

LECTURE 12

3. Banach spaces

DEFINITION. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} . A *Banach space over \mathbb{K}* is a normed \mathbb{K} -vector space $(\mathcal{X}, \|\cdot\|)$, which is complete with respect to the metric

$$d(x, y) = \|x - y\|, \quad x, y \in \mathcal{X}.$$

EXAMPLE 3.1. The field \mathbb{K} , equipped with the absolute value norm, is a Banach space. More generally, the vector space \mathbb{K}^n , equipped with any of the norms

$$\begin{aligned} \|(\lambda_1, \dots, \lambda_n)\|_\infty &= \max\{|\lambda_1|, \dots, |\lambda_n|\}, \\ \|(\lambda_1, \dots, \lambda_n)\|_p &= [|\lambda_1|^p + \dots + |\lambda_n|^p]^{1/p}, \quad p \geq 1, \end{aligned}$$

is a Banach space.

REMARK 3.1. Using the facts from the general theory of metric spaces, we know that for a normed vector space $(\mathcal{X}, \|\cdot\|)$, the following are equivalent:

- (i) \mathcal{X} is a Banach space;
- (ii) given any sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ with $\sum_{n=1}^{\infty} \|x_n\| < \infty$, the sequence $(y_n)_{n \geq 1}$ of partial sums, defined by $y_n = \sum_{k=1}^n x_k$, is convergent;
- (iii) every Cauchy sequence in \mathcal{X} has a convergent subsequence.

This is pretty obvious, since the sequence of partial sums has the property that

$$d(y_{n+1}, y_n) = \|y_{n+1} - y_n\| = \|x_{n+1}\|, \quad \forall n \geq 1.$$

*Exercise 1**. Let \mathcal{X} be a *finite dimensional* normed vector space. Prove that \mathcal{X} is a Banach space.

HINTS: Use induction on $\dim \mathcal{X}$. The case $\dim \mathcal{X} = 1$ is trivial. Assume the statement is true for all normed vector spaces of dimension d , and let us prove it for a normed vector space of dimension $d+1$. Fix such an \mathcal{X} , and a linear basis $\{e_1, e_2, \dots, e_n, e_{d+1}\}$ for \mathcal{X} . Start with a Cauchy sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$. Write each term as

$$x_n = \sum_{k=1}^{d+1} \alpha_n(k) e_k.$$

Prove first that $(\alpha_n(d+1))_{n \geq 1} \subset \mathbb{K}$ is *bounded*. Then extract a subsequence $(x_{n_p})_{p \geq 1}$ such that $(\alpha_{n_p}(d+1))_{p \geq 1}$ is convergent. If we take $\alpha(d+1) = \lim_{p \rightarrow \infty} \alpha_{n_p}(d+1)$, then prove that the sequence $(x_{n_p} - \alpha_{n_p}(d+1)e_{d+1})_{p \geq 1}$ is Cauchy in the space $\text{Span}\{e_1, \dots, e_d\}$. Using the inductive hypothesis, conclude that $(x_{n_p})_{p \geq 1}$ is convergent in \mathcal{X} . Thus, *every Cauchy sequence in \mathcal{X} has a convergent subsequence*, hence \mathcal{X} is Banach.

*Exercise 2**. Let $n \geq 1$ be an integer, and let $\|\cdot\|$ be a norm on \mathbb{K}^n . Prove that there exist constants $C, D > 0$, such that

$$C\|x\|_\infty \leq \|x\| \leq D\|x\|_\infty, \quad \forall x \in \mathbb{K}^n.$$

HINT: Let e_1, \dots, e_n be the standard basis vectors for \mathbb{K}^n , so that

$$\alpha_1 e_1 + \dots + \alpha_n e_n = (\alpha_1, \dots, \alpha_n), \quad \forall (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n.$$

Define $D = \|e_1\| + \dots + \|e_n\|$. The existence of C is equivalent to the existence of some $C' > 0$ such that

$$\|x\|_\infty \leq C' \|x\|, \quad \forall x \in \mathbb{K}^n.$$

(If such a C' exists, then we take $C = 1/C'$.) To prove the existence of C' as above, we consider the set $T = \{x \in \mathbb{K}^n : \|x\| \leq 1\}$, and we need to prove that

$$\sup_{x \in T} \|x\|_\infty < \infty.$$

Argue by contradiction (see also the hint from the preceding exercise).

