Lecture 13

4. The weak dual topology

In this section we examine the topological duals of normed vector spaces. Besides the norm topology, there is another natural topology which is constructed as follows.

Definition. Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}(=\mathbb{R}, \mathbb{C})$. For every $x \in \mathcal{X}$, let $\epsilon_x : \mathcal{X}^* \to \mathbb{K}$ be the linear map defined by

$$\epsilon_x(\phi) = \phi(x), \ \ \forall\, \phi \in \mathcal{X}^*.$$

We equipp the vector space $\mathcal{X}^*$ with the weak topology defined by the family $\Xi = (\epsilon_x)_{x\in\mathcal{X}}$. This topology is called the *weak dual topology*, which is denoted by w^*. Recall (see Section 3) that this topology is characterized by the following property

(w^*) *Given a topological space T, a map $f : T \to \mathcal{X}^*$ is continuous with respect to the w^* topology, if and only if $\epsilon_x \circ f : T \to \mathbb{K}$ is continuous, for each $x \in \mathcal{X}$.*

Remark that all the maps $\epsilon_x : \mathcal{X}^* \to \mathbb{K}$, $x \in \mathcal{X}$ are already continuous with respect to the *norm* topology. This gives the fact that

- *the w^* topology on $\mathcal{X}^*$ is weaker than the norm topology.*

Remark 4.1. The w^* topology is Hausdorff. Indeed, if $\phi, \psi \in \mathcal{X}^*$ are such that $\phi \neq \psi$, then there exists some $x \in \mathcal{X}$ such that

$$\epsilon_x(\phi) = \phi(x) \neq \psi(x) = \epsilon_x(\psi).$$

Proposition 4.1. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For every $\varepsilon > 0$, $\phi \in \mathcal{X}^*$, and $x \in \mathcal{X}$, define the set*

$$W(\phi; x, \varepsilon) = \big\{\psi \in \mathcal{X}^* \ : \ |\psi(x) - \phi(x)| < \varepsilon.$$

Then the collection

$$\mathcal{W} = \big\{W(\phi; x, \varepsilon) \ : \ \varepsilon > 0,\, \phi \in \mathcal{X}^*,\, x \in \mathcal{X}\big\}$$

is a subbase for the w^* topology. More precisely, given $\phi \in \mathcal{X}^*$, a set $N \subset \mathcal{X}^*$ is a neighborhood of ϕ with respect to the w^* topology, if and only if, there exist $\varepsilon > 0$ and $x_1, \dots, x_n \in \mathcal{X}$, such that

$$N \supset W(\phi; \varepsilon, x_1) \cap \dots \cap W(\phi; \varepsilon, x_n).$$

Proof. It is clearly sufficient to prove the second assertion, because it would imply the fact that any w^* open set is a union of finite intersections of sets in $\mathcal{W}$.

If we define the collection

$$\mathcal{S} = \big\{\epsilon_x^{-1}(D) \ : \ x \in \mathcal{X},\ D \subset \mathbb{K} \text{ open } \big\},$$

then we know that $\mathcal{S}$ is a subbase for the w^* topology.

Fix $\phi \in \mathcal{X}^*$. Start with some w^* neighborhood N of ϕ, so there exists some w^* open set E with $\phi \in E \subset N$. Using the fact that $\mathcal{S}$ is a subbase for the w^* topology, there exist open sets $D_1, \dots, D_n \subset \mathbb{K}$, and points $x_1, \dots, x_n$, such that

$$\phi \in \bigcap_{k=1}^{n} \epsilon_{x_k}^{-1}(D_k) \subset E.$$

Fix for the moment $k \in \{1, \dots, n\}$. The fact that $\phi \in \epsilon_{x_k}^{-1}(D_k)$ means that $\phi(x_k) \in D_k$. Since D_k is open in $\mathbb{K}$, there exists some $\varepsilon_k > 0$, such that

$$D_k \supset \mathcal{B}_{\varepsilon_k}\big(\phi(x_k)\big).$$

Then if we have an arbitrary $\psi \in W(\phi; \varepsilon_k, x_k)$, we will have

$$|\psi(x_k) - \phi(x_k)| < \varepsilon_k,$$

which gives $\psi \in \epsilon_{x_k}^{-1}(D_k)$. This proves that

$$W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

Notice that, if one takes $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then we clearly have the inclusions

$$W(\phi; \varepsilon, x_k) \subset W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

