Chapter III

Measure Theory

Lecture 18

1. Set arithmetic: (σ-)rings, (σ-)algebras, and monontone classes

In this section we discuss various types of set collections used in Measure Theory.

NOTATION. Given a (non-empty) set X, we denote by $\mathcal{P}(X)$ the collection of all subsets of X.

DEFINITION. Let X be a non-empty set. For $A \in \mathcal{P}(X)$, we define the function $\varkappa_A : X \to \{0, 1\}$ by

$$\varkappa_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in X \smallsetminus A \end{cases}$$

The function $\varkappa_A$ is called the *characteristic function of* A.

The basic properties of characteristic functions are summarized in the following.

Exercise 1. Let X be a non-empty set. Prove:

(i) $\varkappa_\varnothing = 0$ and $\varkappa_X = 1$.

(ii) For $A, B \in \mathcal{P}(X)$ one has

$$A \subset B \Leftrightarrow \varkappa_A \leq \varkappa_B;$$
$$A = B \Leftrightarrow \varkappa_A = \varkappa_B.$$

(iii) $\varkappa_{A\cap B} = \varkappa_A \cdot \varkappa_B$, $\forall\, A, B \in \mathcal{P}(X)$.

(iv) $\varkappa_{A\smallsetminus B} = \varkappa_A \cdot (1 - \varkappa_B)$, $\forall\, A, B \in \mathcal{P}(X)$.

(v) $\varkappa_{A\cap B} = \varkappa_A + \varkappa_B - \varkappa_A \cdot \varkappa_B$, $\forall\, A, B \in \mathcal{P}(X)$.

(vi) $\varkappa_{A_1\cup\cdots\cup A_n} = \displaystyle\sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} \varkappa_{A_{i_1}} \cdots \varkappa_{A_{i_k}}$, $\forall, A_1, \dots, A_n \in \mathcal{P}(X)$.

(vii) $\varkappa_{A\triangle B} = |\varkappa_A - \varkappa_B| = \varkappa_A + \varkappa_B - 2\varkappa_A \cdot \varkappa_B$, $\forall\, A, B \in \mathcal{P}(X)$. Here $\triangle$ stands for the symmetric set difference, defined by $A\triangle B = (A \smallsetminus B) \cup (B \smallsetminus A)$.

Property (vi) is called the *Inclusion-Exclusion Formula.*

HINT: (vi). Show that the right hand side is equal to $1 - (1 - \varkappa_{A_1}) \cdots (1 - \varkappa_{A_n})$.

REMARK 1.1. The Inclusion-Exclusion formula has an interesting application in Combinatorics. If the ambient set X is finite, then the number of elements of any subset $A \subset X$ is given by

$$|A| = \sum_{x\in X} \varkappa_A(x).$$

Using the Inclusion-Exclusion formula, we then get

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n} |A_{i_1} \cap \cdots A_{i_k}|.$$

This is known as the *Inclusion-Exclusion Principle.*

DEFINITION. Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields[1] $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. An function $\phi : X \to \mathbb{K}$ is said to be *elementary*, if its range $\phi(X)$ is finite. Remark that this gives

$$\phi = \sum_{\lambda \in \phi(X)} \lambda \cdot \varkappa_{\phi^{-1}(\{\lambda\})} = \sum_{\lambda \in \phi(X) \smallsetminus \{0\}} \lambda \cdot \varkappa_{\phi^{-1}(\{\lambda\})}.$$

We define

$$\mathrm{Elem}_{\mathbb{K}}(X) = \{\phi : X \to \mathbb{K} : \phi \text{ elementary}\}.$$

Given a collection $\mathcal{M} \subset \mathcal{P}(X)$, a function $\phi : X \to \mathbb{K}$ is said to be $\mathcal{M}$-*elementary*, if ϕ is elementary, and moreover,

$$\phi^{-1}(\{\lambda\}) \in \mathcal{M}, \ \ \forall \lambda \in \mathbb{K} \smallsetminus \{0\}.$$

We define

$$\mathcal{M}\text{-}\mathrm{Elem}_{\mathbb{K}}(X) = \{\phi : X \to \mathbb{K} : \phi \ \mathcal{M}\text{-elementary}\}.$$

Exercise 2. With the above notations, prove that $\mathrm{Elem}_{\mathbb{K}}(X)$ is a unital $\mathbb{K}$-algebra.