Exercise 3. Let \mathcal{X} and \mathcal{Y} be normed vector spaces. Consider the product $\mathcal{X} \times \mathcal{Y}$, equipped with the natural vector space structure.

- (i) Prove that $\|(x, y)\| = \|x\| + \|y\|$, $(x, y) \in \mathcal{X} \times \mathcal{Y}$ defines a norm on $\mathcal{X} \times \mathcal{Y}$.
- (ii) Prove that, when equipped with the above norm, $\mathcal{X} \times \mathcal{Y}$ is a Banach space, if and only if both \mathcal{X} and \mathcal{Y} are Banach spaces.

There are two key constructions which enable one to construct new Banach space out of old ones.

PROPOSITION 3.1. *Let \mathcal{X} be a normed vector space, and let \mathcal{Y} be a Banach space. Then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space, when equipped with the operator norm.*

PROOF. Start with a Cauchy sequence $(T_n)_{n \geq 1} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. This means that for every $\varepsilon > 0$, there exists some N_ε such that

$$(1) \quad \|T_m - T_n\| < \varepsilon, \quad \forall m, n \geq N_\varepsilon.$$

Notice that, if one takes for example $\varepsilon = 1$, and we define

$$C = 1 + \max\{\|T_1\|, \|T_2\|, \dots, \|T_{N_1}\|\},$$

then we clearly have

$$(2) \quad \|T_n\| \leq C, \quad \forall n \geq 1.$$

Notice that, using (1), we have

$$(3) \quad \|T_m x - T_n x\| \leq \varepsilon \|x\|, \quad \forall m, n \geq N_\varepsilon, \quad x \in \mathcal{X},$$

which proves that

- for every $x \in \mathcal{X}$, the sequence $(T_n x)_{n \geq 1} \subset \mathcal{Y}$ is Cauchy.

Since \mathcal{Y} is a Banach space, for each $x \in \mathcal{X}$, the sequence $(T_n)_{n \geq 1}$ will be convergent. We define the map $T : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad x \in \mathcal{X}.$$

Using (2) we immediately get

$$\|Tx\| \leq C \|x\|, \quad \forall x \in \mathcal{X}.$$

Since T is obviously linear, this prove that T is continuous. Finally, if we fix $n \geq N_\varepsilon$ and we take $\lim_{m \rightarrow \infty}$ in (3), we get

$$\|T_n x - Tx\| \leq \varepsilon \|x\|, \quad \forall n \geq N_\varepsilon, \quad x \in \mathcal{X},$$

which proves precisely that we have the inequality

$$\|T_n - T\| \leq \varepsilon, \quad \forall n \geq N_\varepsilon,$$

hence $(T_n)_{n \geq 1}$ is convergent to T in the norm topology. \square

COROLLARY 3.1. *If \mathcal{X} is a normed vector space, then its topological dual $\mathcal{X}^* = \mathcal{L}(\mathcal{X}, \mathbb{K})$ is a Banach space.*

PROOF. Immediate from the fact that \mathbb{K} is a Banach space. \square

As a direct application of the above result we get

COROLLARY 3.2. *If I is a non-empty set, if $p \in [1, \infty]$, then $\ell_{\mathbb{K}}^p(I)$ is a Banach space.*

PROOF. For $p = 1$ we know that $\ell^1 \simeq (c_0)^*$. For $p \in (1, \infty]$, we know that $\ell^p \simeq (\ell^q)^*$, where q is Hölder conjugate to p . \square

PROPOSITION 3.2. *Let \mathcal{X} be a Banach space, and let $\mathcal{Z} \subset \mathcal{X}$ be a linear subspace. The following are equivalent:*

- (i) \mathcal{Z} is a Banach space, when equipped with the norm from \mathcal{X} ;
- (ii) \mathcal{Z} is closed in \mathcal{X} , in the norm topology.