We then immediately get

$$W(\phi; \varepsilon, x_k) \subset \bigcap_{k=1}^{n} \epsilon_{x_k}^{-1}(D_k) \subset E \subset N,$$

and we are done. □

COROLLARY 4.1. *Let $\mathcal{X}$ be a normed vector space. Then the w^* topology on $\mathcal{X}^*$ is locally convex, i.e.*

- *for every $\phi \in \mathcal{X}^*$ and every w^*-neighborhood N of ϕ, there exists a convex w^*-open set D such that $\phi \in D \subset N$.*

PROOF. Apply the second part of the proposition, together with the obvious fact that each of the sets $W(\phi; \varepsilon, x)$ is convex and w^*-open. □

PROPOSITION 4.2. *Let $\mathcal{X}$ be a normed vector space. When equipped with the w^* topology, the space $\mathcal{X}^*$ is a topological vector space. This means that the maps*

$$\mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \phi + \psi \in \mathcal{X}^*$$
$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \lambda\phi \in \mathcal{X}^*$$

are continuous with respect to the w^* topology on the target space, and the w^* product topology on the domanin.

PROOF. According to the definition of the w^* topology, it suffices to prove that, for every $x \in \mathcal{X}$, the maps

$$\sigma_x : \mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \gamma_x : \epsilon_x(\phi + \psi) \in \mathbb{K}$$
$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \epsilon_x(\lambda\phi) \in \mathbb{K}$$

are continuous. But the continuity of σ_x and γ_x is obvious, since we have

$$\sigma_x(\phi, \psi) = \phi(x) + \phi(x) = \epsilon_x(\phi) + \epsilon_x(\psi), \ \ \forall\, (\phi, \psi) \in \mathcal{X}^* \times \mathcal{X}^*;$$
$$\gamma_x(\lambda, \phi) = \lambda\phi(x) = \lambda\epsilon_x(\phi), \ \ \forall\, (\lambda, \phi, \psi) \in \mathbb{K} \times \mathcal{X}^*.$$

□

Our next goal will be to describe the linear maps $\mathcal{X}^* \to \mathbb{K}$, which are continuous in the w^* topology.

PROPOSITION 4.3. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For a linear map $\omega : \mathcal{X}^* \to \mathbb{K}$, the following are equivalent:*

(i) *ω is continuous with respect to the w^* topology;*
(ii) *there exists some $x \in \mathcal{X}$, such that*

$$\omega(\phi) = \phi(x), \ \ \forall \phi \in \mathcal{X}^*.$$

PROOF. The implication $(ii) \Rightarrow (i)$ is trivial, since condition (ii) gives $\omega = \epsilon_x$

$(i) \Rightarrow (ii)$. Suppose ω is continuous. In particular, ω is continuous at 0, so if we take the set

$$D = \{\lambda \in \mathbb{K} : |\lambda| < 1\},$$

the set

$$\omega^{-1}(D) = \{\phi \in \mathcal{X}^* : |\omega(\phi)| < 1\}$$

is an open neighborhood of 0 in the w^* topology. By Proposition ?? there exist $x_1, \dots, x_n \in \mathcal{X}$, and $\varepsilon > 0$, such that

$$W(0; \varepsilon, x_1) \cap \dots \cap W(0; \varepsilon, x_n) \subset D. \tag{1}$$

Claim 1: *One has the inequality*

$$|\omega(\phi)| \le \varepsilon^{-1} \cdot \max\big\{|\phi(x_1)|, \dots, |\phi(x_n)|\big\}, \ \ \forall \phi \in \mathcal{X}^*.$$

Fix an arbitrary $\phi \in \mathcal{X}^*$, and put $M = \max\big\{|\phi(x_1)|, \dots, |\phi(x_n)|\big\}$. For every integer $k \ge 1$, define

$$\phi_k = \varepsilon\big(M + \tfrac{1}{k}\big)^{-1}\phi,$$

so that

$$|\phi_k(x_j)| = \varepsilon\big(M + \tfrac{1}{k}\big)^{-1}|\phi(x_j)| \le \varepsilon M\big(M + \tfrac{1}{k}\big)^{-1} < \varepsilon, \ \ \forall k \ge 1, \ j \in \{1, \dots, n\}.$$

