$;
- (ii) *There exist sets A_1, A_2, \dots, A_n such that $A = A_1 \Delta A_2 \Delta \dots \Delta A_n$, and each A_k , $k = 1, \dots, n$ is a finite intersection of sets in \mathcal{E} .*

B. *The algebra generated by \mathcal{E} is*

$$\mathbf{A}(\mathcal{E}) = \mathbf{R}(\mathcal{E}) \cup \{X \setminus A : A \in \mathbf{R}(\mathcal{E})\} = \mathbf{R}(\mathcal{E} \cup \{X\}).$$

PROOF. A. Define \mathcal{R} to be the class of all subsets $A \subset X$, which satisfy property (ii), so that what we have to prove is the equality

$$\mathcal{R} = \mathbf{R}(\mathcal{E}).$$

It is clear that $\mathcal{E} \subset \mathcal{R}$. Since every finite intersection of sets in \mathcal{E} belongs to $\mathbf{R}(\mathcal{E})$, and the latter is a ring, it follows that $\mathcal{R} \subset \mathbf{R}(\mathcal{E})$. So in order to prove the desired equality, all we have to do is to prove that \mathcal{R} is a ring. But this is pretty clear, if we think Δ as the sum operation, and \cap as the product operation. More explicitly, let us take $\Pi(\mathcal{E})$ to be the collection of all finite intersections of sets in \mathcal{E} , so that

$$(1) \quad A \cap B \in \Pi(\mathcal{E}), \quad \forall A, B \in \Pi(\mathcal{E}).$$

Now if we start with two sets $A, B \in \mathcal{R}$, written as $A = A_1 \Delta \dots \Delta A_m$ and $B = B_1 \Delta \dots \Delta B_n$, with $A_1, \dots, A_m, B_1, \dots, B_n \in \Pi(\mathcal{E})$, then the equality

$$A \cap B = [(A_1 \cap B_1) \Delta \dots \Delta (A_m \cap B_1)] \Delta [(A_1 \cap B_2) \Delta \dots \Delta (A_m \cap B_2)] \Delta \dots \\ \dots \Delta [(A_1 \cap B_n) \Delta \dots \Delta (A_m \cap B_n)],$$

combined with (1) proves that $A \cap B \in \mathcal{R}$, while the equality

$$A \Delta B = A_1 \Delta \dots \Delta A_m \Delta B_1 \Delta \dots \Delta B_n$$

proves that $A \Delta B$ also belongs to \mathcal{R} .

B. Define

$$\mathcal{A} = \mathbf{R}(\mathcal{E}) \cup \{X \setminus A : A \in \mathbf{R}(\mathcal{E})\}.$$

Since we clearly have $\mathcal{E} \subset \mathcal{A} \subset \mathbf{A}(\mathcal{E})$, all we need to prove is the fact that \mathcal{A} is an algebra. It is clear that, whenever $A \in \mathcal{A}$, it follows that $X \setminus A \in \mathcal{A}$. Therefore (see Section III.1), we only need to show that

$$A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}.$$

There are four cases to examine: (i) $A, B \in \mathbf{R}(\mathcal{E})$; (ii) $A \in \mathbf{R}(\mathcal{E})$ and $X \setminus B \in \mathbf{R}(\mathcal{E})$; (iii) $X \setminus A \in \mathbf{R}(\mathcal{E})$ and $B \in \mathbf{R}(\mathcal{E})$; (iv) $X \setminus A \in \mathbf{R}(\mathcal{E})$ and $X \setminus B \in \mathbf{R}(\mathcal{E})$.

Case (i) is clear, since it will force $A \cup B \in \mathbf{R}(\mathcal{E})$.

In case (ii), we use

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) = (X \setminus B) \setminus A,$$

which proves that $X \setminus (A \cup B) \in \mathbf{R}(\mathcal{E})$.

Case (iii) is proven exactly as case (ii).

In case (iv) we use

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B),$$

which proves that $X \setminus (A \cup B) \in \mathbf{R}(\mathcal{E})$.

The equality $\mathbf{A}(\mathcal{E}) = \mathbf{R}(\mathcal{E} \cup \{X\})$ is trivial. \square

COMMENT. Unfortunately, for σ -rings and σ -algebras, no easy constructive description is available. There is an analogue of Proposition 2.1 uses transfinite induction. In order to formulate such a statement, we introduce the following notations. For every collection \mathcal{C} of subsets of X , we define

$$\mathcal{C}^* = \left\{ \bigcup_{n=1}^{\infty} (A_n \setminus B_n) : A_n, B_n \in \mathcal{C} \cup \{\emptyset\}, \forall n \geq 1 \right\}.$$

Notice that

$$(2) \quad \mathcal{C} \cup \{\emptyset\} \subset \mathcal{C}^* \subset \mathbf{S}(\mathcal{C}).$$

THEOREM 2.1. *Let X be a non-empty set, and let \mathcal{E} be an arbitrary collection of subsets of X . For every ordinal number η define the set*

$$P_\eta = \{\alpha : \alpha \text{ ordinal number with } \alpha < \eta\}.$$

Let Ω denote the smallest uncountable ordinal number, and define the classes \mathcal{E}_α , $\alpha \in P_\Omega$ recursively by $\mathcal{E}_0 = \mathcal{E}$, and

$$\mathcal{E}_\alpha = \left(\bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta \right)^*, \quad \forall \alpha \in P_\Omega \setminus \{0\}.$$

Then the σ -ring generated by \mathcal{E} is

$$\mathbf{S}(\mathcal{E}) = \bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha.$$

PROOF. Denote the union $\bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha$ simply by \mathcal{U} . It is obvious that $\mathcal{E} \subset \mathcal{U}$.