PROOF. This is a particular case of a general result from the theory of complete metric spaces. \square

COROLLARY 3.3. *Let I be a non-empty set, and let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} . Then $c_0^{\mathbb{K}}(I)$ is a Banach space.*

PROOF. Use the fact that $c_0^{\mathbb{K}}(I)$ is closed in $\ell_{\mathbb{K}}^{\infty}(I)$. \square

*Exercise 4**. Let \mathcal{X} be an infinite dimensional Banach space, and let B be a linear basis for \mathcal{X} . Prove that B is uncountable.

HINT: If B is countable, say $B = \{b_n : n \in \mathbb{N}\}$, then

$$\mathcal{X} = \bigcup_{n=1}^{\infty} F_n,$$

where $F_n = \text{Span}(b_1, b_2, \dots, b_n)$. Since the F_n 's are finite dimensional linear subspaces, they will be closed. Use Baire's Theorem to get a contradiction.

COMMENTS. A third method of constructing Banach spaces is the completion. If we start with a normed \mathbb{K} -vector space \mathcal{X} , when we regard \mathcal{X} as a metric space, its completion $\tilde{\mathcal{X}}$ is constructed as follows. One defines

$$\text{CS}(\mathcal{X}) = \{\mathbf{x} = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ Cauchy sequence in } \mathcal{X}\}.$$

Two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$ are said to be equivalent, if $\lim_{n \rightarrow \infty} \|x_n - x'_n\| = 0$. In this case one writes $\mathbf{x} \sim \mathbf{x}'$. The completion $\tilde{\mathcal{X}}$ is then defined as the space

$$\tilde{\mathcal{X}} = \text{CS}(\mathcal{X}) / \sim$$

of equivalence classes. For $\mathbf{x} \in \text{CS}(\mathcal{X})$, one denotes by $\tilde{\mathbf{x}}$ its equivalence class in $\tilde{\mathcal{X}}$. Finally for an element $x \in \mathcal{X}$ one denotes by $\langle x \rangle \in \tilde{\mathcal{X}}$ the equivalence class of the constant sequence x .

We know from general theory that $\tilde{\mathcal{X}}$ is a complete metric space, with the distance \tilde{d} (correctly) defined by

$$\tilde{d}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \lim_{n \rightarrow \infty} \|x_n - x'_n\|,$$

for any two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{x}' = (x'_n)_{n \geq 1}$.

It turns out that, in our situation, the space $\text{CS}(\mathcal{X})$ carries a natural vector space structure, defined by pointwise addition and scalar multiplication. Moreover, the space $\tilde{\mathcal{X}}$ is identified as a quotient vector space

$$\tilde{\mathcal{X}} = \text{CS}(\mathcal{X})/\text{NS}(\mathcal{X}),$$

where

$$\text{NS}(\mathcal{X}) = \{\mathbf{x} = (x_n)_{n \geq 1} : (x_n)_{n \geq 1} \text{ sequence in } \mathcal{X} \text{ with } \lim_{n \rightarrow \infty} x_n = 0\}$$

is the linear subspace of null sequences. It then follows that $\tilde{\mathcal{X}}$ carries a natural vector space structure. More explicitly, if we start with a scalar $\lambda \in \mathbb{K}$, and with two elements $p, q \in \tilde{\mathcal{X}}$, which are represented as $p = \tilde{\mathbf{x}}$ and $q = \tilde{\mathbf{y}}$, for two Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{y} = (y_n)_{n \geq 1}$ in \mathcal{X} , then the sequence

$$\mathbf{w} = (\lambda x_n + y_n)_{n \geq 1}$$

is Cauchy in \mathcal{X} , and the element $\lambda p + q \in \tilde{\mathcal{X}}$ is then defined as $\lambda p + q = \tilde{\mathbf{w}}$.