This proves that $\phi_k \in W(0; \varepsilon, x_j)$, for all $k \ge 1$, and all $j \in \{1, \dots, n\}$. By (1) this will give

$$|\omega(\phi_k)| < 1, \ \ \forall k \ge 1,$$

which reads

$$\varepsilon\big(M + \tfrac{1}{k}\big)^{-1}|\omega(\phi)| < 1, \ \ \forall k \ge 1.$$

This gives

$$|\omega(\phi)| \le \varepsilon^{-1}\big(M + \tfrac{1}{k}\big), \ \ \forall k \ge 1,$$

and it will obviously force

$$|\omega(\phi)| \le \varepsilon^{-1} M.$$

Having proven the Claim, we now define the linear map $T : \mathcal{X}^* \to \mathbb{K}^n$, by

$$T\phi = \big(\phi(x_1), \dots, \phi(x_n)\big), \ \ \forall \phi \in \mathcal{X}^*.$$

Claim 2: *There exists a linear map $\sigma : \mathbb{K}^n \to \mathbb{K}$, such that $\omega = \sigma \circ T$.*

First we show that we have the inclusion

$$\operatorname{Ker}\omega \supset \operatorname{Ker}T.$$

If we start with $\phi \in \operatorname{Ker}T$, then $\phi(x_1) = \dots = \phi(x_n) = 0$, and then by Claim 1 we immediately get $\omega(\phi) = 0$, so ϕ indeed belongs to $\operatorname{Ker}\omega$. We us now a bit of linear algebra. On the one hand, since $\omega\big|_{\operatorname{Ker}T} = 0$, there exists a linear map $\hat{\omega} : \mathcal{X}/\operatorname{Ker}T \to \mathbb{K}$, such that $\omega = \hat{\omega} \circ \pi$, where $\pi : \mathcal{X} \to \mathcal{X}/\operatorname{Ker}T$ denotes the quoatient map. On the other hand, by the Isomorphism Theorem for linear maps, there exists a linear isomorphism $\hat{T} : \mathcal{X}/\text{Ker}\, T \xrightarrow{\sim} \text{Ran}\, T$, such that $\hat{T} \circ \pi = T$. We then define

$$\sigma_0 = \hat{\omega} \circ \hat{T}^{-1} : \text{Ran}\, T \to \mathbb{K},$$

and we will have

$$\sigma_0 \circ T = (\hat{\omega} \circ \hat{T}^{-1}) \circ (\hat{T} \circ \pi) = \hat{\omega} \circ \pi = \omega.$$

We finally extend[1] $\sigma_0 : \text{Ran}\, T \to \mathbb{K}$ to a linear map $\sigma : \mathbb{K}^n \to \mathbb{K}$.

Having proven Claim 2, we choose scalars $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, such that

$$\sigma(\lambda_1, \dots, \lambda_n) = \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n, \ \ \forall\, (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n.$$

We now have

$$\omega(\phi) = \sigma(T\phi) = \sigma\big(\phi(x_1), \dots, \phi(x_n)\big) = \alpha_1\phi(x_1) + \dots + \alpha_n\phi(x_n), \ \ \forall\, \phi \in \mathcal{X}^*,$$

so if we define $x = \alpha_1 x_1 + \dots + \alpha_n x_n$, we claerly have

$$\omega(\phi) = \phi(x), \ \ \forall\, \phi \in \mathcal{X}^*.$$

$(ii) \Rightarrow (i)$. This implication is trivial. □

Corollary 4.2. *Let $\mathcal{X}$ be a normed vector space, let $\mathcal{C} \subset \mathcal{X}^*$ be a convex set, and let $\phi \in \mathcal{X}^* \smallsetminus \overline{\mathcal{C}}^{w^*}$. (Here $\overline{\mathcal{C}}^{w^*}$ denotes the w^*-closure of $\mathcal{C}$.) Then there exists an element $x \in \mathcal{X}$, and a real number α, such that*

$$\text{Re}\, \phi(x) < \alpha \le \text{Re}\, \psi(x), \ \ \forall\, \psi \in \mathcal{C}.$$

Proof. Since the w^* topology on $\mathcal{X}^*$ is locally convex, there exists a convex w^*-open set $\mathcal{A} \subset \mathcal{X}^*$, such that $\phi \in \mathcal{A} \subset \mathcal{X}^* \smallsetminus \overline{\mathcal{C}}^{w^*}$. In particular, we have $\mathcal{A} \cap \mathcal{C} = \varnothing$. Apply the Hahn-Banach separation theorem to find a linear map $\omega : \mathcal{X}^* \to \mathbb{K}$, which is w^*-continuous, and a real number α, such that

$$\text{Re}\, \omega(\rho) < \alpha \le \text{Re}\, \omega(\psi), \ \ \forall\, \rho \in \mathcal{A},\ \psi \in \mathcal{C}.$$

We then apply the above Proposition. □

Comments. The definition of the w^* topology can be used in a more general setting, when $\mathcal{X}$ is just a topological vector space. The above results are still vaild in this general setting.