PROPOSITION 1.1. *Given a non-empty set X, the collection $\mathcal{P}(X)$ is a unital ring, with the operations*

$$A + B = A \triangle B \text{ and } A \cdot B = A \cap B, \ \ A, B \in \mathcal{P}(X).$$

PROOF. First of all, it is clear that $\triangle$ is commutative.

To prove the associativity of $\triangle$, we simply observe that

$$\begin{aligned} \varkappa_{(A\triangle B)\triangle C} &= \varkappa_{A\triangle B} + \varkappa_C - 2\varkappa_{A\triangle B}\varkappa_C = \\ &= \varkappa_A + \varkappa_B - 2\varkappa_A\varkappa_B + \varkappa_C - (\varkappa_A + \varkappa_B - 2\varkappa_A\varkappa_B) \cdot \varkappa_C = \\ &= \varkappa_A + \varkappa_B + \varkappa_C - 2(\varkappa_A\varkappa_B + \varkappa_A\varkappa_C + \varkappa_B\varkappa_C) + 2\varkappa_A\varkappa_B\varkappa_C. \end{aligned}$$

Since the final result is symmetric in A, B, C, we see that we get

$$\varkappa_{A\triangle(B\triangle C)} = \varkappa_{(A\triangle B)\triangle C},$$

so we indeed get

$$(A\triangle B)\triangle C = A\triangle(B\triangle C).$$

The neutral element for $\triangle$ is the empty set $\varnothing$. Since we obviously have $A\triangle A = \varnothing$, it follows that $\big(\mathcal{P}(X), \triangle\big)$ is indeed an abelian group.

The operation $\cap$ is clearly commutative, associative, and has the total set X as the unit.

To check distributivity, we again use characteristic functions:

$$\begin{aligned} \varkappa_{(A\cap C)\triangle(B\cap C)} &= \varkappa_{A\cap C} + \varkappa_{B\cap C} - 2\varkappa_{A\cap C}\varkappa_{B\cap C} = \\ &= \varkappa_A\varkappa_C + \varkappa_B\varkappa_C - 2\varkappa_A\varkappa_B\varkappa_C = (\varkappa_A + \varkappa_B - 2\varkappa_A\varkappa_B)\varkappa_C = \\ &= \varkappa_{A\triangle B}\varkappa_C = \varkappa_{(A\triangle B)\cap C}, \end{aligned}$$

so we indeed have the equality

$$(A \cap C)\triangle(B \cap C) = (A\triangle B) \cap C.$$

□

[1] $\mathbb{K}$ can be *any* field.

Definitions. Let X be a non-empty set. A *ring on* X is a non-empty sub-ring $\mathcal{R} \subset \mathcal{P}(X)$. We do not require the unit X to belong to $\mathcal{R}$, but we do require $\varnothing \in \mathcal{R}$.

An *algebra on* X is a ring $\mathcal{A}$ which contains the unit X.

Rings and algebras of sets are characterized as follows.

Proposition 1.2. *Let X be a non-empty set.*

A. *For a non-empty collection $\mathcal{R} \subset \mathcal{P}(X)$, the following are equivalent:*

(i) *$\mathcal{R}$ is a ring on X;*

(ii) *For any $A, B \in \mathcal{R}$, we have $A \smallsetminus B \in \mathcal{R}$ and $A \cup B \in \mathcal{R}$.*

B. *For a non-empty collection $\mathcal{A} \subset \mathcal{P}(X)$, the following are equivalent:*

(i) *$\mathcal{A}$ is an algebra on X;*

(ii) *For any $A \in \mathcal{A}$, we have $X \smallsetminus A \in \mathcal{A}$, and for any $A, B \in \mathcal{A}$, we have $A \cup B \in \mathcal{A}$.*

Proof. A. $(i) \Rightarrow (ii)$. Assume $\mathcal{R}$ is a ring on X, and let $A, B \in \mathcal{R}$. Then $A \cap B$ belongs to $\mathcal{R}$, so

$$A \smallsetminus B = A \triangle (A \cap B)$$

also belongs to $\mathcal{R}$. It the follows that

$$A \cup B = (A \triangle B) \triangle (A \cap B)$$

again belongs to $\mathcal{R}$.