Finally, there is a natural norm on $\tilde{\mathcal{X}}$, (correctly) defined by

$$\|\tilde{\mathbf{x}}\| = \tilde{d}(\tilde{\mathbf{x}}, \langle 0 \rangle) = \lim_{n \rightarrow \infty} \|x_n\|,$$

for all Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$. These considerations then prove that $\tilde{\mathcal{X}}$ is a Banach space, and the map

$$\mathcal{X} \ni x \longmapsto \langle x \rangle \in \tilde{\mathcal{X}}$$

is linear and isometric, in the sense that

$$\|\langle x \rangle\| = \|x\|, \quad \forall x \in \mathcal{X}.$$

In the context of normed vector spaces, the universality property of the completion is stated as follows:

PROPOSITION 3.3. *Let \mathcal{X} be a normed vector space, let $\tilde{\mathcal{X}}$ denote its completion, and let \mathcal{Y} be a Banach space. For every linear continuous map $T : \mathcal{X} \rightarrow \mathcal{Y}$, there exists a unique linear continuous map $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$, such that*

$$\tilde{T}\langle x \rangle = Tx, \quad \forall x \in \mathcal{X}.$$

Moreover the map

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto \tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$$

is an isometric linear isomorphism.

PROOF. If $T : \mathcal{X} \rightarrow \mathcal{Y}$ is linear and continuous, then T is a Lipschitz map with Lipschitz constant $\|T\|$, because

$$\|Tx - Tx'\| \leq \|T\| \cdot \|x - x'\|, \quad \forall x, x' \in \mathcal{X}.$$

We know, from the theory of metric spaces, that there exists a unique continuous map $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$, such that

$$\tilde{T}\langle x \rangle = Tx, \quad \forall x \in \mathcal{X}.$$

We also know that \tilde{T} is Lipschitz, with Lipschitz constant $\|T\|$. The only thing we need to prove is the fact that \tilde{T} is linear. Start with two points $p, q \in \tilde{\mathcal{X}}$, represented as $p = \tilde{\mathbf{x}}$ and $q = \tilde{\mathbf{z}}$, for some Cauchy sequences $\mathbf{x} = (x_n)_{n \geq 1}$ and $\mathbf{z} = (z_n)_{n \geq 1}$ in \mathcal{X} . If $\lambda \in \mathbb{K}$, then $\lambda p + q = \tilde{\mathbf{w}}$, where $\mathbf{w} = (\lambda x_n + z_n)_{n \geq 1}$. We then have

$$\tilde{T}(\lambda p + q) = \lim_{n \rightarrow \infty} T(\lambda x_n + z_n) = \left[\lambda \cdot \lim_{n \rightarrow \infty} Tx_n \right] + \left[\lim_{n \rightarrow \infty} Tz_n \right] = \lambda \tilde{T}p + \tilde{T}q.$$

Let us prove now that $\|\tilde{T}\| = \|T\|$. Since \tilde{T} is Lipschitz, with Lipschitz constant $\|\tilde{T}\|$, we will have $\|\tilde{T}\| \leq \|T\|$. To prove the other inequality, let us consider the sets

$$\mathcal{B}_0 = \{p \in \tilde{\mathcal{X}} : \|p\| \leq 1\}, \mathcal{B}_1 = \{\langle x \rangle : x \in \mathcal{X}, \|x\| \leq 1\}.$$

By definition, we have

$$\|\tilde{T}\| = \sup_{p \in \mathcal{B}_0} \|\tilde{T}p\|.$$

Since we clearly have $\mathcal{B}_0 \supset \mathcal{B}_1$, we get

$$\begin{aligned} \|\tilde{T}\| &= \sup_{p \in \mathcal{B}_1} \|\tilde{T}p\| \geq \sup \{\|\tilde{T}\langle x \rangle\| : x \in \mathcal{X}, \|x\| \leq 1\} = \\ &= \sup \{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\} = \|T\|. \end{aligned}$$

The fact that the map $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \mapsto \tilde{T} \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$ is linear is obvious.