Let us prove that $\mathcal{U} \subset \mathbf{S}(\mathcal{E})$. We do this by showing that $\mathcal{E}_\alpha \subset \mathbf{S}(\mathcal{E})$, $\forall \alpha \in P_\Omega$. We use transfinite induction. The case $\alpha = 0$ is clear. Assume $\alpha \in P_\Omega$ has the property that $\mathcal{E}_\beta \subset \mathbf{S}(\mathcal{E})$, for all $\beta \in P_\alpha$, and let us show that we also have the inclusion $\mathcal{E}_\alpha \subset \mathbf{S}(\mathcal{E})$. On the one hand, if we take the class

$$\mathcal{C} = \bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta,$$

then $\mathcal{E}_\alpha = \mathcal{C}^*$. On the other hand, by the inductive hypothesis, we have $\mathcal{C} \subset \mathbf{S}(\mathcal{E})$, which clearly forces $\mathbf{S}(\mathcal{C}) \subset \mathbf{S}(\mathcal{E})$. Then the desired inclusion follows from (2)

In order to finish the proof, we only need to prove that \mathcal{U} is a σ -ring. It suffices to prove the equality $\mathcal{U}^* = \mathcal{U}$, which in turn is equivalent to the inclusion $\mathcal{U}^* \subset \mathcal{U}$. Start with some $U \in \mathcal{U}^*$, written as

$$U = \bigcup_{n=1}^{\infty} (A_n \setminus B_n),$$

for two sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ in \mathcal{U} . For each $n \geq 1$ choose $\alpha_n, \beta_n \in P_\Omega$, such that $A_n \in \mathcal{E}_{\alpha_n}$ and $B_n \in \mathcal{E}_{\beta_n}$. Form then the countable set

$$Z = \{\alpha_n : n \in \mathbb{N}\} \cup \{\beta_n : n \in \mathbb{N}\} \subset P_\Omega.$$

Then we clearly have

$$U \in \left(\bigcup_{\nu \in Z} \mathcal{E}_\nu \right)^*.$$

Since Z is countable, there is a strict upper bound for Z in P_Ω , i.e. there exists $\gamma \in P_\Omega$, such that $\alpha_n < \gamma$ and $\beta_n < \gamma$, $\forall n \geq 1$. In other words we have $Z \subset P_\gamma$, so

$$U \in \left(\bigcup_{\nu \in P_\gamma} \mathcal{E}_\nu \right)^* = \mathcal{E}_\gamma,$$

so U indeed belongs to \mathcal{U} . □

COROLLARY 2.1. *Given a non-empty set X , and an arbitrary collection \mathcal{E} of subsets of X , with $\text{card } \mathcal{E} \geq 2$, one has the inequality*

$$\text{card } \mathbf{S}(\mathcal{E}) \leq (\text{card } \mathcal{E})^{\aleph_0}.$$

PROOF. Using the notations from the proof of the above theorem, we will first prove, by transfinite induction, that

$$(3) \quad \text{card } \mathcal{E}_\alpha \leq (\text{card } \mathcal{E})^{\aleph_0}, \quad \forall \alpha \in P_\Omega.$$

The case $\alpha = 0$ is clear. Assume now we have $\alpha \in P_\Omega \setminus \{0\}$, such that

$$\text{card } \mathcal{E}_\beta \leq (\text{card } \mathcal{E})^{\aleph_0}, \quad \forall \beta \in P_\alpha,$$

and let us prove that we also have the inequality $\text{card } \mathcal{E}_\alpha \leq (\text{card } \mathcal{E})^{\aleph_0}$. If we take $\mathcal{C} = \bigcup_{\beta \in P_\alpha} \mathcal{E}_\beta$, we know that \mathcal{C} is a countable union of sets, each having cardinality $\leq (\text{card } \mathcal{E})^{\aleph_0}$, so we immediately get

$$\text{card } \mathcal{C} \leq \aleph_0 \cdot (\text{card } \mathcal{E})^{\aleph_0} = (\text{card } \mathcal{E})^{\aleph_0}.$$

Then the collection

$$D(\mathcal{C}) = \{A \setminus B : A, B \in \mathcal{C}\}$$

has cardinality at most $(\text{card } \mathcal{C})^2$, so we also have

$$\text{card } D(\mathcal{C}) \leq (\text{card } \mathcal{E})^{\aleph_0}.$$

Finally, the collection $\mathcal{E}_\alpha = \mathcal{C}^*$ has cardinality at most $(\text{card } D(\mathcal{C}))^{\aleph_0}$, so we get

$$\text{card } \mathcal{E}_\alpha \leq [(\text{card } \mathcal{E})^{\aleph_0}]^{\aleph_0} = (\text{card } \mathcal{E})^{\aleph_0}.$$

Having proven (3), we now have

$$\text{card } \mathbf{S}(\mathcal{E}) = \text{card} \left(\bigcup_{\alpha \in P_\Omega} \mathcal{E}_\alpha \right) \leq (\text{card } P_\Omega) \cdot (\text{card } \mathcal{E})^{\aleph_0} = \aleph_1 \cdot (\text{card } \mathcal{E})^{\aleph_0}.$$

Since $\aleph_1 \leq \mathfrak{c} = 2^{\aleph_0} \leq (\text{card } \mathcal{E})^{\aleph_0}$, the above estimate gives

$$\text{card } \mathbf{S}(\mathcal{E}) \leq [(\text{card } \mathcal{E})^{\aleph_0}]^2 = (\text{card } \mathcal{E})^{\aleph_0}. \quad \square$$

COMMENT. Suppose Θ is a consistent type. There is a very useful technique for proving results on classes of the form $\Theta(\mathcal{E})$. More explicitly, suppose \mathcal{E} is an arbitrary collection of subsets of X , and (P) is a certain property which refers to subsets of X . Suppose now we want to prove a statement like:

(*) *Every set $A \in \Theta(\mathcal{E})$ has property (P).*

In order to prove such a statement, one defines

$$\mathcal{U} = \{A \in \Theta(\mathcal{E}) : A \text{ has property (P)}\},$$

and it suffices to prove that:

- (i) \mathcal{U} is of type Θ ;
- (ii) $\mathcal{U} \supset \mathcal{E}$, i.e. every set $A \in \mathcal{E}$ has property (P).

Indeed, if we prove the above two facts, that would force $\mathcal{U} \supset \Theta(\mathcal{E})$, and since by construction we have $\mathcal{U} \supset \Theta(\mathcal{E})$, we will in fact get $\mathcal{U} = \Theta(\mathcal{E})$, thus proving (*).

As a first illustration of the above technique, we prove the following.

PROPOSITION 2.2. *Let X be a non-empty set, and let \mathcal{R} be a ring on X . Then the σ -ring generated by \mathcal{R} is the same as the monotone class generated by \mathcal{R} , that is, one has the equality*

$$\mathbf{S}(\mathcal{R}) = \mathbf{M}(\mathcal{R}).$$

PROOF. Since $\mathbf{S}(\mathcal{R})$ is a monotone class, and contains \mathcal{R} , we have the inclusion $\mathbf{S}(\mathcal{R}) \supset \mathbf{M}(\mathcal{R})$.