$(ii) \Rightarrow (i)$. Assume $\mathcal{R}$ satisfies property (ii). Start with $A, B \in \mathcal{R}$. Then $A \smallsetminus B$ belongs to $\mathcal{R}$, and

$$A \cap B = A \smallsetminus (A \smallsetminus B)$$

again belongs to $\mathcal{R}$. Since $A \cup B$ also belongs to $\mathcal{R}$, it follows that the set

$$A \triangle B = (A \cup B) \smallsetminus (A \cap B)$$

again belongs to $\mathcal{R}$.

B. $(i) \Rightarrow (ii)$. This is clear from the implication A.$(i) \Rightarrow (ii)$.

$(ii) \Rightarrow (i)$. Assume $\mathcal{A}$ satisfies property (ii). Start with two sets $A, B \in \mathcal{A}$. Then the complements $X \smallsetminus A$ and $X \smallsetminus B$ both belong to $\mathcal{A}$, hence their union

$$(X \smallsetminus A) \cup (X \smallsetminus B) = X \smallsetminus (A \cap B)$$

belongs to $\mathcal{A}$, and the complement of this union

$$X \smallsetminus \big[X \smallsetminus (A \cap B)\big] = A \cap B$$

will also belong to $\mathcal{A}$.

If $A, B \in \mathcal{A}$, then since $X \smallsetminus B$ belongs to $\mathcal{A}$, by the above considerations, it follows that the intersection

$$A \cap (X \smallsetminus B) = A \smallsetminus B$$

also belongs to $\mathcal{A}$. Likewise, the difference $B \smallsetminus A$ also belongs to $\mathcal{A}$, hence the union

$$(A \smallsetminus B) \cup (B \smallsetminus A) = A \triangle B$$

also belongs to $\mathcal{A}$. By part A, it follows that $\mathcal{A}$ is a ring.

Finally, since $\mathcal{A}$ is non-empty, if we choose some $A \in \mathcal{A}$, then $A \triangle A = \varnothing$ belongs to $\mathcal{A}$, so its complement $X \smallsetminus \varnothing = X$ also belongs to $\mathcal{A}$. □

It will be useful to introduce the following terminology.

Definition. A system of sets $(A_i)_{i \in I}$ is said to be *pair-wise disjoint*, if $A_i \cap A_j = \varnothing$, for all $i, j \in I$ with $i \neq j$.

Lemma 1.1. *Let X be a non-empty set, let $\mathbb{K}$ be one of fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and let $\mathcal{R}$ be a ring on X. For a function $\phi : X \to \mathbb{K}$, the following are equivalent:*

(i) *ϕ is $\mathcal{R}$-elementary;*

(ii) *there exist an integer $n \geq 1$ and sets $A_1, \dots, A_n \in \mathcal{R}$, and numbers $\lambda_1, \dots, \lambda_n \in \mathbb{K}$, such that*

$$\phi = \lambda_1 \varkappa_{A_1} + \cdots + \lambda_n \varkappa_{A_n}.$$

(ii) *there exist an integer $m \geq 1$, and a finite pair-wise disjoint system $(B_j)_{j=1}^m \subset \mathcal{R}$, and numbers $\mu_1, \dots, \mu_m \in \mathbb{K}$, such that*

$$\phi = \mu_1 \varkappa_{B_1} + \cdots + \mu_m \varkappa_{B_m}.$$

Proof. *(i)* $\Rightarrow$ *(ii)*. Assume ϕ is $\mathcal{R}$-elementary. If $\phi = 0$, there is nothing to prove, because we have $\phi = \varkappa_{\varnothing}$. If ϕ is not identically zero, then we can obviously write

$$\phi = \sum_{\lambda \in \phi(X) \smallsetminus \{0\}} \lambda \varkappa_{\phi^{-1}(\{\lambda\})},$$

with all sets $\phi^{-1}(\{\lambda\})$ in $\mathcal{R}$.