To prove the surjectivity, start with some $S \in \mathcal{L}(\tilde{\mathcal{X}}, \mathcal{Y})$. Consider the map

$$\iota : \mathcal{X} \ni x \mapsto \langle x \rangle \in \tilde{\mathcal{X}}.$$

Since ι is linear and isometric, in particular it is continuous, so the composition $T = S \circ \iota$ is linear and continuous. Notice that

$$S\langle x \rangle = S(\iota(x)) = (S \circ \iota)x = Tx, \quad \forall x \in \mathcal{X},$$

so by uniqueness we have $S = \tilde{T}$. \square

COROLLARY 3.4. *Let \mathcal{X} be a normed space, let \mathcal{Y} be a Banach space, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be an isometric linear map.*

- (i) *Let $\tilde{T} : \tilde{\mathcal{X}} \rightarrow \mathcal{Y}$ be the linear continuous map defined in the previous result. Then \tilde{T} is linear, isometric, and $\tilde{T}(\tilde{\mathcal{X}}) = \overline{T(\mathcal{X})}$.*
- (ii) *\mathcal{X} is complete, if and only if $T(\mathcal{X})$ is closed in \mathcal{Y} .*

PROOF. (i). The fact that \tilde{T} is isometric, and has the range equal to $\overline{T(\mathcal{X})}$ is true in general (i.e. for $\tilde{\mathcal{X}}$ metric space, and \mathcal{Y} complete metric space). The linearity follows from the previous result.

(ii). This is obvious. \square

EXAMPLE 3.2. Let \mathcal{X} be a normed vector space. For every $x \in \mathcal{X}$ define the map $\epsilon_x : \mathcal{X}^* \rightarrow \mathbb{K}$ by

$$\epsilon_x(\phi) = \phi(x), \quad \forall \phi \in \mathcal{X}^*.$$

Then ϵ_x is a linear and continuous. This is an immediate consequence of the inequality

$$|\epsilon_x(\phi)| = |\phi(x)| \leq \|x\| \cdot \|\phi\|, \quad \forall \phi \in \mathcal{X}^*.$$

Notice that this also proves

$$\|\epsilon_x\| \leq \|x\|, \quad \forall x \in \mathcal{X}.$$

Interestingly enough, we actually have

$$(4) \quad \|\epsilon_x\| = \|x\|, \quad \forall x \in \mathcal{X}.$$

To prove this fact, we start with an arbitrary $x \in \mathcal{X}$, and we consider the linear subspace

$$\mathcal{Y} = \mathbb{K}x = \{\lambda x : \lambda \in \mathbb{K}\}.$$

If we define $\phi_0 : \mathcal{Y} \rightarrow \mathbb{K}$, by

$$\phi_0(\lambda x) = \lambda \|x\|, \quad \forall \lambda \in \mathbb{K},$$

then it is clear that $\phi_0(x) = \|x\|$, and

$$|\phi_0(y)| \leq \|y\|, \quad \forall y \in \mathcal{Y}.$$

Use then the Hahn-Banach Theorem to find $\phi : \mathcal{X} \rightarrow \mathbb{K}$ such that $\phi|_{\mathcal{Y}} = \phi_0$, and

$$|\phi(z)| \leq \|z\|, \quad \forall z \in \mathcal{X}.$$

This will clearly imply $\|\phi\| \leq 1$, while the first condition will give $\phi(x) = \phi_0(x) = \|x\|$. In particular, we will have

$$\|x\| = |\phi(x)| = |\epsilon_x(\phi)| \leq \|\epsilon_x\| \cdot \|\phi\| \leq \|\epsilon_x\|.$$

Having proven (4), we now have a linear isometric map

$$E : \mathcal{X} \ni x \longmapsto \epsilon_x \in \mathcal{X}^{**}.$$

Since \mathcal{X}^{**} is a Banach space, we now see that $\tilde{E} : \tilde{\mathcal{X}} \rightarrow \overline{E(\mathcal{X})}$ is an isometric linear isomorphism. In particular, \mathcal{X} is Banach, if and only if $E(\mathcal{X})$ is closed in \mathcal{X}^{**} .

We conclude with a series of results, which are often regarded as the “principles of Banach space theory.” These results are consequences of Baire Theorem.

THEOREM 3.1 (Uniform Boundedness Principle). *Let \mathcal{X} be a Banach space, let \mathcal{Y} be normed vector space, and let $\mathcal{M} \subset \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The following are equivalent*

- (i) $\sup \{\|T\| : T \in \mathcal{M}\} < \infty$;
- (ii) $\sup \{\|Tx\| : T \in \mathcal{M}\} < \infty, \forall x \in \mathcal{X}$.