To prove the other inclusion, using the fact that $\mathbf{M}(\mathcal{R})$ contains \mathcal{R} , it suffices to show that $\mathbf{M}(\mathcal{R})$ is a σ -ring. Since $\mathbf{M}(\mathcal{R})$ is already a monotone class, we only need to show that it is a ring. In other words, we need to show that whenever $A, B \in \mathbf{M}(\mathcal{R})$, it follows that both $A \setminus B$ and $A \cup B$ belong to $\mathbf{M}(\mathcal{R})$. Define then, for every $A \in \mathbf{M}(\mathcal{R})$ the set

$$\mathcal{M}_A = \{B \in \mathbf{M}(\mathcal{R}) : A \cap B, A \setminus B, B \setminus A \in \mathbf{M}(\mathcal{R})\},$$

so that what we need to prove is:

(*) $\mathcal{M}_A = \mathbf{M}(\mathcal{A}), \forall A \in \mathbf{M}(\mathcal{R})$.

Before we proceed with the proof of (*), let us first remark that, for $A, B \in \mathbf{M}(\mathcal{R})$, one has

$$(4) \quad B \in \mathcal{M}_A \iff A \in \mathcal{M}_B.$$

Secondly, we have the following

Claim 1: For every $A \in \mathbf{M}(\mathcal{R})$, the collection \mathcal{M}_A is a monotone class.

To prove this, we start with a monotone sequence $(B_n)_{n=1}^\infty$ in \mathcal{M}_A , and we prove that the limit $B = \lim_{n \rightarrow \infty} B_n$ again belongs to \mathcal{M}_A . First of all, clearly B belongs to $\mathbf{M}(\mathcal{R})$. Second, since the sequences $(A \cap B_n)_{n=1}^\infty$, $(A \setminus B_n)_{n=1}^\infty$, and $(B_n \setminus A)_{n=1}^\infty$ are all monotone sequences in $\mathbf{M}(\mathcal{R})$, and since $\mathbf{M}(\mathcal{R})$ is a monotone class, it follows

that the limits $A \cap B = \lim_{n \rightarrow \infty} (A \cap B_n)$, $A \setminus B = \lim_{n \rightarrow \infty} (A \setminus B_n)$, and $B \setminus A = \lim_{n \rightarrow \infty} (B_n \setminus A)$ all belong to $\mathbf{M}(\mathcal{R})$, so B indeed belongs to \mathcal{M}_A .

Having proven Claim 1, we now prove (*) in a particular case:

Claim 2: $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$, $\forall A \in \mathcal{R}$.

Fix $A \in \mathcal{R}$. We know that $\mathcal{M}_A \subset \mathbf{M}(\mathcal{R})$ is a monotone class, so it suffices to prove that $\mathcal{M}_A \supset \mathcal{R}$. But this is obvious, since \mathcal{R} is a ring.

We now proceed with the proof of (*) in the general case. If we define

$$\mathcal{U} = \{A \in \mathbf{M}(\mathcal{R}) : \mathcal{M}_A = \mathbf{M}(\mathcal{R})\},$$

all we need to prove is the equality $\mathcal{U} = \mathbf{M}(\mathcal{R})$. By Claim 2, we know that $\mathcal{U} \supset \mathcal{R}$, so it suffices to prove that \mathcal{U} is a monotone class. Start then with a monotone sequence $(A_n)_{n=1}^{\infty}$, and let us show that the limit $A = \lim_{n \rightarrow \infty} A_n$ again belongs to \mathcal{U} . First of all, A belongs to $\mathbf{M}(\mathcal{R})$. What we then have to prove is that $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$. Start with some arbitrary $B \in \mathbf{M}(\mathcal{R})$. We know that $B \in \mathcal{M}_{A_n}$, $\forall n \geq 1$. Using (4) we have $A_n \in \mathcal{M}_B$, $\forall n \geq 1$, and using the fact that \mathcal{M}_B is a monotone class (see Claim 1), it follows that $A = \lim_{n \rightarrow \infty} A_n$ belongs to \mathcal{M}_A . Using (4) again, this gives $B \in \mathcal{M}_A$. This way we have proven that any $B \in \mathbf{M}(\mathcal{R})$ also belongs to \mathcal{M}_A , so we indeed have the equality $\mathcal{M}_A = \mathbf{M}(\mathcal{R})$. \square

COROLLARY 2.2. *Let X be a non-empty set, and let \mathcal{E} be an arbitrary family of subsets of X . Then the σ -ring, and the σ -algebra generated by \mathcal{E} respectively, are given as the monotone classes generated by the ring, and by the algebra generated by \mathcal{E} respectively. That is, one has the equalities:*

- (i) $\mathbf{S}(\mathcal{E}) = \mathbf{M}(\mathbf{R}(\mathcal{E}))$;
- (ii) $\Sigma(\mathcal{E}) = \mathbf{M}(\mathbf{A}(\mathcal{E}))$.

PROOF. (i). By the above result, since $\mathbf{R}(\mathcal{E})$ is a ring, we have

$$(5) \quad \mathbf{M}(\mathbf{R}(\mathcal{E})) = \mathbf{S}(\mathbf{R}(\mathcal{E})).$$

Since $\mathbf{S}(\mathbf{R}(\mathcal{E}))$ is a σ -ring, and contains \mathcal{E} , it follows that

$$\mathbf{S}(\mathbf{R}(\mathcal{E})) \supset \mathbf{S}(\mathcal{E}).$$

Conversely, since $\mathbf{S}(\mathcal{E})$ is a ring, and contains \mathcal{E} , we get the inclusion

$$\mathbf{S}(\mathcal{E}) \supset \mathbf{R}(\mathcal{E}),$$

and since $\mathbf{S}(\mathcal{E})$ is a σ -ring, we will now get

$$\mathbf{S}(\mathcal{E}) \supset \mathbf{S}(\mathbf{R}(\mathcal{E})),$$

so we get

$$\mathbf{S}(\mathbf{R}(\mathcal{E})) = \mathbf{S}(\mathcal{E}).$$

Using (5), the desired equality follows.

(ii). This follows from Proposition 2.1 and part (i) applied to $\mathcal{E} \cup \{X\}$, combined with the obvious equality $\Sigma(\mathcal{E}) = \mathbf{S}(\mathcal{E} \cup \{X\})$. \square

The σ -ring and the σ -algebra, generated by an arbitrary collection of sets, are related by means of the following result.