In general the unit ball

$$(\mathcal{X}^*)_1 = \{\phi \in \mathcal{X}^* \,:\, \|\phi\| \le 1\},$$

although bounded and closed, is *not* compact in the *norm* topology. However, when the w^* topology is used, we have

Theorem 4.1 (Alaoglu). *If $\mathcal{X}$ is a normed vector space, then the unit ball $(\mathcal{X}^*)_1$, in the topological dual space, is compact in the w^* topology.*

Proof. Let us consider the unit ball in $\mathbb{K}$:

$$\mathbb{B} = \{\lambda \in \mathbb{K} \,:\, |\lambda| \le 1\}.$$

Let us also consider the unital ball in $\mathcal{X}$:

$$(\mathcal{X})_1 = \{x \in \mathcal{X} \,:\, \|x\| \le 1\}.$$

[1] One can invoke the Hahn-Banach Theorem here. In fact this is not necessary, since $\text{Ran}\, T \subset \mathbb{K}^n$ are finite dimensional vector spaces.

Define the product space

$$P = \prod_{x \in (\mathfrak{X})_1} \mathbb{B},$$

identified equivalently as the space of maps $(\mathfrak{X})_1 \to \mathbb{B}$. By Tihonov's Theorem, when we equip P with the *product topology*, it will become a *compact* topological space. We denote by $\pi_x : P \to \mathbb{B}$, $x \in (\mathfrak{X})_1$, the projection onto the factor with label x. By definition of the product topology π_x is continuous.

For any $x, y \in (\mathfrak{X})_1$ define the map $\Delta_{x,y} : P \to \mathbb{K}$ by

$$\Delta_{x,y}(f) = \frac{f(x) + f(y)}{2} - f\Big(\frac{x+y}{2}\Big), \ \ \forall\, f \in P.$$

Note that

$$\Delta_{x,y} = \frac{1}{2}(\pi_x + \pi_y) - \pi_{(x+y)/2},$$

so $\Delta_{x,y} : P \to \mathbb{K}$ is obviously continuous. In particular, the set

$$A_{x,y} = \Delta_{x,y}^{-1}(\{0\}) = \Big\{ f \in P \, : \, \frac{f(x)+f(y)}{2} = f\Big(\frac{x+y}{2}\Big) \Big\}$$

is closed in P, for every $x, y \in (\mathfrak{X})_1$.

Similarly, for every $x \in (\mathfrak{X})_1$ and every $\lambda \in \mathbb{B}$, we define the map $\Sigma_{\lambda,x} : P \to \mathbb{K}$ by

$$\Sigma_{\lambda,x}(f) = f(\lambda x) - \lambda f(x), \ \ \forall\, f \in P,$$

then $\Sigma_{\lambda,x}$ is continuous, so the set

$$B_{x,y} = \Sigma_{x,y}^{-1}(\{0\}) = \big\{ f \in P \, : \, f(\lambda x) = \lambda f(x) \big\}$$

is closed in P, for every $\lambda \in \mathbb{B}$, $x \in (\mathfrak{X})_1$.

Define the set

$$L = \Big(\bigcap_{x,y \in (\mathfrak{X})_1} A_{x,y} \Big) \cap \Big(\bigcap_{\substack{\lambda \in \mathbb{B} \\ x \in (\mathfrak{X})_1}} B_{\lambda,y} \Big).$$

Since L is an intersection of closed sets, it follows that L itself is closed. In particular, L is *compact*. By construction, we have

$$L = \left\{ f : (\mathfrak{X})_1 \to \mathbb{B} \ \middle| \ \begin{array}{c} \frac{1}{2}\big[f(x) + f(y)\big] = f\big(\frac{1}{2}[x+y]\big) \text{ and} \\ \\ f(\lambda x) = \lambda f(x), \ \ \forall\, x, y \in (\mathfrak{X})_1, \ \lambda \in \mathbb{B} \end{array} \right\}.$$