(ii) $\Rightarrow$ *(iii)*. Define

$$\mathcal{E} = \{\psi : X \to \mathbb{K} : \psi \text{ satisfies property (iii)}\}.$$

Assume ϕ satisfies (ii), i.e.

$$\phi = \lambda_1 \varkappa_{A_1} + \cdots + \lambda_n \varkappa_{A_n},$$

with $A_1, \dots, A_n \in \mathcal{R}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. We are going to prove that $\phi \in \mathcal{E}$, by induction on n. The case $n = 1$ is trivial (either $\phi = 0$, so $\phi = \varkappa_{\varnothing} \in \mathcal{E}$, or $\phi = \lambda \varkappa_A$ for some $A \in \mathcal{R}$ and $\lambda \neq 0$, in which case we also have $\phi \in \mathcal{E}$).

Assume

$$\alpha_1 \varkappa_{D_1} + \cdots + \alpha_k \varkappa_{D_k} \in \mathcal{E},$$

for all $D_1, \dots, D_k \in \mathcal{R}$, $\alpha_1, \dots, \alpha_k \in \mathbb{K}$. Start with a function

$$\phi = \lambda_1 \varkappa_{A_1} + \cdots + \lambda_k \varkappa_{A_k} + \lambda_{k+1} \varkappa_{A_{k+1}},$$

with $A_1, \dots, A_{k+1} \in \mathcal{R}$ and $\lambda_1, \dots, \lambda_{k+1} \in \mathbb{K}$, and based on the above inductive hypothesis, let us show that $\phi \in \mathcal{E}$. Using the inductive hypothesis, the function

$$\psi = \lambda_2 \varkappa_{A_2} + \cdots + \lambda_k \varkappa_{A_k} + \lambda_{k+1} \varkappa_{A_{k+1}}$$

belongs to $\mathcal{E}$, so there exist scalars $\eta_1, \dots, \eta_p \in \mathbb{K}$, an integer $p \geq 1$, and a pair-wise disjoint system $(C_j)_{j=1}^p \subset \mathcal{R}$, such that

$$\psi = \eta_1 \varkappa_{C_1} + \cdots + \eta_p \varkappa_{C_p}.$$

With this notation, we have

$$\phi = \lambda_1 \varkappa_{A_1} + \eta_1 \varkappa_{C_1} + \cdots + \eta_p \varkappa_{C_p}.$$

Put then

$$B_{2j} = A_1 \cap C_j \text{ and } B_{2j-1} = C_j \smallsetminus A_1, \text{ for all } j \in \{1, \dots, p\};$$
$$B_{2p+1} = A_1 \smallsetminus (C_1 \cup \cdots \cup C_P).$$

It is clear that $(B_k)_{k=1}^{2p+1} \subset \mathcal{R}$ is pair-wise disjoint. Notice now that the equalities

$$C_j = B_{2j-1} \cup B_{2j}, \ \ \forall\, j \in \{1, \dots, p\},$$
$$A_1 = B_1 \cup B_3 \cup \cdots \cup B_{2p+1},$$

combined with the fact that the B's are pairwise disjoint, give

$$\varkappa_{C_j} = \varkappa_{B_{2j-1}} + \varkappa_{B_{2j}}, \ \ \forall\, j \in \{1,\dots,p\},$$
$$\varkappa_{A_1} = \varkappa_{B_1} + \varkappa_{B_3} + \dots \varkappa_{B_{2p+1}},$$

which give

$$\phi = \sum_{j=1}^{p} \eta_j \varkappa_{B_{2j}} + \sum_{j=1}^{p} (\eta_j + \lambda_1) \varkappa_{B_{2j-1}} + \lambda_1 \varkappa_{B_{2p+1}},$$

which proves that ϕ indeed belongs to $\mathcal{E}$.