PROPOSITION 2.3. *Let X be a non-empty set, and let \mathcal{E} be an arbitrary collection of subsets of X . Define the collection*

$$\mathcal{P}_\sigma^\mathcal{E}(X) = \left\{ A \subset X : \text{there exists } (E_n)_{n=1}^\infty \subset \mathcal{E}, \text{ with } A \subset \bigcup_{n=1}^\infty E_n \right\}.$$

- (i) $\mathcal{P}_\sigma^\mathcal{E}(X)$ is a σ -ring on X ;
- (ii) the σ -ring $\mathbf{S}(\mathcal{E})$ and the σ -algebra $\mathbf{\Sigma}(\mathcal{E})$, generated by \mathcal{E} , satisfy the equality

$$\mathbf{S}(\mathcal{E}) = \mathbf{\Sigma}(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X).$$

PROOF. Part (i) is trivial.

To prove part (ii), we first observe that the intersection $\mathbf{\Sigma}(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$ is a σ -ring, which obviously contains \mathcal{E} , so we immediately get the inclusion

$$\mathbf{S}(\mathcal{E}) \subset \mathbf{\Sigma}(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X).$$

The key ingredient in proving the inclusion “ \supset ” is contained in the following.

Claim: Given a set $E \in \mathcal{E}$, the collection

$$\mathcal{A}_E(X) = \left\{ A \subset X : A \cap E \in \mathbf{S}(\mathcal{E}) \right\}$$

is a σ -algebra on X .

To prove this we need to check:

- (a) if A belongs to $\mathcal{A}_E(X)$, then $X \setminus A$ also belongs to $\mathcal{A}_E(X)$;
- (b) whenever $(A_n)_{n=1}^\infty$ is a sequence of sets in $\mathcal{A}_E(X)$, it follows that the union $\bigcup_{n=1}^\infty A_n$ also belongs to $\mathcal{A}_E(X)$.

To check (a) we simply remark that, since both E and $A \cap E$ belong to $\mathbf{S}(\mathcal{E})$, it follows immediately that $(X \setminus A) \cap E = E \setminus (A \cap E)$, also belongs to $\mathbf{S}(\mathcal{E})$, which means that $X \setminus A$ indeed belongs to $\mathcal{A}_E(X)$.

Property (b) is clear. Since the fact that $A_n \cap E$ belongs to $\mathbf{S}(\mathcal{E})$, for all n , immediately gives the fact that $(\bigcup_{n=1}^\infty A_n) \cap E = \bigcup_{n=1}^\infty (A_n \cap E)$ belongs to $\mathbf{S}(\mathcal{E})$, which means precisely that $\bigcup_{n=1}^\infty A_n$ belongs to \mathcal{A}_E .

Having proven the Claim, we now proceed with the proof of the inclusion $\mathbf{S}(\mathcal{E}) \supset \mathbf{\Sigma}(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$. Start with some set $A \in \mathbf{\Sigma}(\mathcal{E}) \cap \mathcal{P}_\sigma^\mathcal{E}(X)$, and we will show that A belongs to $\mathbf{S}(\mathcal{E})$. First of all, there exists a sequence $(E_n)_{n=1}^\infty \subset \mathcal{E}$, such that

$$(6) \quad A \subset \bigcup_{n=1}^\infty E_n.$$

Using the Claim, we know that for each $n \in \mathbb{N}$, the collection \mathcal{A}_{E_n} is a σ -algebra. This σ -algebra clearly contains \mathcal{E} , so we have

$$\mathbf{\Sigma}(\mathcal{E}) \subset \mathcal{A}_{E_n}, \quad \forall n \in \mathbb{N}.$$

In particular, we get the fact that $A \in \mathcal{A}_{E_n}$, which means that $A \cap E_n$ belongs to $\mathbf{S}(\mathcal{E})$, for all $n \in \mathbb{N}$. But then the inclusion (6) forces the equality

$$A = \bigcup_{n=1}^\infty (A \cap E_n),$$

which then gives the fact that A indeed belongs to $\mathbf{S}(\mathcal{E})$. □

The above result motivates the following.

DEFINITION. A collection \mathcal{E} of subsets of X is said to be σ -total in X , if $X \in \mathcal{P}_\sigma^\mathcal{E}(X)$, i.e. there exists some sequence $(E_n)_{n=1}^\infty \subset \mathcal{E}$ with $\bigcup_{n=1}^\infty E_n = X$. By the above result, this is equivalent to the fact that X belongs to the σ -ring $\mathbf{S}(\mathcal{E})$ generated by \mathcal{E} , which in turn is equivalent to the equality $\Sigma(\mathcal{E}) = \mathbf{S}(\mathcal{E})$.

We discuss now two more methods of constructing (σ) -rings, (σ) -algebras, or monotone classes.

NOTATIONS. Let $f : X \rightarrow Y$ be a function, and let $\mathcal{E} \subset \mathcal{P}(X)$ and $\mathcal{G} \subset \mathcal{P}(Y)$ be two arbitrary collections of sets. We define

$$\begin{aligned} f_*\mathcal{E} &= \{A \in \mathcal{P}(Y) : f^{-1}(A) \in \mathcal{E}\} \subset \mathcal{P}(Y); \\ f^*\mathcal{G} &= \{f^{-1}(G) : G \in \mathcal{G}\} \subset \mathcal{P}(X). \end{aligned}$$

DEFINITIONS. Let Θ be a type of set collections. We say that Θ is *natural*, if for any map $f : X \rightarrow Y$, one has the implications

- (i) \mathcal{C} of type Θ on $X \implies f_*\mathcal{C}$ of type Θ on Y ;
- (ii) \mathcal{D} of type Θ on $Y \implies f^*\mathcal{D}$ of type Θ on X .

EXAMPLES 2.3. The types \mathbf{R} , \mathbf{A} , \mathbf{S} , Σ , and \mathbf{M} are natural.

The term “natural” is justified by the following.

Exercise 1. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be maps.

- (i) Prove that, for any collection $\mathcal{C} \subset \mathcal{P}(X)$, one has the equality $g_*(f_*\mathcal{C}) = (g \circ f)_*\mathcal{C}$.
- (ii) Prove that, for any collection $\mathcal{D} \subset \mathcal{P}(Y)$, one has the equality $f^*(g^*\mathcal{D}) = (g \circ f)^*\mathcal{D}$.