For any $f \in L$, we define the map $\psi_f : \mathfrak{X} \to \mathbb{K}$ by

$$\psi_f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \\ \|x\| \cdot f\Big(\dfrac{x}{\|x\|}\Big) & \text{if } x \neq 0 \end{cases}$$

Claim 1: *For any $f \in L$, the map $\psi_f : \mathfrak{X} \to \mathbb{K}$ is linear, and satisfies $\psi_f\big|_{(\mathfrak{X})_1} = f$.*

Fix $f \in L$. Start with some $x \in \mathfrak{X}$ and some $\lambda \in \mathbb{K}$. We have $\|\lambda x\| = |\lambda| \cdot \|x\|$, so we get

$$\psi_f(\lambda x) = \begin{cases} 0 & \text{if either } x = 0, \text{ or } \lambda = 0 \\ \\ |\lambda| \cdot \|x\| \cdot f\Big(\dfrac{\lambda}{|\lambda|} \cdot \dfrac{x}{\|x\|}\Big) & \text{if } \lambda \neq 0 \text{ and } x \neq 0 \end{cases}$$

If $\lambda \neq 0$ and $x \neq 0$, we put

$$\mu = \frac{\lambda}{|\lambda|} \text{ and } y = \frac{x}{\|x\|},$$

and the fact that $\mu \in \mathbb{B}$, $y \in (\mathfrak{X})_1$, and $f \in B_{\mu,y}$, will give

$$f\big(\frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|}\big) = f(\mu y) = \mu f(y) = \frac{\lambda}{|\lambda|} \cdot f\big(\frac{x}{\|x\|}\big) = \frac{\lambda}{|\lambda| \cdot \|x\|}\psi_f(x),$$

so in this case we get

$$\psi_f(\lambda x) = |\lambda| \cdot \|x\| \cdot f\big(\frac{\lambda}{|\lambda|} \cdot \frac{x}{\|x\|}\big) = |\lambda| \cdot \|x\| \cdot \frac{\lambda}{|\lambda| \cdot \|x\|}\psi_f(x) = \lambda\psi_f(x).$$

In the case when either $\lambda = 0$ or $x = 0$, we also get the equality

$$\psi_f(\lambda x) = 0 = \lambda\psi_f(x).$$

This way we have proven the homeogeneity of ψ_f

(2) $$\psi_f(\lambda x) = \lambda\psi_f(x), \ \forall \lambda \in \mathbb{K}, \, x \in \mathfrak{X}.$$

Let us prove now that $\psi_f\big|_{(\mathfrak{X})_1} = f$. If $x = 0$, then using the property

(3) $$f(\mu y) = \mu f(y), \ \forall \mu \in \mathbb{B}, \ y \in (\mathfrak{X})_1$$

with $\mu = 0$ and $y = 0$, we immediately get $f(x) = 0 = \psi_f(x)$. If $x \neq 0$, we use (3) with $\mu = \|x\|$ and $y = \dfrac{x}{\|x\|}$ and we again get

$$f(x) = f(\|x\| \cdot y) = \|x\| \cdot f(y) = \|x\| \cdot f\big(\frac{x}{\|x\|}\big) = \psi_f(x).$$

We now prove that ψ_f is additive. Start with two elements $x, y \in \mathfrak{X}$. Define

$$v = \frac{x}{\|x\| + \|y\| + 1} \text{ and } w = \frac{y}{\|x\| + \|y\| + 1},$$

so that we obviously have $v, w \in (\mathfrak{X})_1$ and

$$x = \{\|x\| + \|y\| + 1\} \cdot v \text{ and } y = \{\|x\| + \|y\| + 1\} \cdot w.$$

By homogeneity, we have

$$\psi_f(x + y) = \psi_f\big(2\{\|x\| + \|y\| + 1\} \cdot \frac{1}{2}[v + w]\big) = 2\{\|x\| + \|y\| + 1\} \cdot f\big(\frac{1}{2}[v + w]\big).$$

Using the fact that $f \in A_{v,w}$ the above computation can be continued to give:

$$\begin{aligned} \psi_f(x + y) &= 2\{\|x\| + \|y\| + 1\} \cdot f\big(\frac{1}{2}[v + w]\big) = \\ &= 2\{\|x\| + \|y\| + 1\} \cdot \frac{1}{2}[f(v) + f(w)] = \\ &= \{\|x\| + \|y\| + 1\} \cdot f(v) + \{\|x\| + \|y\| + 1\} \cdot f(w). \end{aligned}$$

