Lectures 14-15

5. Banach spaces of continuous functions

In this section we discuss a examples of Banach spaces coming from topology.

Notation. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let Ω be a topological space. We define

$$C_b^{\mathbb{K}}(\Omega) = \{f : \Omega \to \mathbb{K} : f \text{ bounded and continuous}\}.$$

In the case when $\mathbb{K} = \mathbb{C}$ we use the notation $C_b(\Omega)$.

Proposition 5.1. *With the notations above, if we define*

$$\|f\| = \sup_{p\in\Omega} |f(p)|, \ \ \forall\, f \in C_b^{\mathbb{K}}(\Omega),$$

then $C_b^{\mathbb{K}}(\Omega)$ is a Banach space.

Proof. It is obvious that $C_b^{\mathbb{K}}(\Omega)$ is a linear subspace of $\ell^\infty_{\mathbb{K}}(\Omega)$, and the norm is precisely the one coming from $\ell^\infty_{\mathbb{K}}(\Omega)$. Therefore, it suffices to prove that $C_b^{\mathbb{K}}(\Omega)$ *is closed in* $\ell^\infty_{\mathbb{K}}(\Omega)$.

Start with some sequence $(f_n)_{n\geq 1} \subset C_b^{\mathbb{K}}(\Omega)$, which convergens in norm to some $f \in \ell^\infty_{\mathbb{K}}(\Omega)$, and let us prove that $f : \Omega \to \mathbb{K}$ is continuous (the fact that f is bounded is automatic).

Fix some point $p_0 \in \Omega$, and some $\varepsilon > 0$. We need to find some neighborhood V of p_0, such that

$$|f(p) - f(p_0)| < \varepsilon, \ \ \forall\, p \in V.$$

Start by choosing n such that $\|f_n - f\| < \frac{\varepsilon}{3}$. Use the fact that f_n is continuous, to find a neighborhood V of p_0, such that

$$|f_n(p) - f_n(p_0)| < \frac{\varepsilon}{3}, \ \ \forall\, \Omega \in V.$$

Suppose now $\Omega \in V$. We have

$$|f(p) - f(p_0)| \leq |f_n(p) - f(p)| + |f_n(p) - f_n(p_0)| + |f_n(p_0) - f(p_0)| \leq$$
$$|f_n(p) - f_n(p_0)| + 2\big[\sup_{q\in\Omega} |f_n(q) - f(q)|\big] < 2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

A first application of Banach space techniques is the following:

Lemma 5.1 (Urysohn type density). *Let Ω be a topological space, let $\mathcal{C} \subset C_b^{\mathbb{R}}(\Omega)$ be a linear subspace, which contains the constant function* 1. *Assume*

(u) *for any two closed sets $A, B \subset \Omega$, with $A \cap B = \varnothing$, there exists a function $h \in \mathcal{C}$, such that $h\big|_A = 0$, $h\big|B = 1$, and $h(\Omega) \in [0, 1]$, for all $\Omega \in \Omega$.*

Then $\mathcal{C}$ is dense in $C_b^{\mathbb{R}}(\Omega)$, in the norm topology.

PROOF. The key step in the proof will be the following:

Claim: *For any $f \in C_b^{\mathbb{R}}(\Omega)$, there exists $g \in \mathcal{C}$, such that*

$$\|g - f\| \leq \frac{2}{3}\|f\|.$$

To prove this claim we define

$$\alpha = \inf_{p\in\Omega} f(p) \text{ and } \beta = \sup_{p\in\Omega} f(x),$$

so that $f(p) \subset [\alpha, \beta]$, and $\|f\| = \max\{|\alpha|, |\beta|\}$. Define the sets

$$A = f^{-1}\left(\left[\alpha, \frac{2\alpha+\beta}{3}\right]\right) \text{ and } B = f^{-1}\left(\left[\frac{\alpha+2\beta}{3}, \beta\right]\right).$$

so that both A and B are closed, and $A \cap B = \varnothing$. Use the hypothesis, to find a function $h \in \mathcal{C}$, such that $h\big|_A = 0$, $h\big|_B = 1$, and $h(p) \in [0,1]$, for all $p \in \Omega$. Define the function $g \in \mathcal{C}$ by

$$g = \frac{1}{3}\big[\alpha 1 + (\beta - \alpha)k\big].$$

Let us examine the difference $g - f$. Start with some arbitrary point $p \in \Omega$. There are three cases to examine:

CASE I: $p \in A$. In this case we have $h(p) = 0$, so we get $g(p) = \frac{\alpha}{3}$. By the construction of A we also have $\alpha \leq f(p) \leq \frac{2\alpha+\beta}{3}$, so we get

$$\frac{2\alpha}{3} \leq f(p) - g(p) \leq \frac{\alpha+\beta}{3}.$$

CASE II: $p \in B$. In this case we have $h(p) = 1$, so we get $g(p) = \frac{\beta}{3}$. We also have $\frac{2\beta+\alpha}{3} \leq f(p) \leq \beta$, so we get

$$\frac{\alpha+\beta}{3} \leq f(p) - g(p) \leq \frac{2\beta}{3}.$$

CASE III: $p \in \Omega \smallsetminus (A \cup B)$. In this case we have $0 \leq h(p) \leq 1$, so we get $\frac{\alpha}{3} \leq g(p) \leq \frac{\beta}{3}$, and $\frac{2\alpha+\beta}{3} < f(p) < \frac{\alpha+2\beta}{3}$. In particular we get

$$f(p) - g(p) > \frac{2\alpha+\beta}{3} - \frac{\beta}{3} = \frac{2\alpha}{3};$$
$$f(p) - g(p) < \frac{\alpha+2\beta}{3} - \frac{\alpha}{3} = \frac{2\beta}{3}.$$

Since $\frac{2\alpha}{3} \leq \frac{\alpha+\beta}{3} \leq \frac{2\beta}{3}$, we see that in all three cases we have

$$\frac{2\alpha}{3} \leq f(p) - g(p) \leq \frac{2\beta}{3},$$

so we get

$$\frac{2\alpha}{3} \leq \inf_{p\in\Omega}\big[f(p) - g(p)\big] \leq \sup_{p\in\Omega}\big[f(p) - g(p)\big] \leq \frac{2\beta}{3},$$

so we indeed get the desired inequality

$$\|g - f\| \leq \frac{2}{3}\|f\|.$$

Having proven the Claim, we now prove the density of $\mathcal{C}$ in $C_b^{\mathbb{R}}(\Omega)$. Start with some $f \in C_b^{\mathbb{R}}(\Omega)$, and we construct recursively two sequences $(g_n)_{n\geq 1} \subset \mathcal{C}$ and $(f_n)_{n\geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, as follows. Set $f_1 = f$. Apply the Claim to find $g_1 \in \mathcal{C}$ such that

$$\|g_1 - f\| \leq \frac{2}{3}\|f_1\|.$$

Once $f_1, f_2, \dots, f_n$ and $g_1, g_2, \dots, g_n$ have been constructed, we set

$$f_{n+1} = g_n - f_n,$$

and we choose $g_{n+1} \in \mathcal{C}$ such that

$$\|g_{n+1} - f_{n+1}\| \leq \frac{2}{3}\|f_{n+1}\|.$$

It is clear, by construction, that

$$\|f_n\| \leq \left(\frac{2}{3}\right)^{n-1} \|f\|, \ \ \forall\, n \geq 1.$$

Consider the sequence $(s_n)_{n\geq 1} \subset \mathcal{C}$ of partial sums, defined by

$$s_n = g_1 + g_2 + \cdots + g_n, \ \ \forall\, n \geq 1.$$

Using the equalities

$$g_n = f_n - f_{n+1}, \ \ \forall\, n \geq 1,$$

we get

$$s_n - f = g_1 + g_2 + \cdots + g_n - f_1 = f_{n+1},$$

so we have

$$\|s_n - f\| \leq \left(\frac{2}{3}\right)^{n} \|f\|, \ \ \forall\, n \geq 1,$$

which clearly give $f = \lim_{n\to\infty} s_n$, so f indeed belongs to the closure $\overline{\mathcal{C}}$. □

We are now in position to prove the following

Theorem 5.1 (Tietze Extension Theorem). *Let Ω be a normal topological space, let $T \subset \Omega$ be a closed subset. Let $f : T \to [0,1]$ be a continuous function. (Here Y is equipped with the induced topology.) There there exists a continuous function $g : \Omega \to [0,1]$ such that $g\big|_T = f$.*

Proof. Let us introduce the Banach space setting that will make the proof clearer. We consider the Banach spaces $C^{\mathbb{R}}(\Omega)$ and $C_b^{\mathbb{R}}(T)$. To avoid any confusion, the norms on these Banach spaces will be denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_T$. If we define the restriction map

$$R : C_b^{\mathbb{R}}(\Omega) \ni g \longmapsto g\big|_T \in C_b^{\mathbb{R}}(T),$$

then R is obviously linear and continuous.

We define the subspace $\mathcal{C} = R\big(C_b^{\mathbb{R}}(\Omega)\big) \subset C_b^{\mathbb{R}}(T)$.