THEOREM 2.2 (Generating Theorem). *Suppose Θ is a consistent class type, which is natural. Let X and Y be non-empty sets, and let $f : X \rightarrow Y$ be a map. For any collection $\mathcal{G} \subset \mathcal{P}(Y)$, one has the equality*

$$f^*\Theta(\mathcal{G}) = \Theta(f^*\mathcal{G}).$$

PROOF. On the one hand, by naturality, we know that $f^*\Theta(\mathcal{G})$ is of type Θ . On the other hand, it is pretty clear that, since $\Theta(\mathcal{G}) \supset \mathcal{G}$, we also have the inclusion $f^*\Theta(\mathcal{G}) \supset f^*\mathcal{G}$. Since Θ is consistent, it then follows that we have the inclusion

$$f^*\Theta(\mathcal{G}) \supset \Theta(f^*\mathcal{G}).$$

To prove the other inclusion, we consider the class

$$\mathcal{C} = f_*[\Theta(f^*\mathcal{G})] \subset \mathcal{P}(Y).$$

By naturality, it follows that \mathcal{C} is of type Θ on Y . For any $G \in \mathcal{G}$, the obvious relation

$$f^{-1}(G) \in f^*\mathcal{G} \subset \Theta(f^*\mathcal{G})$$

proves that $G \in \mathcal{C}$. This means that we have the inclusion $\mathcal{C} \supset \mathcal{G}$, and since \mathcal{C} is of class Θ , it follows that we have the inclusion

$$\Theta(\mathcal{G}) \subset \mathcal{C}.$$

This means that, for every $A \in \Theta(\mathcal{G})$, we have $f^{-1}(A) \in \Theta(f^*\mathcal{G})$, which means precisely that we have the desired inclusion

$$f^*\Theta(\mathcal{G}) \subset \Theta(f^*\mathcal{G}). \quad \square$$

EXAMPLE 2.4. Let Θ be a consistent class type, which is both covariant and contravariant. Let Y be some set, and let \mathcal{C} be a collection of type Θ on Y . Given a subset $X \subset Y$, we consider the inclusion map $\iota : X \hookrightarrow Y$. The collection $\iota^*\mathcal{C}$ is then of type Θ on X . It will be denoted by $\mathcal{C}|_X$. Since $\iota^{-1}A = A \cap X, \forall A \in \mathcal{P}(Y)$, we have

$$\mathcal{C}|_X = \{A \cap X : A \in \mathcal{C}\}.$$

If $\mathcal{E} \subset \mathcal{P}(Y)$ is a collection with $\mathcal{C} = \Theta(\mathcal{E})$, then by the Generating Theorem we have the equality

$$(7) \quad \Theta(\mathcal{E})|_X = \Theta(\{E \cap X : E \in \mathcal{E}\}).$$

COMMENT. The exercise below shows that a “forward” version of the Generating Theorem does not hold in general. In other words, an equality of the type $f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G})$ may fail. The reason is the fact that the collection $f_*\mathcal{G}$ may be relatively “small.”

Exercise 2. Consider the sets $X = \{1, 2, 3\}, Y = \{1, 2\}$, the function $f : X \rightarrow Y$, defined by $f(1) = f(2) = 1, f(3) = 2$, and the collection $\mathcal{C} = \{\{1\}, \{2\}, \emptyset\}$. Describe the collection $f_*\mathcal{C}$, the algebra $\mathbf{A}(\mathcal{C})$ generated by \mathcal{C} (on X), and the algebra $\mathbf{A}(f_*\mathcal{C})$ generated by $f_*\mathcal{C}$ (on Y). Prove that one has a strict inclusion $\mathbf{A}(f_*\mathcal{C}) \subsetneq f_*\mathbf{A}(\mathcal{C})$.

Exercise 3. Let Θ be a consistent natural type, let $f : X \rightarrow Y$ be a *surjective* map, and let \mathcal{G} be a collection of subsets of X . Assume one has the inclusion

$$(8) \quad \mathcal{G} \subset f^*\Theta(f_*\mathcal{G}).$$

Prove that one has the equality

$$f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G}).$$

(One instance when (8) holds is for example when $f^{-1}(f(G)) = G, \forall G \in \mathcal{G}$.)

Exercise 4.* Let Θ be one of the types $\mathbf{A}, \mathbf{R}, \mathbf{S}, \mathbf{\Sigma},$ or \mathbf{M} . Let $f : X \rightarrow Y$ be an *injective map*, and let $\mathcal{G} \subset \mathcal{P}(X)$ be some arbitrary collection. Prove the equality

$$f_*\Theta(\mathcal{G}) = \Theta(f_*\mathcal{G}).$$

Natural consistent types are useful, because it is possible to construct product structures.

DEFINITION. Let Θ be a consistent type which is natural. Let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{E}_i \subset \mathcal{P}(X_i)$ of type Θ is given. Consider the product cartesian product $X = \prod_{i \in I} X_i$, together with the projection maps $\pi_i : X \rightarrow X_i, i \in I$. The collection

$$\Theta\text{-}\bigtimes_{i \in I} \mathcal{E}_i = \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{E}_i\right)$$

is a collection of type Θ on X , which is called the Θ -*product*. When there is no danger of confusion, we use the notation \bigtimes .

REMARK 2.1. Use the notations from the above definition. Assume that, for each $i \in I$, a collection $\mathcal{G}_i \subset \mathcal{P}(X_i)$ is given. Then one has the equality

$$\bigtimes_{i \in I} \Theta(\mathcal{G}_i) = \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right).$$

Indeed, if we define $\mathcal{E}_i = \Theta(\mathcal{G}_i)$, the inclusion \supset follows from the obvious inclusions

$$\bigtimes_{i \in I} \mathcal{E}_i \supset \pi_i^* \mathcal{E}_i \supset \pi_i^* \mathcal{G}_i.$$

The inclusion \subset , follows from the inclusions

$$\pi_i^* \mathcal{G}_i \subset \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right),$$

which combined with the fact that the right hand side is of type Θ , and the Generating Theorem, give the inclusions

$$\pi_i^* \mathcal{E}_i = \pi_i^* \Theta(\mathcal{G}_i) = \Theta(\pi_i^* \mathcal{G}_i) \subset \Theta\left(\bigcup_{i \in I} \pi_i^* \mathcal{G}_i\right).$$

Natural consistent types also allow one to define disjoint union structures.

DEFINITIONS. Let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{C}_i \subset \mathcal{P}(X_i)$ is given. On the disjoint union $X = \bigsqcup_{i \in I} X_i$ one defines the collection

$$\bigvee_{i \in I} \mathcal{C}_i = \{C \subset X : C \cap X_i \in \mathcal{C}_i, \forall i \in I\}.$$

Assume now Θ is a natural consistent, and \mathcal{C}_i is of type Θ on X_i , for each $i \in I$. If we consider the inclusion maps $\epsilon_i : X_i \rightarrow X$, $i \in I$, then one clearly has the equality

$$\bigvee_{i \in I} \mathcal{C}_i = \bigcap_{i \in I} \epsilon_{i*} \mathcal{C}_i,$$

which means that $\bigvee_{i \in I} \mathcal{C}_i$ is a collection of type Θ on X .