Using the fact that $\psi_f\big|_{(\mathfrak{X})_1} = f$, the above equality gives

$$\psi_f(x + y) = \{\|x\| + \|y\| + 1\} \cdot \psi_f(v) + \{\|x\| + \|y\| + 1\} \cdot \psi_f(w).$$

Finally, using the homogeneity property (2) we get

$$\psi_f(x + y) = \psi_f\big(\{\|x\| + \|y\| + 1\} \cdot v\big) + \psi_f\big(\{\|x\| + \|y\| + 1\} \cdot w\big) = \psi_f(x) + \psi_f(y).$$

Having proven the Claim, let us now observe that, for $f \in L$, the fact that

$$\psi_f(x) = f(x) \in \mathbb{B}, \ \forall x \in (\mathfrak{X})_1,$$

shows that ψ_f is *continuous*, and $\|\psi_f\| \leq 1$. Therefore we have a correctly defined map

$$\Psi : L \ni f \longmapsto \psi_f \in (\mathcal{X}^*)_1.$$

Claim 2: *When $(\mathcal{X}^*)_1$ is equipped with the w^* topology, the map Ψ is continuous.*

By the definition of the w^* topology, we need to prove that $\epsilon_x \circ \Psi : L \to \mathbb{K}$ is continuous, for avery $x \in \mathcal{X}$. If $x = 0$, the composition $\epsilon_x \circ \Psi$ is the constant map 0, so there is nothing to prove. If $x \neq 0$, we define

$$y = \frac{x}{\|x\|} \in (\S)_1,$$

and using Claim 1, we have

$$(\epsilon_x \circ \Psi)(f) = \epsilon_x(\psi_f) = \psi_f(x) = \|x\| \cdot \psi_f\big(\frac{x}{\|x\|}\big) = \|x\| \cdot \psi_f(y) = \|x\| \cdot f(y), \ \ \forall\, f \in L.$$

This proves that

$$\epsilon_x \circ \Psi = \|x\| \cdot \pi_y,$$

and since $\pi_y : P \to \mathbb{B}$ is continuous, the continuity of $\epsilon_x \circ \Psi$ follows.

In order to finish the proof of the Theorem, it then suffices to prove

Claim 3: *The map $\Psi : L \to (\mathcal{X}^*)_1$ is surjective.*

Start with an arbitrary $\phi \in (\mathcal{X}^*)_1$, which means that $\phi : \mathcal{X} \to \mathbb{K}$ is linear, continuous, and

$$|\phi(x)| \leq 1, \ \ \forall\, x \in (\mathcal{X})_1.$$

In particular, if we define $f = \phi\big|_{(\mathcal{X})_1}$, then

$$f(x) \in \mathbb{B}, \ \ \forall\, x \in (\mathcal{X})_1,$$

which means that $f \in P$. Using the fact that ϕ is linear, it is obvious that $f \in L$. Using Claim 1, we have

$$\psi_f(x) = f(x) = \phi(x), \ \ \forall\, x \in (\mathcal{X})_1.$$

Now, since $\psi_f\big|_{(\mathcal{X})_1} = \phi\big|_{(\mathcal{X})_1}$, and both ψ_f and ϕ are linear, we immediately get $\psi_f = \phi$. □

Remarks 4.2. Using the notations from the above proof, the continuous map $\Psi : L \to (\mathcal{X}^*)_1$ is in fact bijective. The only thing we need to prove is the injectivity. Suppose $\psi_f = \psi_g$, for some $f, g \in L$. Then

$$f = \psi_f\big|_{(\mathcal{X})_1} = \psi_g\big|_{(\mathcal{X})_1} = g.$$

Since $\Psi : (\mathcal{X}^*)_1 \to L$ is bijective, continuous, and the spaces $(\mathcal{X}^*)_1$ and L are compact Hausdorff, it follows that Ψ is in fact a *homeomorphism*. The inverse map $\Psi^{-1} : (\mathcal{X}^*)_1 \to L$ is simply defined by

$$\Psi^{-1}(\phi) = \phi\big|_{(\mathcal{X})_1}, \ \ \forall\, \phi \in (\mathcal{X}^*)_1.$$