Claim: *For every $f \in \mathcal{C}$, there exists some $g \in C_b^{\mathbb{R}}(\Omega)$ such that $f = Rg$, and*

$$\inf_{q\in T} f(q) \leq g(p) \leq sup_{q\in T} f(q), \ \ \forall\, p \in \Omega.$$

To prove this fact, we start first with some arbitrary $g_0 \in C_b^{\mathbb{R}}(\Omega)$, such that $f = Rg_0 = g_0\big|_Y$. Put

$$\alpha = \inf_{q\in T} f(q) \text{ and } \beta = \sup_{q\in T} f(q),$$

so that $\|f\|_T = \max\big\{|\alpha|, |\beta|\big\}$. Define the function $\theta : \mathbb{R} \to [\alpha, \beta]$ by

$$\theta(t) = \begin{cases} \alpha & \text{if } t < \alpha \\ t & \text{if } \alpha \leq t \leq \beta \\ \beta & \text{if } t > \beta \end{cases}$$

Then obviously θ is continuous, and the composition $g = \theta \circ g_0 : \Omega \to [\alpha, \beta]$ will still satisfy $g\big|_T = f$, and we will clearly have

$$\alpha \leq g(p) \leq \beta, \ \ \forall\, p \in \Omega.$$

Having proven the Claim, we are going to prove that $\mathcal{C}$ is *closed*. We do this by showing that $\mathcal{C}$ is a *Banach space*, in the norm $\|\cdot\|_Y$. To get this, we use Remark ??. Start with some sequence $(f_n)_{n\geq 1} \subset \mathcal{C}$, with $\sum_{n=1}^{\infty} \|f_n\|_T < \infty$. Apply the Claim, to construct a sequence $(g_n)_{n\geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, such that $Rg_n = f_n$, and

$$\inf_{q\in T} f_n(q) \leq g_n(p) \leq \sup_{q\in T} f_n(q), \ \ \forall\, p \in \Omega,$$

for each $n \geq 1$. Notice that this forces

$$\|g_n\|_\Omega \leq \|f_n\|_T, \ \ \forall\, n \geq 1.$$

Define the sequences of partial sums $(h_n)_{n\geq 1} \subset \mathcal{C}$ and $(s_n)_{n\geq 1} \subset C_b^{\mathbb{R}}(\Omega)$, by

$$h_n = f_1 + \cdots + f_n \text{ and } s_n = g_1 + \cdots + g_n, \ \ \forall\, n \geq 1.$$

Since

$$\sum_{n=1}^{\infty} \|g_n\|_\Omega \leq \sum_{n=1}^{\infty} \|f_n\|_T < \infty,$$

and $C_b^{\mathbb{R}}(\Omega)$ is a Banach space, it follows that the sequence $(s_n)_{n\geq 1}$ is convergent to some point $g \in C_b^{\mathbb{R}}(\Omega)$. Since $R : C_b^{\mathbb{R}}(\Omega) \to C_b^{\mathbb{R}}(T)$ is linear an continuous, we will have

$$Rs = \lim_{n\to\infty} [Rg_1 + \cdots + Rg_n] = \lim_{n\to\infty} [f_1 + \cdots + f_n] = \lim_{n\to\infty} h_n,$$

which proves that the sequence of partial sums $(h_n)_{n\geq 1} \subset \mathcal{C}$ is indeed convergent to $Rs \in \mathcal{C}$.

Let us remark now that obviously $\mathcal{C}$ contains the constant function $1 = R1$. Using Urysohn Lemma (applied to T) it is clear that $\mathcal{C}$ satifies the condition (U) in the above lemma. Using the Lemma ??, it follows that $\mathcal{C} = C_b^{\mathbb{R}}(T)$, i.e. R is surjective.

To finish the proof, start with some arbitrary continuous function $f : Y \to [0, 1]$. Use surjectivity of R, combined with the Claim, to find $g \in C_b^{\mathbb{R}}(\Omega)$, such that $Rg = f$, and

$$\inf_{q\in T} f(q) \leq g(p) \leq \sup_{q\in T} f(q), \ \ \forall\, p \in \Omega.$$

This clearly forces g to take values in $[0, 1]$. □

Next we concentrate on the case when Ω is a compact Hausdorff space. In this case, every continuous function $F : \Omega \to \mathbb{K}$ is automatically bounded, and the Banach space $C_b^{\mathbb{K}}(\Omega)$ will be denoted simply by $C^{\mathbb{K}}(\Omega)$. (When $\mathbb{K} = \mathbb{C}$ this space will be denoted simply by $C(\Omega)$.)

THEOREM 5.2 (Dini). *Let K be a compact Hausdorff space, let $(f_n)_{n\geq 1} \subset C^{\mathbb{R}}(K)$ be a monotone sequence. Assume there is some $f \in C^{\mathbb{R}}(K)$, such that*

$$\lim_{n\to\infty} f_n(p) = f(p), \ \ \forall\, p \in K.$$

Then $\lim_{n\to\infty} f_n = f$, in the norm topology.

PROOF. Replacing f_n with $f_n - f$, we can assume that $\lim_{n\to\infty} f_n(p) = 0$, $\forall\, p \in K$. Replacing (if necessary) f_n with $-f_n$, we can also assume that the sequence $(f_n)_{n\geq 1}$ is *decreasing.* In particular, each f_n is non-negative.

We need to prove that $\lim_{n\to\infty} \|f_n\| = 0$. Assume this is not true, so there exists some $\varepsilon > 0$, such that the set

$$M = \{m \in \mathbb{N} \,:\, \|f_m\| \geq \varepsilon\}$$

is infinite. For each integer $n \geq 1$, let us define the set

$$F_n = \{p \in K \,:\, f_n(p) \geq \varepsilon\}.$$

Then by the definition of M, we have

$$F_m \neq \varnothing, \ \ \forall\, m \in M.$$

*Claim***:** *One has the inclusion $F_n \supset F_{n+1}$, $\forall\, n \geq 1$.*

Indeed, if $p \in F_{n+1}$, then

$$\varepsilon \leq f_{n+1}(p) \leq f_n(p),$$

which proves that $p \in F_n$.

Using the claim, plus the fact that the set M is infinite, it follows that, $F_n \neq \varnothing$, $\forall\, n \geq 1$. (Indeed, if we start with some arbitrary n, then since M is infinite, we can find $m \in M$, with $m \geq n$, and then using the Claim we have $\varnothing \neq F_m \subset F_n$.)

Since K is compact, and the sets $F_1 \supset F_2 \supset \dots$ are closed and non-empty, by the finite intersection property, it follows that

$$\bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

But this leads to a contradiction, because if we pick an element $p \in \bigcap_{n=1}^{\infty} F_n$, then we will have $f_n(p) \geq \varepsilon$, $\forall\, n \geq 1$, and then the equality $\lim_{n\to\infty} f_n(p) = 0$ is impossible. □

Exercise 1. Define the sequence $(P_n)_{n\geq 1}$ of polynomials, by $P_1(t) = 0$, and

$$P_{n+1}(t) = \frac{1}{2}\big[t - P_n(t)^2\big] + P_n(t), \ \ \forall\, n \geq 1.$$

Prove that

$$\lim_{n\to\infty} \Big(\max_{t\in[0,1]} \big|P_n(t) - \sqrt{t}\big| \Big) = 0.$$

HINT: Define the functions $f_n, f : [0,1] \to \mathbb{R}$ by $f_n(t) = P_n(t)$ and $f(t) = \sqrt{t}$. Prove that, for every $t \in [0,1]$, the sequence $\big(f_n(t)\big)_{n\geq 1}$ is incresing, bounded, and $\lim_{n\to\infty} f_n(t) = f(t)$. Then apply Dini's Theorem.

THEOREM 5.3 (Stone-Weierstrass). *Let K be a compact Hausdorff space. Let $\mathcal{A} \subset C^{\mathbb{R}}(K)$ be a unital subalgebra, i.e.*

- *$\mathcal{A} \ni 1$ - the constant function 1;*
- *$\mathcal{A}$ is a linear subspace;*
- *if $f, g \in \mathcal{A}$, then $fg \in \mathcal{A}$.*

Assume $\mathcal{A}$ separates the points of K, i.e. for any $p, q \in K$, with $p \neq q$, there exists $f \in \mathcal{A}$ such that $f(p) \neq f(q)$.

Then $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$, in the norm topology.

Proof. Let $\mathcal{C}$ denote the closure of $\mathcal{A}$. Remark that $\mathcal{C}$ is again a unital subalgebra and it still separates the points.

The proof will eventually use the Urysohn density Lemma. Before we get to that point, we need several preparations.