Exercise 5. Let I be countable, and let $(X_i)_{i \in I}$ be a collection of non-empty sets. Assume that, for each $i \in I$, a collection $\mathcal{C}_i \subset \mathcal{P}(X_i)$ is given, such that $\emptyset \in \mathcal{C}_i$. Prove the equalities

$$\bigvee_{i \in I} \mathbf{S}(\mathcal{C}_i) = \mathbf{S}\left(\bigvee_{i \in I} \mathcal{C}_i\right) \text{ and } \bigvee_{i \in I} \mathbf{\Sigma}(\mathcal{C}_i) = \mathbf{\Sigma}\left(\bigvee_{i \in I} \mathcal{C}_i\right).$$

We conclude with a discussion on certain constructions related to topology.

DEFINITIONS. Let X be a topological Hausdorff space. We consider the collection \mathcal{T} of all open sets in X . The σ -algebra $\mathbf{\Sigma}(\mathcal{T})$ on X , generated by \mathcal{T} is denoted by $Bor(X)$. The sets in $Bor(X)$ are called *Borel sets*.

Remark that singleton sets are Borel, since they are closed. Moreover

- every countable set $B \subset X$ is Borel.

One also defines the σ -algebra $Bor_c(X) = \mathbf{\Sigma}(\mathcal{C}_X)$ generated by the class \mathcal{C}_X of all compact subsets of X .

Another class of sets will also be of interest. Its construction uses the following terminology.

A subset $A \subset X$ is said to be σ -compact, if there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of X , such that $A = \bigcup_{n=1}^{\infty} K_n$. A set $B \subset X$ is said to be *relatively σ -compact*, if there exists a σ -compact set A with $B \subset A$. We set

$$\mathcal{P}_{\sigma c}(X) = \{B \in \mathcal{P}(X) : B \text{ relatively } \sigma\text{-compact}\},$$

and we define

$$Bor_{\sigma c}(X) = Bor(X) \cap \mathcal{P}_{\sigma c}(X).$$

PROPOSITION 2.4. *Let X be a topological Hausdorff space.*

- (i) $\mathcal{P}_{\sigma c}(X)$ is a σ -ring on X ;
- (ii) the σ -ring $Bor_{\sigma c}(X)$ coincides with the σ -ring $\mathbf{S}(\mathcal{C}_X)$ generated by the collection \mathcal{C}_X of all compact subsets of X .

PROOF. Using the notations from Proposition 2.3, we have $\mathcal{P}_{\sigma c}(X) = \mathcal{P}_{\sigma}^{\mathcal{C}_X}(X)$, so part (i) is a consequence of Proposition 2.3.(i). By Proposition 2.3.(ii) we also know that

$$\mathbf{S}(\mathcal{C}_X) = \Sigma(\mathcal{C}_X) \cap \mathcal{P}_{\sigma c}(X) = Bor_c(X) \cap \mathcal{P}_{\sigma c}(X),$$

and since $Bor_c(X) \subset Bor(X)$, we have the inclusion

$$\mathbf{S}(\mathcal{C}_X) \subset Bor_{\sigma c}(X).$$

To prove the other inclusion, all we need to show is the inclusion

$$Bor_{\sigma c}(X) \subset Bor_c(X).$$

Start with some arbitrary set $B \in Bor_{\sigma c}(X)$, and let us prove that $B \in Bor_c(X)$. Since B is relatively σ -compact, there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact sets, such that $B \subset \bigcup_{n=1}^{\infty} K_n$. Define, for each integer $n \geq 1$, the set $B_n = B \cap K_n$. Since $B = \bigcup_{n=1}^{\infty} B_n$, It suffices to show that

$$(9) \quad B_n \in Bor_c(X), \quad \forall n \in \mathbb{N}.$$

Fix n , and let us analyze the inclusion $\iota_n : K_n \hookrightarrow X$. Denote by \mathcal{T} the collection of all open sets in X , and denote by \mathcal{T}_{K_n} the collection of all sets $D \subset K_n$, which are open in the induced topology, that is,

$$\mathcal{T}_{K_n} = \{D \cap K_n : D \in \mathcal{T}\}.$$

By the Generating Theorem (Example 2.4), we know that

$$Bor(X)|_{K_n} = \Sigma(\mathcal{T})|_{K_n} = \Sigma_{K_n}(\{D \cap K_n : D \in \mathcal{T}\}) = \Sigma_{K_n}(\mathcal{T}_{K_n}) = Bor(K_n).$$

(Here the notation Σ_{K_n} indicates that the σ -algebra is taken on K_n .) In particular, we get

$$(10) \quad B_n = B \cap K_n \in Bor(X)|_{K_n} = Bor(K_n), \quad \forall n \in \mathbb{N}.$$

Since K_n is compact, the σ -ring $\mathbf{S}(\mathcal{C}_{K_n})$, generated by all compact subsets of K_n , is a σ -algebra on K_n (simply because it contains K_n .) Notice that every set $D \in \mathcal{T}_{K_n}$ is of the form $K_n \setminus F$, with $F \subset K_n$ compact (in X), therefore D belongs to $\mathbf{S}(\mathcal{C}_{K_n})$. Since $\mathbf{S}(\mathcal{C}_X)$ is a σ -algebra, which contains \mathcal{T}_{K_n} , we have

$$Bor(K_n) = \Sigma_{K_n}(\mathcal{T}_{K_n}) \subset \mathbf{S}(\mathcal{C}_{K_n}) \subset \mathbf{S}(\mathcal{C}_X) \subset Bor_c(X), \quad \forall n \in \mathbb{N}.$$

Now (9) immediately follows from the above inclusions, combined with (10). \square

REMARK 2.2. For a topological Hausdorff space, we always have the inclusions

$$Bor_{\sigma c}(X) \subset Bor_c(X) \subset Bor(X).$$

The following are equivalent

- (i) $Bor_{\sigma c}(X) = Bor(X)$;
- (ii) X is σ -compact.

The following result explains when a minimal set of generators can be chosen for the Borel sets.