Proposition 4.4. *Suppose $\mathcal{X}$ is a normed vector space, which is separable in the norm topology. When equipped with the w^* topology, the compact space $(\mathcal{X}^*)_1$ is metrizable.*

Proof. Fix a countable dense subset $\mathcal{M} \subset \mathcal{X}$, and define $(\mathcal{M})_1 = (\mathcal{X})_1 \cap \mathcal{M}$. Notice that $(\mathcal{M})_1$ *is dense in* $(\mathcal{X})_1$. Indeed, if we start with some $x \in (\mathcal{X})_1$, and some $\varepsilon > 0$, then we set $x_\varepsilon = (1-\frac{\varepsilon}{2})x$, and we choose $y \in \mathcal{M}$ such that $\|x_\varepsilon - y\| < \frac{\varepsilon}{2}$. On the one hand, we have

$$\|y\| \leq \|x_\varepsilon - y\| + \|x_\varepsilon\| < \frac{\varepsilon}{2} + \big(1 - \frac{\varepsilon}{2}\big) \cdot \|x\| \leq \frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2} = 1,$$

so $y \in (\mathcal{M})_1$. On the other hand, we have

$$\|y - x\| \leq \|y - x_\varepsilon\| + \|x - x_\varepsilon\| < \frac{\varepsilon}{2} + \Big\|\frac{\varepsilon}{2}x\Big\| \leq \frac{\varepsilon}{2} \cdot \big(1 + \|x\|\big) \leq \varepsilon.$$

Let us use the notations from the proof of Theorem 4.1. Let us then define the product space

$$\prod_{x \in (\mathcal{M})_1} \mathbb{B},$$

equipped with the product topology. Define also the map

$$\Upsilon : \prod_{x \in (\mathcal{X})_1} \mathbb{B} \ni f \longmapsto f\big|_{(\mathcal{M})_1} \in \prod_{x \in (\mathcal{M})_1} \mathbb{B}.$$

It is obvious that Υ is continuous. Let

$$\varkappa : (\mathcal{X}^*)_1 \ni \phi \longmapsto \phi\big|_{(\mathcal{X})_1} \in \prod_{x \in (\mathcal{X})_1} \mathbb{B}.$$

We know that $\varkappa$ is continuous and injective (being the inverse of $\Psi : L \to (\mathcal{X}^*)_1$).

Claim: *The composition* $\Upsilon \circ \varkappa : (\mathcal{X}^*)_1 \to \prod_{x \in (\mathcal{M})_1} \mathbb{B}$ *is injective.*

Indeed, if $\phi, \psi \in (\mathcal{X}^*)_1$ satisfy $(\Upsilon \circ \varkappa)(\phi) = (\Upsilon \circ \varkappa)(\psi)$, then we get $\phi\big|_{(\mathcal{M})_1} = \psi\big|_{(\mathcal{M})_1}$. Since $(\mathcal{M})_1$ is dense in $(\mathcal{X})_1$, this will force $\phi\big|_{(\mathcal{X})_1} = \psi\big|_{(\mathcal{X})_1}$, which finally forces $\phi = \psi$.

Using the above Claim, we see that if we define $Q = (\Upsilon \circ \varkappa)\big((\mathcal{X}^*)_1\big)$, then $Q \subset \prod_{x \in (\mathcal{M})_1} \mathbb{B}$ is compact, and $\Upsilon \circ \varkappa : (\mathcal{X}^*)_1 \to Q$ is a homeomorphism. Notice that $\prod_{x \in (\mathcal{M})_1} \mathbb{B}$ is a countable product of metric spaces, so it is metrizable. Therefore Q is also metrizable, and so will be $(\mathcal{X}^*)_1$. $\square$

Remark 4.3. Assuming $\mathcal{X}$ is separable, and $\mathcal{M} \subset \mathcal{X}$ is a countable dense subset. If we enumerate the countable set $(\mathcal{M})_1$ as

$$(\mathcal{M})_1 = \{y_n \,:\, n \geq 1\},$$

then a metric d that defines the w^* topology on $(\mathcal{X}^*)_1$ can be constructed as

$$d(\phi, \psi) = \sum_{n=1}^{\infty} \frac{|\phi(y_n) - \psi(y_n)|}{2^n}, \ \ \forall\, \phi, \psi \in (\mathcal{X}^*)_1.$$