Step 1. *If $f \in \mathcal{C}$, then $|f| \in \mathcal{C}$.*

To prove this fact, we define $g = f^2 \in \mathcal{C}$, and we set $h = \|g\|^{-1}g$, so that $h \in \mathcal{C}$, and $h(p) \in [0, 1]$, for all $p \in K$. Let $P_n(t)$, $n \geq 1$ be the polynominals defined in the above exercise. The functions $h_n = P_n \circ h$, $n \geq 1$ are clearly all in $\mathcal{C}$. By the above Exercise, we clearly get

$$\lim_{n\to\infty} \big(\max_{p\in K} |h_n(p) - \sqrt{h(p)}| \big) = 0,$$

which means that $\lim_{n\to\infty} h_n = \sqrt{h}$, in the norm topology. In particular, $\sqrt{h}$ belongs to $\mathcal{C}$. Obviously we have

$$\sqrt{h} = \|f\|^{-1} \cdot |f|,$$

so $|f|$ indeed belongs to $\mathcal{C}$.

Step 2: *Given two functions $f, g \in \mathcal{C}$, the continuous functions $\max\{f, g\}$ and $\min\{f, g\}$ both belong to $\mathcal{C}$.*

This follows immediately from Step 1, and the equalities

$$\max\{f, g\} = \frac{1}{2}\big(f + g + |f - g|\big) \text{ and } \min\{f, g\} = \frac{1}{2}\big(f + g - |f - g|\big).$$

Step 3: *For any two points $p, q \in K$, $p \neq q$, there exists $h \in \mathcal{C}$, such that $h(p) = 0$, $h(q) = 1$, and $h(s) \in [0, 1]$, $\forall\, s \in K$.*

Use the assumption on $\mathcal{A}$, to find first a function $f \in \mathcal{A}$, such that $f(p) \neq f(q)$. Put $\alpha = f(p)$ and $\beta = f(q)$, and define

$$g = \frac{1}{\beta - \alpha}\big(f - \alpha 1\big).$$

The function g still belongs to $\mathcal{A}$, but now we have $g(p) = 0$ and $g(q) = 1$. Define the function $h = \min\{g^2, 1\}$. By Step 3, $h \in \mathcal{C}$, and it clearly satisfies the required properties.

Step 4: *Given a closed subset $A \subset K$, and a point $p \in K \smallsetminus A$, there exists a function $h \in \mathcal{C}$, such that $h(p) = 0$, $h\big|_A = 1$, and $h(q) \in [0, 1]$, $\forall\, q \in K$.*

For every $q \in A$, we use Step 3 to find a function $h_q \in \mathcal{C}$, such that $h_q(p) = 0$, $h_q(q) = 1$, and $h_q(s) \in [0, 1]$, $\forall\, s \in K$, and we define the open set

$$D_q = \{s \in K : h_q(s) > 0\}.$$

Using the compactness of A, we find points $q_1, \dots, q_n \in A$, such that

$$A \subset D_{q_1} \cup \dots \cup D_{q_n}.$$

Define the function $f = h_{q_1} + \dots + h_{q_n} \in \mathcal{C}$, so that $f(p) = 0$, $f(q) > 0$, for all $q \in A$, and $f(s) \geq 0$, $\forall\, s \in K$. If we define

$$m = \min_{q\in A} f(q),$$

then the function $g = m^{-1}f$ again belongs to $\mathcal{C}$, and it satisfies $g(p) = 0$, $g(q) \geq 1$, $\forall\, q \in A$, and $g(s) \geq 0$, $\forall\, s \in K$. Finally, the function

$$h = \min\{g, 1\}$$

will satisfy the required properties.

STEP 5: *Given closed sets $A, B \subset K$ with $A \cap B = \varnothing$, there exists $h \in \mathcal{C}$, such that $h\big|_A = 1$, $h\big|_B = 0$, and $h(q) \in [0,1]$, $\forall\, q \in K$.*

Use Step 4, to find for every $p \in B$, a function $h_p \in \mathcal{C}$, such that $h_p\big|_B = 1$, $h_p(p) = 0$, and $h_p(s) \in [0,1]$, $\forall\, s \in K$. Put $g_p = 1 - h_p$, so that $g_p(p) = 1$, $g_p\big|_B = 0$, and $g_p(s) \in [0,1]$, $\forall\, s \in K$. We the proceed as above. For each $p \in A$ we define the open set

$$D_p = \{s \in K \,:\, g_p(s) > 0\}.$$

Using the compactness of A, we find points $p_1, \dots, p_n \in A$, such that

$$A \subset D_{p_1} \cup \dots \cup D_{p_n}.$$

Define the function $f = g_{p_1} + \dots + g_{p_n} \in \mathcal{C}$, so that $f\big|_B = 0$, $f(q) > 0$, for all $q \in A$, and $f(s) \geq 0$, $\forall\, s \in K$. If we define

$$m = \min_{q \in A} f(q),$$

then the function $g = m^{-1}f$ again belongs to $\mathcal{C}$, and it satisfies $g\big|_B = 0$, $g(q) \geq 1$, $\forall\, q \in A$, and $g(s) \geq 0$, $\forall\, s \in K$. Finally, the function

$$h = \min\{g, 1\}$$

will satisfy the required properties.

We now apply the Urysohn density Lemma, to conclude that $\mathcal{C}$ is dense in $C^{\mathbb{R}}(K)$. Since $\mathcal{C}$ is already closed, this forces $\mathcal{C} = C^{\mathbb{R}}(K)$, i.e. $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$. ☐

COROLLARY 5.1 (Complex version of Stone-Weierstrass Theorem). *Let K be a compact Hausdorff space. Let $\mathcal{A} \subset C(K)$ be a unital subalgebra, which satisfies;*

- *if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$.*

Assume $\mathcal{A}$ separates the points of K. Then $\mathcal{A}$ is dense in $C(K)$, in the norm topology.

PROOF. Consider the sub-algebra

$$\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} \,:\, f = \overline{f}\}.$$

It is clear that

$$\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}},$$

and $\mathcal{A}_{\mathbb{R}}$ is a unital sub-algebra of $C^{\mathbb{R}}(K)$, which separates the points of K. Using the real version, we know that $\mathcal{A}_{\mathbb{R}}$ is dense in $C^{\mathbb{R}}(K)$. Then $\mathcal{A}$ is clearly dense in $C(K)$. ☐

EXAMPLE 5.1. Consider the unit disk

$$\mathbb{D} = \{\lambda \in \mathbb{C} \,:\, |\lambda| < 1\},$$

and let $\overline{\mathbb{D}}$ denote its closure. Consider the algebra $\mathcal{A} \subset C(\overline{\mathbb{D}})$ consisting of all polynomial functions. Notice that, although $\mathcal{A}$ is unital and separates the points of $\overline{\mathbb{D}}$, it does not have the property

$$f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}.$$

In fact, one way to see that this property fails is by inspecting the closure of $\mathcal{A}$ in $C(\overline{\mathbb{D}})$. This closure is denoted by $\mathbf{A}(\mathbb{D})$ and is called the *disk algebra*. The main feature of $\mathbf{A}(\mathbb{D})$ is the following:

*Exercise 2**. Prove that

$$\mathbf{A}(\mathbb{D}) = \big\{ f : \overline{\mathbb{D}} \to \mathbb{C} \,:\, f \text{ continuous, and } f\big|_{\mathbb{D}} \text{ holomorphic}\big\}.$$

We now examine the topological dual of $C(K)$.

NOTATIONS. Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. We define the space

$$\mathcal{M}^{\mathbb{K}}(K) = C^{\mathbb{K}}(K)^* = \{\phi : C^{\mathbb{K}}(K) \to \mathbb{K} \,:\, \phi \text{ } \mathbb{K}\text{-linear continuous}\}.$$

The unit ball will be denoted by $\mathcal{M}^{\mathbb{K}}(K)_1$. When $\mathbb{K} = \mathbb{C}$, the superscript $\mathbb{C}$ will be omitted from the notation.

REMARKS 5.1. Let K be a compact Hausdorff space. The space $\mathcal{M}(K) = C(K)^*$ carries a natural involution, defined as follows. For $\phi \in \mathcal{M}(K)$, we define the map $\phi^\star : C(K) \to \mathbb{C}$ by

$$\phi^\star(f) = \overline{\phi(\overline{f})},\ \ \forall\, f \in C(K).$$

For every $\phi \in \mathcal{M}(K)$, the map $\phi^\star : C(K) \to \mathbb{C}$ is again linear, continuous, and has

$$\|\phi^\star\| = \|\phi\|.$$

The map $\phi^\star$ will be called the *adjoint of* ϕ. We used the term *involution*, because the map

$$\mathcal{M}(K) \ni \phi \longmapsto \phi^\star \in \mathcal{M}(K)$$

has the following properties:

- $(\phi^\star)^\star = \phi$, $\forall\, \phi \in \mathcal{M}(K)$;
- $(\phi + \psi)^\star = \phi^\star + \psi^\star$, $\forall\, \phi, \psi \in \mathcal{M}(K)$;
- $(\lambda\phi)^\star = \overline{\lambda}\phi^\star$, $\forall\, \phi, \in \mathcal{M}(K),\ \lambda \in \mathbb{C}$.