PROPOSITION 2.5. *Let X be a topological space which is second countable, i.e. there is a countable base for the topology. If \mathcal{S} is any sub-base for the topology (countable or not), then*

$$\text{Bor}(X) = \Sigma(\mathcal{S}).$$

PROOF. Denote by \mathcal{T} the collection of all open sets in X . Denote by \mathcal{V} the collection of all subsets of X , which can be written as finite intersections of sets in \mathcal{S} . It is obvious that $\mathcal{S} \subset \mathcal{V} \subset \Sigma(\mathcal{S})$, so we have the equality $\Sigma(\mathcal{S}) = \Sigma(\mathcal{V})$. This means that it suffices to prove the equality

$$(11) \quad \Sigma(\mathcal{T}) = \Sigma(\mathcal{V}).$$

Notice that \mathcal{V} is a base for the topology, which means that every open subset $D \subsetneq X$ can be written as a union of sets in \mathcal{V} . What we want to prove is

Claim: Every open set $D \subsetneq X$ is a countable union of sets in \mathcal{V} .

To prove this fact, we fix an open set $D \subsetneq X$, as well as a countable base $\mathcal{B} = \{B_n\}_{n=1}^\infty$ for the topology. For every $x \in D$ we define the set

$$M_x = \{n \in \mathbb{N} : \text{there exists } V \in \mathcal{V}, \text{ such that } x \in B_n \subset V \subset D\}.$$

It is pretty clear that $M_x \neq \emptyset, \forall x \in D$. (First use the fact that \mathcal{V} is a base, to find $V \in \mathcal{V}$ such that $x \in V \subset D$, and then use the fact that \mathcal{B} is a base to find n such that $x \in B_n \subset V$.) If we put $M = \bigcup_{x \in D} M_x$, then it is pretty obvious that $\bigcup_{n \in M} B_n = D$. For every $n \in M$ we choose some $V_n \in \mathcal{V}$ with $B_n \subset V_n \subset D$ (use the fact that n must belong to some M_x). It is then clear that $D = \bigcup_{n \in M} V_n$, and the claim follows.

As a consequence of the Claim, we see that any open set $D \subsetneq X$ automatically belongs to $\Sigma(\mathcal{V})$, and then we have the inclusion $\mathcal{T} \subset \Sigma(\mathcal{V}) \subset \Sigma(\mathcal{T})$. This clearly forces the equality (11). \square

COROLLARY 2.3. *Let I be a set which is at most countable, and let $(X_i)_{i \in I}$ be a collection of second countable topological spaces. Then one has the equality*

$$(12) \quad \text{Bor}\left(\prod_{i \in I} X_i\right) = \Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i),$$

where the product space $\prod_{i \in I} X_i$ is equipped with the product topology.

PROOF. By the definition of the product σ -algebra, we know that

$$\Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i) = \Sigma\left(\bigcup_{j \in I} \pi_j^* \text{Bor}(X_j)\right),$$

where $\pi_j : \prod_{i \in I} X_i \rightarrow X_j, j \in I$, denote the projection maps. Choose, for each $j \in I$, a countable sub-base \mathcal{S}_j for X_j , so that we have the equalities

$$\text{Bor}(X_j) = \Sigma(\mathcal{S}_j), \quad \forall j \in I.$$

By Remark 2.1 we have the equality

$$\Sigma\text{-}\bigtimes_{i \in I} \text{Bor}(X_i) = \Sigma\left(\bigcup_{j \in I} \pi_j^* \mathcal{S}_j\right),$$

where $\pi_j : \prod_{i \in I} X_i \rightarrow X_j, j \in I$, denote the projection maps. Since the collection $\bigcup_{i \in I} \pi_i^* \mathcal{S}_i$ is a countable sub-base for the product topology, the above equality, combined with Proposition 2.5 immediately gives (12). \square

Exercise 6. A. Prove that, if X is second countable, and \mathfrak{S} is a sub-base for its topology (countable or not), with $\bigcup_{S \in \mathfrak{S}} S = X$, then we have in fact the equality

$$\text{Bor}(X) = \mathbf{S}(\mathfrak{S}).$$

B. Prove that, if X is Hausdorff, second countable, with $\text{card } X \geq 2$, then for *any* sub-base \mathfrak{S} (countable or not), we have the equality

$$\text{Bor}(X) = \mathbf{S}(\mathfrak{S}).$$

HINTS: Follow the proof above. Remark that every open set $D \subset X$, which is a countable union of sets in \mathcal{V} , belongs in fact to the σ -ring $\mathbf{S}(\mathcal{V}) = \mathbf{S}(\mathfrak{S})$. So in either case, we only have to show that X is a countable union of sets in \mathcal{V} .

In case A, we trace the proof of the Claim, and we notice that the only property that we used was the fact that, for every $x \in D$, there exists $V \in \mathcal{V}$ with $x \in V \subset D$, i.e. D is a (possibly uncountable) union of sets in \mathcal{V} . Since X itself satisfies this property, it follows that X is also a countable union of sets in \mathcal{V} .

In case B, we use the Hausdorff property to write $X = D_1 \cup D_2$, with $D_1, D_2 \subsetneq X$ open.

COROLLARY 2.4. *If X is a topological Hausdorff space, which is second countable, and X is infinite (as a set), then $\text{card } \text{Bor}(X) = \mathfrak{c}$.*

PROOF. First of all, since X is infinite, one can choose an infinite countable subset $A \subset X$. Then A , and all its subsets are Borel, i.e. we have the inclusion $\mathcal{P}(A) \subset \text{Bor}(X)$, thus proving the inequality

$$\text{card } \text{Bor}(X) \geq \text{card } \mathcal{P}(A) = 2^{\aleph_0} = \mathfrak{c}.$$

Secondly, one can choose a base \mathcal{V} for the topology, which is countable. We now have $\text{Bor}(X) = \mathbf{S}(\mathcal{V} \cup \{X\})$, so by Corollary 2.1. we get

$$\text{card } \text{Bor}(X) \leq \text{card}(\mathcal{V} \cup \{X\})^{\aleph_0} \leq \aleph_0^{\aleph_0} = \mathfrak{c},$$

and the desired equality follows. □

EXAMPLES 2.5. A. Consider the extended real line $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, thought as a compact space, homeomorphic to the interval $[-\pi/2, \pi/2]$, via the map $f : [-\pi/2, \pi/2] \rightarrow [-\infty, \infty]$, defined by

$$f(t) = \begin{cases} -\infty & \text{if } t = -\pi/2 \\ \tan t & \text{if } -\pi/2 < t < \pi/2 \\ \infty & \text{if } t = \pi/2 \end{cases}$$

Notice that, when restricted to $\mathbb{R} = (-\infty, \infty)$, this topology agrees with the usual topology. In particular, this gives the equality $\text{Bor}([-\infty, \infty])|_{\mathbb{R}} = \text{Bor}(\mathbb{R})$.