PROOF. The implication (i) \Rightarrow (ii) is trivial, because if we define

$$M = \sup \{\|T\| : T \in \mathcal{M}\},$$

then by the definition of the norm, we clearly have

$$\sup \{\|Tx\| : T \in \mathcal{M}\} \leq M\|x\|, \quad \forall x \in \mathcal{X}.$$

(ii) \Rightarrow (i). Assume \mathcal{M} satisfies condition (ii). For each integer $n \geq 1$, let us define the set

$$\mathcal{F}_n = \{x \in \mathcal{X} : \|Tx\| \leq n, \forall T \in \mathcal{M}\}.$$

It is obvious that \mathcal{F}_n is a closed subset of \mathcal{X} , for each $n \geq 1$. Moreover, by (ii) we clearly have $\bigcup_{n=1}^{\infty} \mathcal{F}_n = \mathcal{X}$. Using Baire’s Theorem, there exists some $n \geq 1$, such that $\text{Int}(\mathcal{F}_n) \neq \emptyset$. This means that there exists some $x_0 \in \mathcal{X}$ and some $r > 0$, such that

$$\mathcal{F}_n \supset \bar{\mathcal{B}}_r(x_0) = \{y \in \mathcal{X} : \|x - x_0\| \leq r\}.$$

Put $M_0 = \sup \{\|Tx_0\| : T \in \mathcal{M}\}$. Fix for the moment some arbitrary $x \in \mathcal{X}$, with $\|x\| \leq 1$, and some arbitrary element $T \in \mathcal{M}$. The vector $y = x_0 + rx$ clearly belongs to $\bar{\mathcal{B}}_r(x_0)$, so we have $\|Ty\| \leq n$. We then get

$$\|Tx\| = \left\| T\left(\frac{1}{r}(y - x_0)\right) \right\| = \frac{1}{r} \|Ty - Tx_0\| \leq \frac{1}{r} (\|Ty\| + \|Tx_0\|) \leq \frac{1}{r} (n + M_0).$$

Keep T fixed, and use the above estimate, which gives

$$\sup \{\|Tx\| : x \in \mathcal{X}, \|x\| \leq 1\} \leq \frac{n + M_0}{r},$$

to conclude that $\|T\| \leq \frac{n + M_0}{r}$. Since $T \in \mathcal{M}$ is arbitrary, we finally get

$$\sup \{\|T\| : T \in \mathcal{M}\} \leq \frac{n + M_0}{r} < \infty. \quad \square$$

THEOREM 3.2 (Inverse Mapping Theorem). *Let \mathcal{X} and \mathcal{Y} be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a bijective linear continuous map. Then the linear map $T^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$ is also continuous.*

PROOF. Let us denote by \mathcal{A} the open unit ball in \mathcal{X} centered at the origin, i.e.

$$\mathcal{A} = \{x \in \mathcal{X} : \|x\| < 1\}.$$

The first step in the proof is contained in the following.

Claim 1: The closure $\overline{T(\mathcal{A})}$ is a neighborhood of 0 in \mathcal{Y} .

Consider the sequence of closed sets $(\overline{kT(\mathcal{A})})_{k=1}^{\infty}$. (Here we use the notation $k\mathcal{M} = \{kv : v \in \mathcal{M}\}$.) Since the map $v \mapsto kv$ is a homeomorphism, one has the equalities

$$\overline{kT(\mathcal{A})} = \overline{kT(\mathcal{A})} = \overline{T(k\mathcal{A})}, \quad \forall k \geq 1.$$

In particular, we have

$$\bigcup_{k=1}^{\infty} \overline{kT(\mathcal{A})} = \bigcup_{k=1}^{\infty} \overline{T(k\mathcal{A})} \supset \bigcup_{k=1}^{\infty} T(k\mathcal{A}) = T\left(\bigcup_{k=1}^{\infty} [k\mathcal{A}]\right).$$