If we define the space of self-adjoint maps

$$\mathcal{M}^{sa}(K) = \{\phi \in \mathcal{M}(K) \,:\, \phi^\star = \phi\},$$

then is clear that, for any $\phi \in \mathcal{M}^{sa}(K)$, the restriction $\phi\big|C^{\mathbb{R}}(K)$ is real-valued. In fact, for $\phi \in \mathcal{M}(K)$, one has

$$\phi^\star = \phi \iff \phi\big|_{C^{\mathbb{R}}(K)} \text{ is real-valued.}$$

Moreover, one has a map

$$\mathcal{M}^{sa}(K) \ni \phi \longmapsto \phi\big|_{C^{\mathbb{R}}(K)} \in \mathcal{M}^{\mathbb{R}}(K), \tag{1}$$

which is an *isomorphism of* $\mathbb{R}$*-vector spaces*. The inverse of this map is defined as follows. Start with some $\phi \in \mathcal{M}^{\mathbb{R}}(K)$, i.e. $\phi : C^{\mathbb{R}}(K) \to \mathbb{R}$ is $\mathbb{R}$-linear and continuous, and we define $\hat{\phi} : C(K) \to \mathbb{C}$ by

$$\hat{\phi}(f) = \phi(\operatorname{Re} f) + i\phi(\operatorname{Im} f),\ \ \forall\, f \in C(K).$$

It turns out that $\hat{\phi}$ is again linear, continuous, and self-adjoint. Moreover, the correspondence

$$\mathcal{M}^{\mathbb{R}}(K) \ni \phi \longmapsto \hat{\phi} \in \mathcal{M}^{sa}(K)$$

is the inverse of (1).

Proposition 5.2. *Let K be a compact Hausdorff space. Then the map*

$$\mathcal{M}^{sa}(K) \ni \phi \longmapsto \phi\big|_{C^{\mathbb{R}}(K)} \in \mathcal{M}^{\mathbb{R}}(K)$$

is isometric. Moreover, when the two spaces are equipped with the w^* topology, this map is a homeomorphism.

Proof. To prove the first statement, fix $\phi \in \mathcal{M}^{sa}(K)$. It is obvious that $\|\phi\big|_{C^{\mathbb{R}}(K)}\| \leq \|\phi\|$. To prove the other inequality, fix for the moment $\varepsilon > 0$, and choose $f \in C(K)$ such that $\|f\| \leq 1$, and

$$|\phi(f)| \geq \|\phi\| - \varepsilon.$$

Choose a complex number λ with $|\lambda| = 1$, such that

$$|\phi(f)| = \lambda\phi(f) = \phi(\lambda f).$$

If we write $\lambda f = g + ih$, with $g, h \in C^{\mathbb{R}}(K)$, then using the fact that ϕ is self-adoint, we will have

$$|\phi(f)| = \phi(g).$$

Since $\|g\| \leq \|\lambda f\| = \|f\| \leq 1$, we will get

$$|\phi(f)| \leq \|\phi\big|_{C^{\mathbb{R}}(K)}\|,$$

so our choice of f will give

$$\|\phi\| - \varepsilon \leq \|\phi\big|_{C^{\mathbb{R}}(K)}\|.$$

Since this holds for all $\varepsilon > 0$, we get

$$\|\phi\| \leq \|\phi\big|_{C^{\mathbb{R}}(K)}\|.$$

The w^* continuity (both ways) is obvious. □

Convention. From now on, we will identify the space $\mathcal{M}^{\mathbb{R}}(K)$ with $\mathcal{M}^{sa}(K)$.

Proposition 5.3. *Let K be a compact Hausdorff space. For every $p \in K$, let $\gamma_p : C(K) \to \mathbb{C}$ be the map*

$$\gamma_p : C(K) \ni f \longmapsto f(p) \in \mathbb{C}.$$

- (i) *For every $p \in K$, the maps γ_p and $\gamma_p^{\mathbb{R}} = \gamma_p\big|_{C^{\mathbb{R}}(K)} : C^{\mathbb{R}}(K) \to \mathbb{R}$ are linear and continuous.*
- (ii) *For every $p \in K$, one has $\|\gamma_p\| = \|\gamma_p^{\mathbb{R}}\| = 1$.*
- (ii) *The maps*

$$\Gamma_K : K \ni p \longmapsto \gamma_p \in \mathcal{M}(K)_1$$
$$\Gamma_K^{\mathbb{R}} : K \ni p \longmapsto \gamma_p^{\mathbb{R}} \in \mathcal{M}^{\mathbb{R}}(K)_1$$

are injcetive and continuous, when the target spaces $\mathcal{M}(K)_1$ and $\mathcal{M}^{\mathbb{R}}(K)_1$ are equipped with the w^* topology.

Proof. (i)-(ii). The fact that γ_p is $\mathbb{C}$-linear is obvious. This will also give the $\mathbb{R}$-linearity of $\gamma_p^{\mathbb{R}}$. The continuity follows from the obvious inequality

$$|\gamma_p(f)| = |f(p)| \leq \max_{q \in K} |f(q)| = \|f\|, \ \ \forall\, f \in C(K).$$

AMong other things, the above inequality also proves

$$\|\gamma_p\| \leq 1 \text{ and } \|\gamma_p^{\mathbb{R}}\| \leq 1.$$

The fact that we have in fact equalities follows from $\gamma_p(1) = 1$.

(iii) Let us first prove the injectivity. Assume we have two point $p, q \in K$, with $p \neq q$. Use Urysohn Lemma to find $f : K \to [0, 1]$ continuous, such that $f(p) = 0$ and $f(q) = 1$. Then $f \in C^{\mathbb{R}}(K)$ and $\gamma_p^{\mathbb{R}}(f) = f(p) = 0$, and $\gamma_q^{\mathbb{R}}(f) = f(q) = 1$, so we indeed have $\gamma_p^{\mathbb{R}} \neq \gamma_q^{\mathbb{R}}$. (This will also imply $\gamma_p \neq \gamma_q$.

To prove the continuity of the maps $\Gamma_K : K \to \mathcal{M}(K)_1$ and $\Gamma_K^{\mathbb{R}} : K \to \mathcal{M}^{\mathbb{R}}(K)_1$, we need to prove the continuity of the maps $\epsilon_f \circ \Gamma_K : K \to \mathbb{C}$, $f \in C(K)$, and of the maps $\epsilon_f \circ \Gamma_K^{\mathbb{R}} : K \to \mathbb{R}$, $f \in C^{\mathbb{R}}(K)$. (Recall that $\epsilon_f(\phi) = \phi(f)$, $\forall \phi \in C^{\mathbb{K}}(K)^*$.) Notice hoewver that we have in fact equalities

$$\epsilon_f \circ \Gamma_K = f, \ \ \forall f \in C(K),$$
$$\epsilon_f \circ \Gamma_K^{\mathbb{R}} = f, \ \ \forall f \in C^{\mathbb{R}}(K),$$

so the desired continuity is automatic. □

Corollary 5.2. *With the above notations, the spaces*

$$\Gamma(K) = \{\gamma_p : \in K\} \subset \mathcal{M}(K)_1 \text{ and } \Gamma^{\mathbb{R}}(K) = \{\gamma_p^{\mathbb{R}} : \in K\} \subset \mathcal{M}^{\mathbb{R}}(K)_1$$

are w^* compact, and the maps

$$\Gamma_K : K \to \Gamma(K) \text{ and } \Gamma_K^{\mathbb{R}} : K \to \Gamma^{\mathbb{R}}(K)$$

are homeomorphisms.

Here is an interesting application of the above result to topology.

Theorem 5.4 (Urysohn Metrizatbility Theorem). *Let K be a compact Hausdorff space. The following are equivalent:*

- (i) *K is metrizable;*
- (ii) *K is second countable, i.e. the topology has a countable base;*
- ($\text{iii}_{\mathbb{R}}$) *the Banach space $C^{\mathbb{R}}(K)$ is separable;*
- ($\text{iii}_{\mathbb{C}}$) *the Banach space $C(K)$ is separable.*

Proof. *(i)* ⇒ *(ii)*. We already know this fact. (See the section on metric spaces).

(ii) ⇒ *($iii_{\mathbb{R}}$)*. Assume K is second countable. Fix a countable base $\{D_n : n \in \mathbb{N}\}$ for the topology. Consider the countable set

$$\Delta = \{(m, n) \in \mathbb{N}^2 : \overline{D}_m \cap \overline{D}_n = \varnothing\}.$$

> *Claim*: *For any two points $p, q \in K$, with $p \neq q$, there exists a pair $(m, n) \in \Delta$ with $p \in D_m$ and $q \in D_n$.*

Indeed, since K is Hausdorff, there exist open sets $U_0, V_0 \subset K$ with $p \in U_0$, $q \in V_0$, and $U_0 \cap V_0 = \varnothing$. Since K is (locally) compact, there exist open sets $U, V \subset K$, such that $p \in U \subset \overline{U} \subset U_0$ and $q \in V \subset \overline{V} \subset V_0$. Finally, since $\{D_n : n \in \mathbb{N}\}$ is a basis for the topology, there exist $m, n \in \mathbb{N}$ such that $p \in D_m \subset U$ and $q \in D_n \subset V$. Then clearly we have $\overline{D}_m \subset \overline{U} \subset U_0$, and $\overline{D}_n \subset \overline{V} \subset V_0$, which forces $\overline{D}_m \cap \overline{D}_n = \varnothing$.