Let $A \subset \mathbb{R}$ be a dense subset. Consider the collections

$$\begin{aligned} \mathcal{E}_1 &= \{(a, \infty] : a \in A\}; & \mathcal{E}_2 &= \{[a, \infty] : a \in A\}; \\ \mathcal{E}_3 &= \{[-\infty, a) : a \in A\}; & \mathcal{E}_4 &= \{[-\infty, a] : a \in A\}. \end{aligned}$$

With these notations we have the equalities

$$\text{Bor}([-\infty, \infty]) = \Sigma(\mathcal{E}_1) = \Sigma(\mathcal{E}_2) = \Sigma(\mathcal{E}_3) = \Sigma(\mathcal{E}_4).$$

First of all, we notice that each set in $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$ is either open or closed, which means that

$$\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \subset \text{Bor}([-\infty, \infty]),$$

thus giving the inclusions

$$\Sigma(\mathcal{E}_k) \subset \text{Bor}([-\infty, \infty]), \quad \forall k \in \{1, 2, 3, 4\}.$$

Second, we observe that $\mathcal{E}_1 \cup \mathcal{E}_3$ is a sub-base for the topology, and since $[-\infty, \infty]$ is obviously second countable, we will have the equality

$$\text{Bor}([-\infty, \infty]) = \Sigma(\mathcal{E}_1 \cup \mathcal{E}_3).$$

So, in order to finish the proof we only need to show the inclusions

$$(13) \quad \mathcal{E}_1 \cup \mathcal{E}_3 \subset \Sigma(\mathcal{E}_k), \quad \forall k \in \{1, 2, 3, 4\}.$$

Since every set in \mathcal{E}_2 has its complement in \mathcal{E}_3 , and viceversa, we have the inclusions

$$\mathcal{E}_2 \subset \Sigma(\mathcal{E}_3) \text{ and } \mathcal{E}_3 \subset \Sigma(\mathcal{E}_2),$$

which prove the equality

$$(14) \quad \Sigma(\mathcal{E}_2) = \Sigma(\mathcal{E}_3).$$

Likewise, we have the equality

$$(15) \quad \Sigma(\mathcal{E}_1) = \Sigma(\mathcal{E}_4).$$

This means that we only have to prove (13) for $k = 2$ and $k = 4$. The case $k = 2$ amounts to proving that $\mathcal{E}_1 \subset \Sigma(\mathcal{E}_2)$. Fix some $a \in A$. For every integer $n \geq 1$ we choose $a_n \in (a, a + \frac{1}{n}) \cap A$. Then the equality

$$(a, \infty] = \bigcup_{n=1}^{\infty} [a_n, \infty]$$

clearly shows that $(a, \infty] \in \Sigma(\mathcal{E}_2)$.

The case $k = 4$ amounts to proving that $\mathcal{E}_3 \subset \Sigma(\mathcal{E}_4)$. Fix some $a \in A$. For every integer $n \geq 1$ we choose $a_n \in (a - \frac{1}{n}, a) \cap A$. Then the equality

$$[-\infty, a) = \bigcup_{n=1}^{\infty} [-\infty, a_n]$$

clearly shows that $[-\infty, a) \in \Sigma(\mathcal{E}_4)$.

B. If we work on \mathbb{R} , and we consider the collections

$$\mathcal{E}_k^0 = \{E \cap \mathbb{R} : E \in \mathcal{E}_k\}, \quad k = 1, 2, 3, 4,$$

then by the Generating Theorem (Example 2.4) we have the equalities

$$\text{Bor}(\mathbb{R}) = \Sigma(\mathcal{E}_k^0) = \mathbf{S}(\mathcal{E}_k^0), \quad k = 1, 2, 3, 4.$$

(The fact that the σ -algebra $\Sigma(\mathcal{E}_k^0)$ and the σ -ring $\mathbf{S}(\mathcal{E}_k^0)$ coincide is a consequence of the fact that \mathcal{E}_k^0 is σ -total in \mathbb{R} .)

C. Let X be a separable metric space. Let $A \subset X$ be a dense set, and let $R \subset (0, \infty)$ be a subset with $\inf R = 0$. Then the collection

$$\mathcal{S}_{A,R} = \{\mathcal{B}_r(a) : r \in R, a \in A\}$$

is clearly a base for the metric topology. Since X is separable, one can choose both A and R to be countable, which proves that X is automatically second countable. Then for *any* choice of A and R , we will have the equality

$$(16) \quad \text{Bor}(X) = \Sigma(\{\mathcal{B}_r(a) : r \in R, a \in A\}) = \mathbf{S}(\{\mathcal{B}_r(a) : r \in R, a \in A\}).$$

(The equality between the generated σ -algebra and σ -ring follows from Exercise 1.A.) As particular cases when the equality (16) holds, one has the metric spaces which are σ -compact.

*Exercise 7**. Let I be an *uncountable* set, and let $(X_i)_{i \in I}$ be a collection of topological spaces. Assume that for each $i \in I$, there exists at least one non-empty closed subset $F_i \subsetneq X_i$. (This is the case for example when X_i is Hausdorff, and $\text{card } X_i \geq 2$.) Prove that one has a strict inclusion

$$\text{Bor}\left(\prod_{i \in I} X_i\right) \supsetneq \prod_{i \in I} \text{Bor}(X_i).$$

HINT: For every subset $J \subset I$, define the projection map $\pi_J : \prod_{i \in I} X_i \rightarrow \prod_{i \in J} X_i$. Consider the collection

$$\mathcal{A} = \left\{ A \subset \prod_{i \in I} X_i : \text{there exists } J \subset I \text{ countable, such that } A = \pi_J^{-1}(\pi_J(A)) \right\}.$$

Prove that $\mathcal{A} \cup \{\emptyset\}$ is a σ -algebra, which contains $\bigcup_{i \in I} \pi_i^* \text{Bor}(X_i)$. Prove that one has a strict inclusion $\text{Bor}\left(\prod_{i \in I} X_i\right) \supsetneq \mathcal{A} \cup \{\emptyset\}$, by constructing a non-empty closed set $F \subset \prod_{i \in I} X_i$, which does not belong to \mathcal{A} .