$(iii) \Rightarrow (i)$. Assume there exists a finite pair-wise disjoint system $(B_j)_{j=1}^m \subset \mathcal{R}$, and numbers $\mu_1, \dots, \mu_m \in \mathbb{K}$, such that

$$\phi = \mu_1 \varkappa_{B_1} + \dots + \mu_m \varkappa_{B_m},$$

and let us prove that ϕ is $\mathcal{R}$-elemntary.

If all the μ's are zero, there is noting to prove, since $\phi = 0$.

Assume the μ's are not all equal to zero. Since the μ's that are equal to zero do not have any contribution, we can in fact assume that all the μ's are non-zero. Notice that

$$\phi(X) \smallsetminus \{0\} = \{\mu_j \, : \, 1 \leq j \leq m\}.$$

In particular ϕ is elementary.

If we start with an arbitrary $\lambda \in \mathbb{K} \smallsetminus \{0\}$, then either $\lambda \notin \phi(X)$, or $\lambda \in \phi(X) \smallsetminus \{0\}$. In the first case we clearly have $\phi^{-1}(\{\lambda\}) = \varnothing \in \mathcal{R}$. In the second case, we have the equality

$$\phi^{-1}(\{\lambda\}) = \bigcup_{j \in M_\lambda} B_j,$$

where

$$M_\lambda = \{j \, : \, 1 \leq j \leq m \text{ and } \mu_j = \lambda\}.$$

Since all B's belong to $\mathcal{R}$, it follows that $\phi^{-1}(\{\lambda\})$ again belongs to $\mathcal{R}$. Having shown that ϕ is elementary, and $\phi^{-1}(\{\lambda\}) \in \mathcal{R}$, for all $\lambda \in \mathbb{K} \smallsetminus \{0\}$, it follows that ϕ is indeed $\mathcal{R}$-elementary. □

PROPOSITION 1.3. *Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.*

A. *For a non-empty collection $\mathcal{R} \subset \mathcal{P}(X)$, the following are equivalent:*

- (i) *$\mathcal{R}$ is a ring on X;*
- (ii) *$\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$-subalgebra of $\text{Elem}_{\mathbb{K}}(X)$.*

B. *For a non-empty collection $\mathcal{A} \subset \mathcal{P}(X)$, the following are equivalent:*

- (i) *$\mathcal{A}$ is an algebra on X;*
- (ii) *$\mathcal{A}$-$\text{Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$-subalgebra of $\text{Elem}_{\mathbb{K}}(X)$, which contains the constant function 1.*

PROOF. A. $(i) \Rightarrow (ii)$. Assume $\mathcal{R}$ is a ring on X. Using Lemma 1.1 we see that we have the equality:

$$\mathcal{R}\text{-Elem}_{\mathbb{K}}(X) = \text{Span}\{\varkappa_A \, : \, A \in \mathcal{R}\}.$$

In particular, this shows that $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$-linear subspace of $\text{Elem}_{\mathbb{K}}(X)$. Moreover, in order to prove that $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$-subalgebra, it suffices to prove the implication

$$A, B \in \mathcal{R} \Longrightarrow \varkappa_A \cdot \varkappa_B \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X).$$

But this implication is trivial, since $\varkappa_A \cdot \varkappa_B = \varkappa_{A\cap B}$, and $A \cap B$ belongs to $\mathcal{R}$.

$(ii) \Rightarrow (i)$. Assume $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$ is a $\mathbb{K}$-subalgebra of $\text{Elem}_{\mathbb{K}}(X)$. First of all, since $\varkappa_\varnothing = 0 \in \mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$, it follows that $\varnothing \in \mathcal{R}$.

Start now with two sets $A, B \in \mathcal{R}$. Then $\varkappa_A$ and $\varkappa_B$ belong to $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$.

Since $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$ is an algebra, the function

$$\varkappa_{A\cap B} = \varkappa_A \cdot \varkappa_B$$

belongs to $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$, so we immediately see that $A \cap B \in \mathcal{R}$.

Likewise, the function

$$\varkappa_{A\triangle B} = \varkappa_A + \varkappa_B - 2\varkappa_A\varkappa_B$$

belongs to $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$, so we also get $A \triangle B \in \mathcal{R}$.