Having proven the Claim, for every pair $(m, n) \in \Delta$ we choose (use Urysohn Lemma) a continuous function $h_{mn} : K \to [0, 1]$ such that $h_{mn}\big|_{\overline{D}_m} = 0$ and $h_{mn}\big|_{\overline{D}_n} = 1$, and we define the countable family

$$\mathcal{F} = \{h_{mn} : (m, n) \in \Delta\}.$$

Using the Claim, we know that $\mathcal{F}$ *separates the points of* K. We set

$$\mathcal{P} = \{h \in C^{\mathbb{R}}(K) : h \text{ is a finite product of functions in } \mathcal{F}\}.$$

Notice that $\mathcal{P}$ is still countable, it also separates the points of K, but also has the property:

$$f, g \in \mathcal{P} \Rightarrow fg \in \mathcal{P}.$$

If we define

$$\mathcal{A} = \text{Span}(\{1\} \cup \mathcal{P}),$$

then $\mathcal{A} \subset C^{\mathbb{R}}(K)$ satisfies the hypothesis of the Stone-Weierstrass Theorem, hence $\mathcal{A}$ is dense in $C^{\mathbb{R}}(K)$. Notice that if we define

$$\mathcal{A}_{\mathbb{Q}} = \text{Span}_{\mathbb{Q}}(\{1\} \cup \mathcal{P}),$$

i.e. the set of linear combinations of elements in $\{1\} \cup \mathcal{P}$ with *rational* coefficients, then clearly $\mathcal{A}_{\mathbb{Q}}$ is dense in $\mathcal{A}$, and so $\mathcal{A}_{\mathbb{Q}}$ is dense in $C^{\mathbb{R}}(K)$. But now we are done, since $\mathcal{A}_{\mathbb{Q}}$ is obviously countable.

$(iii_{\mathbb{R}}) \Rightarrow (iii_{\mathbb{C}})$. Assume $C^{\mathbb{R}}(K)$ is separable. Let $\mathcal{S} \subset C^{\mathbb{R}}(K)$ be a countable dense set. Then the set

$$\mathcal{S} + i\mathcal{S} = \{f + ig \,:\, f, g \in \mathcal{S}\}$$

is clearly countable, and dense in $C(K)$.

$(iii_{\mathbb{C}}) \Rightarrow (i)$. Assume $C(K)$ is separable. By the results from the previous section, it follows that, when equipped with the w^* topology, the compact space $\mathcal{M}(K)_1$ is metrizable. Then the compact subset $\Gamma(K) \subset \mathcal{M}(K)_1$ is also metrizable. Since K is homeomorphic to $\Gamma(K)$, it follows that K itself is metrizable. □

DEFINITION. Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-linear map $\phi : C^{\mathbb{K}}(K) \to \mathbb{K}$ is said to be *positive*, if it has the property

$$f \in C^{\mathbb{R}}(K),\ f \geq 0 \Longrightarrow \phi(f) \geq 0.$$

PROPOSITION 5.4 (Automatic continuity for positive linear maps). *Let K be a compact Hausdorff space, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Any positive $\mathbb{K}$-linear map $\phi : C^{\mathbb{K}}(K) \to \mathbb{K}$ is continuous. Moreover, one has the equality $\|\phi\| = \phi(1)$.*

PROOF. In the case when $\mathbb{K} = \mathbb{C}$, it suffices to prove that $\phi\big|_{C^{\mathbb{R}}(K)}$ is continuous. Therefore, it suffices to prove the statement for $\mathbb{K} = \mathbb{R}$. Start with some arbitrary $f \in C^{\mathbb{R}}(K)$, and define the function $f_{\pm} \in C^{\mathbb{R}}(K)$ by

$$f_+ = \max\{f, 0\} \text{ and } f_- = \max\{-f, 0\},$$

so that $f_{\pm} \geq 0$, $f = f + -f_-$, and $\|f\| = \max\{\|f_+\|, \|f_-\|\}$. On the one hand, by positivity, we have the inequalities $\phi(f_{\pm}) \geq 0$, so we get

$$-\phi(f_-) \leq \phi(f_+) - \phi(f_-) \leq \phi(f_+),$$

which give

$$|\phi(f)| = |\phi(f_+) - \phi(f_-)| \leq \max\{\phi(f_+), \phi(f_-)\}. \tag{2}$$

On the other hand, we have

$$\|f_{\pm}\| \cdot 1 - f_{\pm} \geq 0,$$

so by positivity we get

$$\|f_{\pm}\| \cdot \phi(1) \geq \phi(f_{\pm}).$$

Using this in (2) gives

$$|\phi(f)| \leq \phi(1) \cdot \max\{\|f_+\|, \|f_-\|\} = \phi(1) \cdot \|f\|.$$

Since this holds for all $f \in C^{\mathbb{R}}(K)$, the continuity of ϕ follows, together with the estimate

$$\|\phi\| \leq \phi(1).$$

Since $\phi(1) \leq \|\phi\| \cdot \|1\| = \|\phi\|$, the desired norm equality follows. $\square$

NOTATIONS. Let K be a compact Hausdorff space. We define

$$\mathcal{M}_+^{\mathbb{K}}(K) = \{\phi : C^{\mathbb{K}}(K) \to \mathbb{K} \,:\, \phi \ \mathbb{K}\text{-linear, positive}\};$$
$$\mathcal{M}_+^{\mathbb{K}}(K)_1 = \{\phi \in \mathcal{M}_+^{\mathbb{K}}(K) \,:\, \|\phi\| \leq 1\} = \mathcal{M}_+^{\mathbb{K}}(K) \cap \mathcal{M}^{\mathbb{K}}(K)_1.$$

When $\mathbb{K} = \mathbb{C}$, the superscript $\mathbb{C}$ will be ommitted.

REMARKS 5.2. Let K be a compact Hausdorff space. We have the inclusion $\mathcal{M}_+(K) \subset \mathcal{M}^{sa}(K)$. Indeed, if we start with $\phi \in \mathcal{M}_+(K)$, then using the fact that every real-valued continuous function $f \in C(K)$ is a difference of non-negative continuous functions $f = f_+ - f_-$, it follows that $\phi(f) = \phi(f_+) - \phi(f_-)$ is a difference of two non-negative (hence real) numbers, so $\phi(f) \in \mathbb{R}$. This implies $\phi^\star = \phi$.

The set $\mathcal{M}_+^{\mathbb{R}}(K)$ is w^*-closed in $\mathcal{M}^{\mathbb{R}}(K)$, and the set $\mathcal{M}_+(K)$ is w^*-closed in $\mathcal{M}(K)$. This follows from the fact that, for each $f \in C^{\mathbb{R}}(K)$, the set

$$\mathcal{A}_f^{\mathbb{K}} = \{f \in \mathcal{M}^{\mathbb{K}}(K) \,:\, \phi(f) \geq 0\} = \epsilon_f^{-1}\big([0,\infty)\big)$$

is w^*-closed, being the preimage of a closed set, under a w^*-continuous map. Then everything is a consequence of the equality

$$\mathcal{M}_+^{\mathbb{K}}(K) = \bigcap_{\substack{f \in C^{\mathbb{R}}(K)\\ f \geq 0}} \mathcal{A}_f^{\mathbb{K}}.$$

In particular, the sets $\mathcal{M}_+^{\mathbb{R}}(K)_1$ and $\mathcal{M}_+(K)_1$ are w^*-compact.

The sets $\mathcal{M}_+^{\mathbb{R}}(K)_1$ and $\mathcal{M}_+(K)_1$ are convex.

Using the identification $\mathcal{M}^{\mathbb{R}}(K) \simeq \mathcal{M}^{sa}(K)$, we have the following hierarchies:

$$\begin{array}{ccc} \mathcal{M}_+^{\mathbb{R}}(K) & \simeq & \mathcal{M}_+(K) \\ \cap & & \cap \\ \mathcal{M}^{\mathbb{R}}(K) & \simeq & \mathcal{M}^{sa}(K) \\ & & \cap \\ & & \mathcal{M}(K) \end{array} \qquad \begin{array}{ccc} \mathcal{M}_+^{\mathbb{R}}(K)_1 & \simeq & \mathcal{M}_+(K)_1 \\ \cap & & \cap \\ \mathcal{M}^{\mathbb{R}}(K)_1 & \simeq & \mathcal{M}^{sa}(K)_1 \\ & & \cap \\ & & \mathcal{M}(K)_1 \end{array}$$

with $\simeq$ isometric and w^*-homeomorphism.

PROPOSITION 5.5. *Let K be a compact Hausdorff space. Then one has the equality*

$$\mathcal{M}^{sa}(K)_1 = \operatorname{conv}\big(\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1\big).$$

(Here conv *denotes the convex cover.)*

PROOF. Denote the set $\operatorname{conv}\big(\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1\big)$ simply by $\mathcal{C}$.

Claim: *One has the equality:*

$$\mathcal{C} = \{t\phi - (1-t)\psi \,:\, \phi, \psi \in \mathcal{M}_+(K)_1,\ t \in [0,1]\}. \tag{3}$$

In particular, the set $\mathcal{C}$ is w^*-compact.