Since we obviously have $\bigcup_{k=1}^{\infty} [k\mathcal{A}] = \mathcal{X}$, and T is surjective, the above equality shows that $\bigcup_{k=1}^{\infty} \overline{kT(\mathcal{A})} = \mathcal{Y}$. Using Baire's Theorem, there exists some $k \geq 1$, such that $\text{Int}\left(\overline{kT(\mathcal{A})}\right) \neq \emptyset$. Again using the fact that $v \mapsto kv$ is a homeomorphism, this gives $\text{Int}\left(\overline{T(\mathcal{A})}\right) \neq \emptyset$. Fix now some point $y \in \text{Int}\left(\overline{T(\mathcal{A})}\right)$, and some $r > 0$, such that $\overline{T(\mathcal{A})}$ contains the open ball

$$(5) \quad \mathcal{B}_r(y) = \{z \in \mathcal{Y} : \|z - y\| < r\}.$$

The proof of the Claim is then finished, once we prove the inclusion

$$\overline{T(\mathcal{A})} \supset \mathcal{B}_{\frac{r}{2}}(0).$$

To prove this inclusion, start with some arbitrary $v \in \mathcal{B}_{\frac{r}{2}}(0)$, i.e. $v \in \mathcal{Y}$ and $\|v\| < \frac{r}{2}$. Since $\|(2v + y) - y\| = 2\|v\| < r$, using (5) it follows that $2v + y \in \overline{T(\mathcal{A})}$. i.e. there exists a sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{X}$ with $\|x_n\| < 1$, $\forall n \geq 1$, and $2v + y = \lim_{n \rightarrow \infty} Tx_n$. Since y itself belongs to $\overline{T(\mathcal{A})}$, there also exists some sequence $(z_n)_{n=1}^{\infty} \subset \mathcal{X}$, with $\|z_n\| < 1$, $\forall n \geq 1$, and $y = \lim_{n \rightarrow \infty} Tz_n$. On the one hand, if we consider the sequence $(u_n)_{n=1}^{\infty} \subset \mathcal{X}$ given by $u_n = \frac{1}{2}(x_n - z_n)$, then it is clear that

$$\|u_n\| \leq \frac{1}{2}(\|x_n\| + \|z_n\|) < 1, \quad \forall n \geq 1,$$

i.e. $(u_n)_{n=1}^{\infty} \subset \mathcal{A}$. On the other hand, we have

$$\lim_{n \rightarrow \infty} Tu_n = \lim_{n \rightarrow \infty} \frac{1}{2}(Tx_n - Tz_n) = \frac{1}{2}(2v + y - y) = v,$$

so v indeed belongs to $\overline{T(\mathcal{A})}$.

The next step is a slight (but crucial) improvement of Claim 1.

Claim 2: $T(\mathcal{A})$ is a neighborhood of 0.

Start off by choosing $\varepsilon > 0$, such that

$$(6) \quad \overline{T(\mathcal{A})} \supset \mathcal{B}_{\varepsilon}(0).$$

The Claim will follow, once we prove the inclusion

$$(7) \quad T(\mathcal{A}) \supset \mathcal{B}_{\frac{\varepsilon}{2}}(0).$$

To prove this inclusion, we start with some arbitrary $y \in \mathcal{B}_\varepsilon(0)$. We want to construct a sequence of vectors $(x_n)_{n=1}^\infty \subset \mathcal{A}$, such that, for every $n \geq 1$, we have the inequality

$$(8) \quad \left\| y - \sum_{k=1}^n T\left(\frac{1}{2^k} x_k\right) \right\| \leq \frac{\varepsilon}{2^{n+1}}.$$

This sequence is constructed inductively as follows. We start by using (6), and we pick $x_1 \in \mathcal{A}$ such that $\|2y - Tx_1\| < \frac{\varepsilon}{2}$. Once x_1, \dots, x_p are constructed, such that (8) holds with $n = p$, we consider the vector

$$z = 2^{p+1} \left[y - \sum_{k=1}^p T\left(\frac{1}{2^k} Tx_k\right) \right] \in \mathcal{B}_\varepsilon(0),$$

and we use again (6) to find $x_{p+1} \in \mathcal{A}$, such that $\|z - Tx_{p+1}\| \leq \frac{\varepsilon}{2}$. We then clearly have

$$\left\| y - \sum_{k=1}^{p+1} T\left(\frac{1}{2^k} x_k\right) \right\| = \frac{\|z - Tx_{p+1}\|}{2^{p+1}} \leq \frac{\varepsilon}{2^{p+2}},$$