B. This equivalence is clear from part A, plus the identity $\varkappa_X = 1$. □

Algebras of elementary functions give in fact a complete description for rings or algebras of sets, as indicated in the result below.

Proposition 1.4. *Let X be a non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$.*

A. *The map*

$$\mathcal{R} \longmapsto \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$$

is a bijective correspondence from the collection of all rings on X, and the collection of all $\mathbb{K}$-subalgebras of $\text{Elem}_{\mathbb{K}}(X)$.

B. *The map*

$$\mathcal{A} \longmapsto \mathcal{A}\text{-Elem}_{\mathbb{K}}(X)$$

is a bijective correspondence from the collection of all algebras on X, and the collection of all $\mathbb{K}$-subalgebras of $\text{Elem}_{\mathbb{K}}(X)$ *that contain* 1.

Proof. A. We start by proving surjectivity. Let $\mathcal{E} \subset \text{Elem}_{\mathbb{K}}(X)$ be an arbitrary $\mathbb{K}$-subalgebra. Define the collection

$$\mathcal{R} = \{A \subset X : \varkappa_A \in \mathcal{E}\}.$$

If $A, B \in \mathcal{R}$, then the equalities

$$\varkappa_{A\cap B} = \varkappa_A\varkappa_B \text{ and } \varkappa_{A\triangle B} = \varkappa_A + \varkappa_B - 2\varkappa_A\varkappa_B,$$

combined with the fact that $\mathcal{E}$ is a subalgebra, prove that $\varkappa_{A\cap B}$ and $\varkappa_{A\triangle B}$ both belong to $\mathcal{E}$, hence $A \cap B$ and $A \triangle B$ both belong to $\mathcal{R}$. This shows that $\mathcal{R}$ is a ring.

It is pretty clear (see Lemma 1.1) that $\mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X) \subset \mathcal{E}$. To prove the other inclusion, start with some arbitrary function $\phi \in \mathcal{E}$, and let us prove that $\phi \in \mathcal{R}$-$\text{Elem}_{\mathbb{K}}(X)$. If $\phi = 0$, there is nothing to prove. Assume ϕ is not identically zero. We write $\phi(X) \smallsetminus \{0\}$ as $\{\lambda_1, \dots, \lambda_n\}$, with $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. For each $i \in \{1, \dots, n\}$, we set $A_i = \phi^{-1}(\{\lambda_i\})$, so that

$$\phi = \sum_{i=1}^{n} \lambda_i \cdot \varkappa_{A_i}.$$

Since all λ's are different, the matrix

$$T = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^n & \lambda_2^n & \dots & \lambda_n^n \end{bmatrix}$$

is invertible. Take $\big[\alpha_{ij}\big]_{i,j=1}^{n}$ to be the inverse of T. The obvious equalities

$$\phi^k = \sum_{j=1}^{n} \lambda_j^k \varkappa_{A_j}, \ \ \forall\, k = 1, \dots, n$$

can be written in matrix form as

$$\begin{bmatrix} \phi \\ \phi^2 \\ \vdots \\ \phi^n \end{bmatrix} = T \cdot \begin{bmatrix} \varkappa_{A_1} \\ \varkappa_{A_2} \\ \vdots \\ \varkappa_{A_n} \end{bmatrix},$$

so multiplying by T^{-1} yields

$$\varkappa_{A_j} = \sum_{k=1}^{n} \alpha_{jk} \phi^k, \ \ \forall\, j = 1, \dots, n,$$

which proves that $\varkappa_{A_1}, \dots, \varkappa_{A_n} \in \mathcal{E}$, so $A_1, \dots, A_n \in \mathcal{R}$. This then shows that $\phi \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$.