Comments. Let $\mathcal{X}$ be a normed vector space. One can extend the map $\varkappa$ to a map

$$\tilde{\varkappa} : \mathcal{X}^* \ni \phi \longmapsto \phi\big|_{(\mathcal{X})_1} \in \prod_{x \in (\mathcal{X})_1} \mathbb{K}.$$

This map will still be injective and continuous, and one can show that

$$\tilde{\varkappa} : \mathcal{X}^* \to \tilde{\varkappa}(\mathcal{X}^*)$$

is a homeomorphism, when $\tilde{\varkappa}(\mathcal{X}^*)$ is equipped with the induced topology from the product space $\prod_{x\in(\mathcal{X})_1}\mathbb{K}$. In general however, the set $\tilde{\varkappa}(\mathcal{X}^*)$ is *not* closed in the product space $\prod_{x\in(\mathcal{X})_1}\mathbb{K}$.

If $\mathcal{X}$ is separable, and if one takes a countable dense set $\mathcal{M}\subset\mathcal{X}$, then as before, one also still has a continuous map

$$\tilde{\Upsilon}:\prod_{x\in(\mathcal{X})_1}\mathbb{K}\ni f\longmapsto f\big|_{(\mathcal{M})_1}\in\prod_{x\in(\mathcal{M})_1}\mathbb{K},$$

and the composition

$$\tilde{\Upsilon}\circ\tilde{\varkappa}:\mathcal{X}^*\to\prod_{x\in(\mathcal{M})_1}\mathbb{B}$$

will still be continuous and injective. In general however, it turns out that the map

$$\tilde{\Upsilon}\circ\tilde{\varkappa}:\mathcal{X}^*\to\tilde{\Upsilon}\circ\tilde{\varkappa}(\mathcal{X}^*)$$

is *not* a homeomrphism. The exercise below explains exactly when this is the case.

Exercise 1.* Let $\mathcal{X}$ be a normed vector space, which is of *uncountable* dimension (for example, a Banach space). Prove that the topological space $(\mathcal{X}^*,w^*)$ is not metrizable.

HINT: Assume $(\mathcal{X}^*,w^*)$ is metrizable. Let d be a metric which gives the w^*-topology. Then $0\in\mathcal{X}^*$ will have a countable basic system of neighborhoods. In particular, there exist sequences $(x_n)_{n\geq 1}\subset\mathcal{X}$, and $(\varepsilon_n)_{n\geq 1}\in(0,\infty)$, such that the sets

$$B_n=\bigcap_{k=1}^{n}W(0;\varepsilon_n,x_k)$$

satisfy $B_n\subset\mathcal{B}_{1/n}(0)$, $\forall\, n\geq 1$, where $\mathcal{B}_{1/n}(0)$ denotes the d-open ball of center 0 and radius $1/n$. Consider the set $\mathcal{M}=\{x_n : n\in\mathbb{N}\}$. We know that $\operatorname{Span}\mathcal{M}\subsetneq\mathcal{X}$. Choose some vector $y\in\mathcal{X}\smallsetminus\operatorname{Span}\mathcal{M}$. For every $n\geq 1$, choose a linear map $\psi_n:\operatorname{Span}\{y,x_1,\dots,x_n\}\to\mathbb{K}$, such that $\psi_n(y)=1$, and $\psi_n(x_k)=0$, $\forall\, k\in\{1,\dots,n\}$. Extend (use Hahn-Banach) ψ_n to a linear continuous map $\phi_n:\mathcal{X}\to\mathbb{K}$. Notice now that $\phi_n\in B_n$, for all $n\geq 1$, which would then force $d\text{-}\lim_{n\to\infty}\phi_n=0$. In particular, this would force $\lim_{n\to\infty}\phi_n(x)=0$, $\forall\, x\in\mathcal{X}$. But this is impossible, since $\phi_n(y)=1$, $\forall\, n\geq 1$.

COMMENT. If $\mathcal{X}$ is a normed vector space of *countable* dimension, then $(\mathcal{X}^*,w^*)$ is metrizable. Indeed, if we take a linear basis $\{b_n : n\in\mathbb{N}\}$ for $\mathcal{X}$, then the w^* topology on $\mathcal{X}^*$ is clearly defined by the metric

$$d(\phi,\psi)=\sum_{n=1}^{n}\frac{1}{2^n}\cdot\frac{|\phi(b_n)-\psi(b_n)|}{1+|\phi(b_n)-\psi(b_n)|},\ \ \phi,\psi\in\mathcal{X}^*.$$