Denote the set on the right hand side of (3) simply by $\mathcal{D}$. The inclusion $\mathcal{C} \supset \mathcal{D}$ is clear. To prove the inclusion $\mathcal{C} \subset \mathcal{D}$, we only need to prove that $\mathcal{D}$ is convex and it contains $\mathcal{M}_+(K)_1 \cup -\mathcal{M}_+(K)_1$. The second property is clear. The convexity of $\mathcal{D}$ is also clear, being a consequence of the convexity of $\pm\mathcal{M}_+(K)_1$.

The w^*-compactness of $\mathcal{C}$ is then a consequence of the compatness of the product space

$$\mathcal{M}_+(K)_1 \times \mathcal{M}_+(K)_1 \times [0,1],$$

and of the fact that $\mathcal{C}$ is the range of the continuous map

$$\mathcal{M}_+(K)_1 \times \mathcal{M}_+(K)_1 \times [0,1] \ni (\phi, \psi, t) \longmapsto t\phi - (1-t)\psi \in \mathcal{M}^{sa}(K).$$

Having proven the Claim, we now proceed with the equality

$$\mathcal{M}^{sa}(K)_1 = \mathcal{C}.$$

The inclusion $\supset$ is clear, since $\mathcal{M}^{sa}(K)_1$ is convex, and it contains both $\mathcal{M}_+(K)_1$ and $-\mathcal{M}_+(K)_1$.

We prove the other inclusion by contradiction. Assume there is some $\phi \in \mathcal{M}^{sa}(K)_1 \smallsetminus \mathcal{C}$. Apply Corollary II.4.2 to find some $f \in C(K)$ and a real number α, such that

$$\operatorname{Re}\phi(f) < \alpha \leq \operatorname{Re}\sigma(f), \ \ \forall\, \sigma \in \mathcal{C}.$$

If we take $g = \operatorname{Re} f$, then this gives

$$\phi(g) < \alpha \leq \sigma(g), \ \ \forall\, \sigma \in \mathcal{C}.$$

Notice that $0 \in \mathcal{C}$, so we get $\alpha \leq 0$. If we define $\beta = -\alpha(\geq 0)$, and $h = -g$, the above inequality gives

$$\phi(h) > \beta \geq \sigma(h), \ \ \forall\, \sigma \in \mathcal{C}.$$

Using the obvious inclusions $\pm\Gamma(K) \subset \mathcal{C}$, we get

$$\beta \geq \pm\gamma_p(h) = \pm h(p), \ \ \forall\, p \in K.$$

Since h is real-valued, this will force $\|h\| \leq \beta$. But then we get a contradiction, because we also have

$$\beta < \phi(h) \leq \|\phi\| \cdot \|h\| \leq \|h\|.$$

□

Corollary 5.3. *Let K be a compact Hausdorff space, and let $\phi \in \mathcal{M}^{sa}(K)$. Then there exist $\phi_1, \phi_2 \in \mathcal{M}_+(K)$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$.*

Proof. If $\phi \in \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, there is nothing to prove. Assume $\phi \notin \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, in particular $\phi \neq 0$. We define $\psi = \dfrac{\phi}{\|\phi\|}$, so that $\psi \in \mathcal{M}^{sa}(K)_1$. Find $\psi_1, \psi_2 \in \mathcal{M}_+(K)_1$ and $t \in [0,1]$, such that

$$\psi = t\psi_1 - (1-t)\psi_2.$$

Since $\psi \notin \mathcal{M}_+(K) \cup -\mathcal{M}_+(K)$, it follows that $0 < t < 1$. Notice that

$$1 = \|\psi\| = \|t\psi_1 - (1-t)\psi_2\| \leq t\|\psi_1\| + (1-t)\|\psi_2\|.$$

If $\|\psi_1\| < 1$, or $\|\psi_2\| < 1$, then this would imply $t\|\psi_1\| + (1-t)\|\psi_2\| < 1$, which is impossible by the above estimate. This argument proves that we must have $\|\psi_1\| = \|\psi_2\| = 1$. If we define

$$\phi_1 = t\|\phi\|\psi_1 \text{ and } \phi_2 = (1-t)\|\phi\|\psi_2,$$

then $\|\phi_1\| = t\|\phi\|$ and $\|\phi_2\| = (1-t)\|\phi\|$, so we indeed have $\|\phi_1\| + \|\phi_2\| = \|\phi\|$. Obviously ϕ_1 and ϕ_2 are positive, and

$$\phi_1 - \phi_2 = \|\phi\| \cdot \big[t\psi_1 - (1-t)\psi_2\big] = \|\phi\| \cdot \psi = \phi.$$

□

PROPOSITION 5.6. *Let K be a compact Hausdorff space. The set*

$$\mathrm{conv}\big(\Gamma(K) \cup \{0\}\big)$$

is w^*-dense in $\mathcal{M}_+(K)_1$.

PROOF. Let $\mathcal{C}$ be the w^*-closure of $\mathrm{conv}\big(\Gamma(K) \cup \{0\}\big)$. It is obvious that $\mathcal{C} \subset \mathcal{M}_+(K)_1$, so we only need to prove the inclusion $\mathcal{M}_+(K)_1 \subset \mathcal{C}$. We do this by contardiction. Assume there exists some $\phi \in \mathcal{M}_+(K)_1 \smallsetminus \mathcal{C}$. Since $\mathcal{C}$ is w^*-closed and convex, there exists some $f \in C(K)$ and a real number α, such that

$$\mathrm{Re}\,\phi(f) < \alpha \leq \mathrm{Re}\,\sigma(f), \;\; \forall\, \sigma \in \mathcal{C}.$$

In particular, if we take $h = -\mathrm{Re}\, f$, and $\beta = -\alpha$, we get

$$\phi(h) > \beta \geq \sigma(h), \;\; \forall\, \sigma \in \mathcal{C}. \tag{4}$$

Sinc $0 \in \mathcal{C}$, we have $\beta \geq 0$. Since $\Gamma(K) \subset \mathcal{C}$, we also get

$$\beta \geq \gamma_p(h) = h(p), \;\; \forall\, p \in K,$$

which menas that $\beta 1 - h \geq 0$. Since ϕ is positive, this will force $\phi(\beta 1 - h) \geq 0$, which gives

$$\phi(h) \leq \phi(\beta 1) = \beta\phi(1) = \beta\|\phi\|.$$

Finally, since $\|\phi\| \leq 1$, this gives

$$\phi(h) \leq \beta,$$

thus contradicting (4). □

The results for the Banach spaces of the form $C(K)$, with K compact Hausdorff space, can be generalized, with suitable modifications, to the situation when K is replaced with a locally compact space. The following result in fact reduces the analysis to the compact case.

THEOREM 5.5. *Let Ω be a locally compact space, and let Ω^β be the Stone-Cech compactification of Ω. Then the restriction map*

$$R : C^{\mathbb{K}}(\Omega^\beta) \ni f \longmapsto f\big|_\Omega \in C_b^{\mathbb{K}}(\Omega)$$

is an isometric linear isomorphism.

PROOF. The linearity is obvious.

Let us show that R is surjective. We show that R is bijective, by exhibiting an inverse for it. For every $h \in C_b^{\mathbb{K}}(\Omega)$, we consider the compact set

$$K_h = \{z \in \mathbb{K} \,:\, |z| \leq \|h\|\},$$

so that we can regard h as a continuous map $\Omega \to K_h$. We know from the functoriality of the Stone-Cech compactification that there exists a unique continuous map $h^\beta : \Omega^\beta \to K_h^\beta$, with $h^\beta\big|_\Omega = h$. Since K_h is compact, we have $K_h^\beta = K_h$. In particular, this gives the inequality

$$|h^\beta(x)| \leq \|h\|, \;\; \forall\, x \in \Omega^\beta. \tag{5}$$

Define the map $T : C_b^{\mathbb{K}}(\Omega) \ni h \longmapsto h^\beta \in C^{\mathbb{K}}(\Omega^\beta)$, and let us show that T is an inverse for R. The equality $R \circ T = \text{Id}$ is trivial, by construction. To prove the equality $T \circ R = \text{Id}$, we start with some $f \in C_b^{\mathbb{K}}(\Omega)$, and we consider $h = Rf$. Then $Th = h^\beta$, and since $h^\beta\big|_\Omega = h = f\big|_\Omega$, the denisty of Ω in Ω^β clearly forces $f = h^\beta = Th = T(Rf)$.

The fact that R is isometric is now clear, because on the one hand we clearly have $\|Rf\| \leq \|f\|$, $\forall\, f \in C^{\mathbb{K}}(\Omega^\beta)$, and on the other hand, by (5), we also have $\|Th\| \leq \|h\|$, $\forall\, h \in C_b^{\mathbb{K}}(\Omega)$. □

If Ω is a locally compact space, the above result suggests that the space $C_b^{\mathbb{K}}(\Omega)$ is quite "large." It is then natural to look at smaller spaces.