We now prove injectivity. Suppose first that $\mathcal{R}$ and $\mathcal{S}$ are rings such that $\mathcal{R}\text{-Elem}_{\mathbb{K}}(X) = \mathcal{S}\text{-Elem}_{\mathbb{K}}(X)$, and let us prove that $\mathcal{R} = \mathcal{S}$. For every $A \in \mathcal{R}$, the function $\varkappa_A \in \mathcal{R}\text{-Elem}_{\mathbb{K}}(X)$ is also $\mathcal{S}$-elementary, which means that $A \in \mathcal{S}$. This proves the inclusion $\mathcal{R} \subset \mathcal{S}$. By symmetry we also have the inclusion $\mathcal{S} \subset \mathcal{R}$, so indeed $\mathcal{R} = \mathcal{S}$.

B. This part is obvious from A. $\square$

Definitions. Let X be a (non-empty) set. A collection $\mathcal{U} \subset \mathcal{P}(X)$ is called a *σ-ring*, if it is a ring, and it has the property:

(σ) *Whenever $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{U}$, it follows that $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{U}$.*

A collection $\mathcal{S} \subset \mathcal{P}(X)$ is called a *σ-algebra*, if it is an algebra, and it has property (σ).

Clearly, every σ-algebra is a σ-ring.

Remarks 1.2. A. For σ-rings and σ-algebras, one of the properties in the definition of rings and algebras is redundant. More explicitly:

(i) A collection $\mathcal{U} \subset \mathcal{P}(X)$ is a σ-ring, if and only if it has the property (σ) and the property: $A, B \in \mathcal{U} \Longrightarrow A \smallsetminus B \in \mathcal{U}$.

(ii) A collection $\mathcal{S} \subset \mathcal{P}(X)$ is a σ-algebra, if and only if it has the property (σ) and the property: $A \in \mathcal{S} \Longrightarrow X \smallsetminus A \in \mathcal{S}$.

B. If $\mathcal{U}$ is a σ-ring, then it also has the property

(δ) $(A_n)_{n=1}^{\infty} \subset \mathcal{U} \Longrightarrow \bigcap_{n=1}^{\infty} \in \mathcal{U}$.

Since σ-algebras are σ-rings, they will also have property (δ).

Definitions. Let X be a non-empty set. A sequence $(A_n)_{n\geq 1}$ of subsets of X is said to be *monotone*, if it satisfies one of the following conditions:

($\uparrow$) $A_n \subset A_{n+1}$, $\forall\, n \geq 1$,

($\downarrow$) $A_n \supset A_{n+1}$, $\forall\, n \geq 1$.

In the case ($\uparrow$) the sequence is said to be *increasing*, and we define

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

In the case (↓) the sequence is said to be *decreasing*, and we define

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

A collection $\mathcal{M} \subset \mathcal{P}(X)$ is said to be a *monotone class on* X, if it satisfies the condition:

(M) *whenever* $(A_n)_{n\geq 1}$ *is a monotone sequence in* $\mathcal{M}$*, it follows that its limit* $\lim_{n\to\infty} A_n$ *also belongs to* $\mathcal{M}$*.*

PROPOSITION 1.5. *Let* $\mathcal{R}$ *be a ring on* X*. Then the following are equivalent:*

(i) $\mathcal{R}$ *is a* σ*-ring;*
(ii) $\mathcal{R}$ *is a monotone class.*

PROOF. *(i)* ⇒ *(ii)*. This is immediate from the definition and Remark 1.2.B.

(ii) ⇒ *(i)*. Assume $\mathcal{R}$ is a monotone class, an let us prove that it is a σ-ring. By Remark 1.2.A, we only need to prove that $\mathcal{R}$ has property (σ). Start with an arbitrary sequence $(A_n)_{n\geq 1}$ in $\mathcal{R}$, and let us prove that $\bigcup_{n=1}^{\infty} A_n$ again belongs to $\mathcal{R}$. For every integer $n \geq 1$, we define $B_n = \bigcup_{k=1}^{n} A_n$. Since $\mathcal{R}$ is a ring, it follows that $B_n \in \mathcal{R}$, $\forall\, n \geq 1$. Moreover, the sequence $(B_n)_{n\geq 1}$ is increasing, so by assumption, the set $\bigcup_{n=1}^{\infty} A_n = \lim_{n\to\infty} B_n$ indeed belongs to $\mathcal{R}$. □