Consider now the series $\sum_{k=1}^\infty \frac{1}{2^k} x_k$. Since $\|x_k\| < 1, \forall k \geq 1$, and \mathcal{X} is a Banach space, by Remark 3.1, the sequence of $(w_n)_{n=1}^\infty \subset \mathcal{X}$ of partial sums

$$w_n = \sum_{k=1}^n \frac{1}{2^k} x_k, \quad n \geq 1,$$

is convergent to some point $x \in \mathcal{X}$. Moreover, since we have

$$\|w_n\| \leq \sum_{k=1}^n \frac{\|x_k\|}{2^k} \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k}, \quad \forall n \geq 1,$$

we get the inequality

$$\|x\| \leq \sum_{k=1}^\infty \frac{\|x_k\|}{2^k} < 1,$$

which means that $x \in \mathcal{A}$. Note also that using these partial sums, the inequality (8) reads

$$\|y - Tw_n\| \leq \frac{\varepsilon}{2^{n+2}}, \quad \forall n \geq 1,$$

so by the continuity of T , we have $y = Tx \in T(\mathcal{A})$.

Let us show now that T^{-1} is continuous. Use Claim 2, to find some $r > 0$ such that

$$(9) \quad T(\mathcal{A}) \supset \mathcal{B}_r(0),$$

and let $y \in \mathcal{Y}$ be an arbitrary vector with $\|y\| \leq 1$. Consider the vector $v = \frac{r}{2}y$, which has $\|v\| \leq \frac{r}{2} < r$. By (9), there exists $x \in \mathcal{A}$, such that $Tx = v$, which means that $T^{-1}y = \frac{2}{r}x$. This forces $\|T^{-1}y\| \leq \frac{2}{r}$. This argument shows that

$$\sup \{ \|T^{-1}y\| : y \in \mathcal{Y}, \|y\| \leq 1 \} \leq \frac{2}{r} < \infty,$$

and the continuity of T^{-1} follows from Proposition 2.4. \square

The following two exercises deal with two more ‘‘principles of Banach space theory.’’

Exercise 5[◇]. (Closed Graph Theorem). Let \mathcal{X} and \mathcal{Y} be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a linear map. Prove that the following are equivalent:

- (i) T is continuous.
- (ii) The *graph of T*

$$\mathcal{G}_T = \{(x, Tx) : x \in \mathcal{X}\}$$

is a closed subset of $\mathcal{X} \times \mathcal{Y}$, in the product topology.

HINT: For the implication (ii) \Rightarrow (i), use Exercise 3, to get the fact that \mathcal{G}_T is a Banach space. Then T is exactly the inverse of $\pi_{\mathcal{X}}|_{\mathcal{G}_T}$, where $\pi_{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X}$ is the projection onto the first coordinate. Use Theorem 3.2.

Exercise 6[◇]. (Open Mapping Theorem). Let \mathcal{X} and \mathcal{Y} be Banach spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a *surjective* linear continuous map. Prove that T is an *open map*, in the sense that

- whenever $\mathcal{D} \subset \mathcal{X}$ is open, it follows that $T(\mathcal{D})$ is open in \mathcal{Y} .

HINT: Consider the linear map

$$S : \mathcal{X} \times \mathcal{Y} \ni (x, y) \longmapsto (x, Tx + y) \in \mathcal{X} \times \mathcal{Y}.$$

Prove that S is linear, continuous, bijective, hence by Theorem 3.2, it is a homeomorphism. Use this fact to prove that for every open set $\mathcal{D} \subset \mathcal{X}$, there exists some open set $\mathcal{E} \subset \mathcal{X} \times \mathcal{Y}$, such that $T(\mathcal{D}) = \pi_{\mathcal{Y}}(\mathcal{E})$, where $\pi_{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ is the projection onto the second coordinate. This reduces the problem to proving the fact that $\pi_{\mathcal{Y}}$ is an open map.