Definitions. Let Ω be a locally compact space. If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $f : \Omega \to \mathbb{K}$ is a continuous function, we define the *support of f* by

$$\operatorname{supp} f = \overline{\{\omega \in \Omega \,:\, f(\omega) \neq 0\}}.$$

We define the space

$$C_c^{\mathbb{K}}(\Omega) = \big\{ f : \Omega \to \mathbb{K} \,:\, f \text{ continuous, with compact support} \big\}.$$

When $\mathbb{K} = \mathbb{C}$, this space will be denoted simply by $C_c(\Omega)$. Remark that, when equipped with pointwise addition and multiplication, the space $C_c^{\mathbb{K}}(\Omega)$ becomes a $\mathbb{K}$-algebra. One has obviously the inclusion $C_c^{\mathbb{K}}(\Omega) \subset C_b^{\mathbb{K}}(\Omega)$.

We define $C_0^{\mathbb{K}}(\Omega) = \overline{C_c^{\mathbb{K}}(\Omega)}$, the closure of $C_c^{\mathbb{K}}(\Omega)$ in $C_b^{\mathbb{K}}(\Omega)$. (When $\mathbb{K} = \mathbb{C}$, we will denote this space simply by $C_0(\Omega)$.) The Banach space $C_0^{\mathbb{K}}(\Omega)$ can be regarded as the completion of $C_c^{\mathbb{K}}(\Omega)$. Of course, when Ω is compact, we have the equality $C_0^{\mathbb{K}}(\Omega) = C^{\mathbb{K}}(\Omega)$.

The following result characterizes the Banach space $C_0^{\mathbb{K}}(\Omega)$.

Proposition 5.7. *Let Ω be a locally compact space. For a function $f \in C_b^{\mathbb{K}}(\Omega)$, the following are equivalent:*

(i) *$f \in C_0^{\mathbb{K}}(\Omega)$;*
(ii) *for every $\varepsilon > 0$, there exists some compact subset $K_\varepsilon \subset \Omega$, such that*

$$\sup_{\omega \in \Omega \smallsetminus K_\varepsilon} |f(\omega)| \leq \varepsilon.$$

Proof. *(i)* ⇒ *(ii)*. Suppose $f \in C_0^{\mathbb{K}}(\Omega)$, which means that there exists some sequence $(f_n)_{n=1}^\infty \subset C_c^{\mathbb{K}}(\Omega)$, such that $\lim_{n\to\infty} f_n = f$, in the norm topology in $C_b^{\mathbb{K}}(\Omega)$. Fix some $\varepsilon > 0$, and choose $k \geq 1$, such that $\|f - f_k\| \leq \varepsilon$. If we define $K_\varepsilon = \operatorname{supp} f_k$, then, for every $\omega \in \Omega \smallsetminus K_\varepsilon$, we have $f_k(\omega) = 0$, so the inequality $\|f - f_k\| \leq \varepsilon$ forces $|f(\omega)| \leq \varepsilon$.

(ii) ⇒ *(i)*. Suppose f satisfies property (ii). Fix for the moment an integer $n \geq 1$. Use condition (ii) to find a compact subset $K_n \subset \Omega$, such that

$$|f(\omega)| \leq \frac{1}{n},\ \ \forall\, \omega \in \Omega \smallsetminus K_n.$$

Use Urysohn Lemma to choose some continuous function $h_n : \Omega \to [0,1]$, with compact support, such that $h_n\big|_{K_n} = 1$. Define the function $f_n = h_n f$, so that $f_n \in C_c^{\mathbb{K}}(\Omega)$. If $\omega \in \Omega \smallsetminus K_n$, then, using the inequality $0 \leq h_n \leq 1$, and the choice of K_n, we have

$$|f(\omega) - f_n(\omega)| = |f(\omega)| \cdot [1 - h_n(\omega)] \leq |f(\omega)| \leq \frac{1}{n}.$$

Using the fact that $f_n\big|_{K_n} = f\big|_{K_n}$, the above equality proves that $\|f - f_n\| \le \frac{1}{n}$. This way we have constructed a sequence $(f_n)_{n=1}^\infty \subset C_c^{\mathbb{K}}(\Omega)$, such that $\lim_{n\to\infty} f_n = f$, in $C_b^{\mathbb{K}}(\Omega)$, so by the definition it follows that $f \in C_0^{\mathbb{K}}(\Omega)$. ☐

The following establishes an interesting connection with the Alexandrov compactification.

Proposition 5.8. *Let Ω be a locally compact space, which is non-compact, and let $\Omega^\alpha = \Omega \sqcup \{\infty\}$ denote the Alexandrov compactification.*

(i) *For every function $f \in C_0^{\mathbb{K}}(\Omega)$, the function $f^\alpha : \Omega^\alpha \to \mathbb{K}$, defined by $f^\alpha\big|_\Omega = f$, and $f^\alpha(\infty) = 0$, is continuous.*
(ii) *The correspondence $U : C_0^{\mathbb{K}}(\Omega) \ni f \longmapsto f^\alpha \in C^{\mathbb{K}}(\Omega^\alpha)$ is an isometric linear map.*
(iii) *One has the equality*

$$\operatorname{Ran} U = \big\{ g \in C^{\mathbb{K}}(\Omega^\alpha) \,:\, g(\infty) = 0 \big\}. \tag{6}$$

Proof. (i). We know that Ω is open in Ω^α, which immediately gives the fact that f^α is continuous at every point $\omega \in \Omega$. So all we need to show is the continuity of f^α at ∞. This amounts to showing that for every neighborhood N of $f^\alpha(\infty) = 0$ in $\mathbb{K}$, there exists a neighborhood V of ∞ in Ω^α, such that $f^\alpha(V) \subset N$. Start with a neighborhood N of 0, and choose $\varepsilon > 0$, such that the set $B_\varepsilon = \{z \in \mathbb{K} : |z| \le \varepsilon\}$ is contained in N. Choose some compact set $K_\varepsilon \subset \Omega$, such that

$$\sup_{\omega \in \Omega \smallsetminus K_\varepsilon} |f(\omega)| \le \varepsilon.$$

Define the set $D = (\Omega \smallsetminus K_\varepsilon) \cup \{\infty\}$. By the definition of the topology on Ω^α, the set D is an open neigborhood of ∞. We are now done, because we clearly have

$$|f^\alpha(x)| \le \varepsilon, \ \ \forall\, x \in D,$$

which gives the inclusion $f^\alpha(D) \subset B_\varepsilon \subset N$.

(ii). This part is trivial.

(iii). Denote the right hand side of (6) by $\mathcal{A}$. The inclusion $\operatorname{Ran} U \subset \mathcal{A}$ is trivial, by definition. Conversely, let us start with some $g \in \mathcal{A}$, and let us consider the function $f = g\big|_\Omega$. Let us show that $f \in C_0^{\mathbb{K}}(\Omega)$, using Proposition 5.7. Start with some $\varepsilon > 0$, and choose some open neighborhood D_ε of ∞, in Ω^α, such that

$$|g(x)| \le \varepsilon, \ \ \forall\, x \in D_\varepsilon.$$

By definition, there exists a compact subset $K_\varepsilon \subset \Omega$, such that $D_\varepsilon = \Omega^\alpha \smallsetminus K_\varepsilon$, so it is immediate that f satisfies condition (ii) from Proposition 5.7. Notice now that, by construction we have $f^\alpha\big|_\Omega = g\big|_\Omega$, and $f^\alpha(\infty) = 0 = g(\infty$, so we indeed get $g = Uf$. ☐

Remark 5.3. Let Ω be a locally compact space, which is non-compact. Use the map U defined above, to identify $C_0^{\mathbb{K}}(\Omega)$ with the subspace $\operatorname{Ran} U \subset C^{\mathbb{K}}(\Omega^\alpha)$. With this identification, we have the equality

$$C^{\mathbb{K}}(\Omega^\alpha) = \mathbb{K}1 + C_0^{\mathbb{K}}(\Omega) = \big\{ \lambda 1 + f \,:\, \lambda \in \mathbb{K},\ f \in C_0^{\mathbb{K}}(\Omega) \big\}.$$

Indeed, if we start with some function $g \in C^{\mathbb{K}}(\Omega^\alpha)$ and we take $\lambda = g(\infty)$ and $f = g - \lambda 1$, then $f(\infty) = 0$. Note that this argument proves that in fact every $g \in C^{\mathbb{K}}(\Omega^\alpha)$, can be *uniquely* represented as $g = \lambda 1 + f$, with $\lambda \in \mathbb{K}$, and $f \in C_0^{\mathbb{K}}(\Omega)$.

We conclude with a couple of generalizations of the various results in this section. The first two ones are proven, the rest are stated as exercises. The following result is a generalization of Proposition 5.4.

PROPOSITION 5.9. *Let Ω be a locally compact space, and let $\phi : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$ be a positive linear map. Then ϕ is continuous, and one has the equality*

$$\|\phi\| = \sup\{\phi(f) \,:\, f \in C_0^{\mathbb{R}}(\Omega),\ 0 \leq f \leq 1\}. \tag{7}$$

PROOF. Let us denote the right hand side of (7) by M. First we show that $M < \infty$. If $M = \infty$, there exists a sequence $(f_n)_{n=1}^{\infty} \subset C_0^{\mathbb{R}}(\Omega)$, such that

$$0 \leq f_n \leq 1 \text{ and } \phi(f_n) \geq 4^n,\ \ \forall\, n \geq 1.$$

Consider then the function $f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$. Since $\sum_{n=1}^{\infty} \left\| \frac{1}{2^n} f_n \right\| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, it follows that $f \in C_0^{\mathbb{R}}(\Omega)$. Notice however that, since we obviously have $\frac{1}{2^n} f_n \leq f$, by the positivity of ϕ, we get

$$\phi(f) \geq \phi\big(\frac{1}{2^n} f_n\big) = \frac{1}{2^n}\phi(f_n) \geq 2^n,\ \ \forall\, n \geq 1,$$

which is clearly impossible. Let us show now that ϕ is continuous, by proving the inequality

$$|\phi(f)| \leq M,\ \ \forall\, f \in C_0^{\mathbb{R}}(\Omega), \text{ with } \|f\| \leq 1. \tag{8}$$

Start with some arbitrary function $f \in C_0^{\mathbb{R}}(\Omega)$. The functions $g^{\pm} = |f| \pm f \in C_0^{\mathbb{R}}(\Omega)$, clearly satisfy $g \geq 0$, so we get $\phi(|f| \pm f) \geq 0$, so we get $\phi(|f|) \geq \pm\phi(f)$. This gives $|\phi(f)| \leq \phi(|f|)$, and since $0 \leq |f| \leq 1$, we immediately get (8).

The inequality (8) proves the inequality $\|\phi\| \leq M$. Since we obviously have $M \leq \|\phi\|$, we get in fact the equality (7). □

COROLLARY 5.4. *Let Ω be a locally compact space, which is non-compact, and let Ω^α be the Alexandrov compactification of Ω. Using the inclusion $C_0^{\mathbb{R}}(\Omega) \subset C^{\mathbb{R}}(\Omega^\alpha)$, given by Proposition 5.8, every positive linear map $\phi : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$ can be uniquely extended to a positive linear map $\psi : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$, such that $\|\psi\| = \|\phi\|$.*

PROOF. For every $g \in C^{\mathbb{R}}(\Omega^\alpha)$, we know that there exists a unique $\lambda \in \mathbb{R}$ and $f \in C_0^{\mathbb{R}}(\Omega)$, such that $g = \lambda 1 + f$ (namely $\lambda = g(\infty)$ and $f = g - \lambda 1$). We then define $\psi(g) = \lambda\|\phi\| + \phi(f)$. Notice that $\psi(1) = \|\phi\|$. It is obvious that $\psi : C^{\mathbb{R}}(\Omega^\alpha) \to \mathbb{R}$ is linear, and $\psi\big|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Let us show that ψ is positive. Start with some $g \in C^{\mathbb{R}}(\Omega^\alpha)$ with $g \geq 0$, and let us prove that $\psi(g) \geq 0$. Write $g = \lambda 1 + f$ with $\lambda \in \mathbb{R}$ and $f \in C_0^{\mathbb{R}}(\Omega)$. We know that $\lambda = g(\infty) \geq 0$. If $\lambda = 0$, there is nothing to prove. If $\lambda > 0$, we define the function $h = \lambda^{-1} f \in C_0^{\mathbb{R}}(\Omega)$, so that $g = \lambda(1 + h)$. The positivity of g forces $1 + h \geq 0$, which means if we consider the function $h^- = \max\{-h, 0\} \in C_0^{\mathbb{R}}(\Omega)$, then we have $0 \leq h^- \leq 1$, as well as $h^- + h \geq 0$. Using the above result, this will then give

$$\|\phi\| + \phi(h) \geq \phi(h^-) + \phi(h) = \phi(h^- + h) \geq 0,$$

which means that $\psi(1 + h) \geq 0$. Consequently we also get

$$\psi(g) = \psi(\lambda(1 + h)) = \lambda\psi(h) \geq 0.$$

Having shown the positivity of ψ, we know that

$$\|\psi\| = \psi(1) = \|\phi\|.$$

To prove uniqueness, start with another positive linear map $\xi : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$, such that $\|\xi\| = \|\phi\|$, with $\xi\big|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Since ξ is positive, this forces $\xi(1) = \|\xi\| = \|\phi\| = \psi(1)$. But then we have

$$\xi(\lambda 1 + f) = \lambda\|\phi\| + \phi(f) = \psi(\lambda 1 + f),\ \ \forall\, \lambda \in \mathbb{R},\ f \in C_0^{\mathbb{R}}(\Omega),$$

which proves that $\xi = \psi$. □

REMARK 5.4. Let Ω be a locally compact space, which is not compact, and let $\phi : C_c^{\mathbb{R}}(\Omega) \to \mathbb{R}$ be a positive linear map. Then the following are equivalent:

(i) ϕ is continuous;

(ii) $\sup\big\{\phi(f) \,:\, f \in C_c^{\mathbb{R}}(\Omega),\ \ 0 \leq f \leq 1\big\} < \infty$.

The implication $(i) \Rightarrow (ii)$ is trivial. To prove the implication $(ii) \Rightarrow (i)$ we follow the exact same steps as in the proof of the equality (7) in Proposition 5.9. Denote the quantity in (ii) by M, and using the inequality $|\phi(f)| \leq \phi(|f|)$, we immediately get $|\phi(f)| \leq M,\ \ \forall\, f \in C_c^{\mathbb{R}}(\Omega)$, with $\|f\| \leq 1$.

Remark also that if ϕ is as above, then we have in fact the equality

$$\|\phi\| = \sup\big\{\phi(f) \,:\, f \in C_c^{\mathbb{R}}(\Omega),\ \ 0 \leq f \leq 1\big\}.$$

The following is a generalization of Corollary 5.3.

PROPOSITION 5.10. *Let Ω be a locally compact space, and let $\phi : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$ be a linear continuous map. Then there exist positive linear maps $\phi_1, \phi_2 : C_0^{\mathbb{R}}(\Omega) \to \mathbb{R}$, such that $\phi = \phi_1 - \phi_2$, and $\|\phi\| = \|\phi_1\| + \|\phi_2\|$.*

PROOF. If Ω is compact there is nothing to prove (this is Corollary 5.3). Assume Ω is non-compact. Use Hahn-Banach Theorem to find a linear continuous map $\psi : C^{\mathbb{R}}(\Omega^\alpha) \to \mathbb{R}$, with $\|\psi\| = 1$ and $\psi\big|_{C_0^{\mathbb{R}}(\Omega)} = \phi$. Apply Corollary 5.3 to find two positive linear maps $\psi_1, \psi_2 : C^{\mathbb{R}}(\Omega^\alpha) \to \mathbb{R}$ such that $\psi = \psi_1 - \psi_2$ and $\|\psi\| = \|\psi_1\| + \|\psi_2\|$. Define the positive linear maps $\phi_k = \psi_k\big|_{C_0^{\mathbb{R}}(\Omega)}$, $k = 1, 2$. We clearly have $\phi = \phi_1 - \phi_2$, and

$$\|\phi_1\| + \|\phi_2\| \leq \|\psi_1\| + \|\psi_2\| = \|\psi\| = \|\phi\| = \|\phi_1 - \phi_2\| \leq \|\phi_1\| + \|\phi_2\|,$$

which forces $\|\phi\| = \|\phi_1\| + \|\phi_2\|$. □

Exercise 3. (Dini's Theorem for locally compact spaces) Let Ω be a locally compact space, let $(f_n)_{n\geq 1} \subset C_0^{\mathbb{R}}(\Omega)$ be a monotone sequence. Assume there is some $f \in C_0^{\mathbb{R}}(\Omega)$, such that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega),\ \ \forall\, \omega \in \Omega.$$

Then $\lim_{n\to\infty} f_n = f$, in the norm topology.

Exercise 4. (Stone-Weierstrass Theorems) Let Ω be a locally compact space, which is non-compact, and let $\mathcal{A} \subset C_0^{\mathbb{K}}(\Omega)$ be a subalgebra, with the following separation properties

- For any two points $\omega_1, \omega_2 \in \Omega$, with $\omega_1 \neq \omega_2$, there exists $f \in \mathcal{A}$ such that $f(\omega_1) \neq f(\omega_2)$.
- For any $\omega \in \Omega$, there exists $f \in \mathcal{A}$ with $f(\omega) \neq 0$.

A. Prove that, if $\mathbb{K} = \mathbb{R}$, then $\mathcal{A}$ is dense in $C_0^{\mathbb{R}}(\mathcal{A})$, in the norm topology.

B. Prove that, if $\mathbb{K} = \mathbb{C}$, and if $\mathcal{A}$ has the property $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$, then $\mathcal{A}$ is dense in $C_0(\Omega)$.

HINT: Work in Ω^α (use Remark 5.3), and prove that $\mathbb{K}1 + \mathcal{A}$ is dense in $C^{\mathbb{K}}(\Omega^\alpha)$.