

LECTURES 9-11

2. Normed vector spaces

DEFINITION. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let \mathcal{X} be a \mathbb{K} -vector space. A *norm on \mathcal{X}* is a map

$$\mathcal{X} \ni x \longmapsto \|x\| \in [0, \infty)$$

with the following properties

- (i) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathcal{X}$;
- (ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$, $\forall x \in \mathcal{X}$, $\lambda \in \mathbb{K}$;
- (iii) $\|x\| = 0 \implies x = 0$.

(Note that conditions (i) and (ii) state that $\|\cdot\|$ is a seminorm.)

EXAMPLE 2.1. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} . Fix some non-empty set I , and define

$$c_0^{\mathbb{K}}(I) = \left\{ \alpha : I \rightarrow \mathbb{K} : \inf_{\substack{F \subset I \\ \text{finite}}} \left[\sup_{i \in I \setminus F} |\alpha(i)| \right] = 0 \right\}.$$

Remark that for a function $\alpha : I \rightarrow \mathbb{K}$, the fact that α belongs to $c_0^{\mathbb{K}}(I)$ is equivalent to the following condition:

- For every $\varepsilon > 0$, there exists some finite set $F \subset I$, such that

$$|\alpha(i)| < \varepsilon, \quad \forall i \in I \setminus F.$$

We equip the space $c_0^{\mathbb{K}}(I)$ with the \mathbb{K} -vector space structure defined by point-wise addition and point-wise scalar multiplication. We also define the norm $\|\cdot\|_{\infty}$ by

$$\|\alpha\| = \sup_{i \in I} |\alpha(i)|, \quad \alpha \in c_0^{\mathbb{K}}(I).$$

When $\mathbb{K} = \mathbb{C}$, the space $c_0^{\mathbb{C}}(I)$ is simply denoted by $c_0(I)$. When $I = \mathbb{N}$ - the set of natural numbers - the space $c_0^{\mathbb{K}}(\mathbb{N})$ can be equivalently described as

$$c_0^{\mathbb{K}}(\mathbb{N}) = \{ \alpha = (\alpha_n)_{n \geq 1} \subset \mathbb{K} : \lim_{n \rightarrow \infty} \alpha_n = 0 \}.$$

In this case instead of $c_0^{\mathbb{R}}(\mathbb{N})$ we simply write $c_0^{\mathbb{R}}$, and instead of $c_0(\mathbb{N})$ we simply write c_0 .

Exercise 1. Prove that $\|\cdot\|_{\infty}$ is indeed a norm on $c_0^{\mathbb{K}}(I)$.

EXAMPLE 2.2. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , and let I be a non-empty set. We define the space

$$fn_{\mathbb{K}}(I) = \{ \alpha : I \rightarrow \mathbb{K} : \text{the set } \{i \in I : \alpha(i) \neq 0\} \text{ is finite} \}.$$

Then $fn_{\mathbb{K}}(I)$ is a linear subspace in $c_0^{\mathbb{K}}(I)$.

DEFINITION. Suppose \mathcal{X} is a normed vector space, with norm $\|\cdot\|$. Then there is a natural metric d on \mathcal{X} , defined by

$$d(x, y) = \|x - y\|, \quad x, y \in \mathcal{X}.$$

The topology on \mathcal{X} , defined by this metric, is called the *norm topology*.

Exercise 2. Let \mathcal{X} be a normed vector space, over $\mathbb{K}(= \mathbb{R}, \mathbb{C})$. Prove that, when equipped with the norm topology, \mathcal{X} becomes a topological vector space. That is, the maps

$$\begin{aligned} \mathcal{X} \times \mathcal{X} \ni (x, y) &\longmapsto x + y \in \mathcal{X} \\ \mathbb{K} \times \mathcal{X} \ni (\lambda, x) &\longmapsto \lambda x \in \mathcal{X} \end{aligned}$$

are continuous.

Exercise 3. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. Prove that $\text{fin}_{\mathbb{K}}(I)$ is dense in $c_0^{\mathbb{K}}(I)$ in the norm topology.

EXAMPLE 2.3. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. Define

$$\ell_{\mathbb{K}}^{\infty}(I) = \left\{ \alpha : I \rightarrow \mathbb{K} : \sup_{i \in I} |\alpha(i)| < \infty \right\}.$$

We equip the space $\ell_{\mathbb{K}}^{\infty}(I)$ with the \mathbb{K} -vector space structure defined by point-wise addition and point-wise scalar multiplication. We also define the norm $\|\cdot\|_{\infty}$ by

$$\|\alpha\|_{\infty} = \sup_{i \in I} |\alpha(i)|, \quad \alpha \in \ell_{\mathbb{K}}^{\infty}(I).$$

When $\mathbb{K} = \mathbb{C}$, the space $\ell_{\mathbb{C}}^{\infty}(I)$ is simply denoted by $\ell^{\infty}(I)$. When $I = \mathbb{N}$ - the set of natural numbers - instead of $\ell_{\mathbb{R}}^{\infty}(\mathbb{N})$ we simply write $\ell_{\mathbb{R}}^{\infty}$, and instead of $\ell^{\infty}(\mathbb{N})$ we simply write ℓ^{∞} .

Exercise 4. Prove that $\|\cdot\|_{\infty}$ is indeed a norm on $\ell_{\mathbb{K}}^{\infty}(I)$.

Exercise 5. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. Prove that $c_0^{\mathbb{K}}(I)$ is a linear subspace in $\ell_{\mathbb{K}}^{\infty}(I)$, which is *closed* in the norm topology.

In preparation for the next class of examples, we introduce the following:

DEFINITION. A map $\alpha : I \rightarrow \mathbb{K}$ is said to be *summable*, if there exists some number $s \in \mathbb{K}$ such that

(S) *for every $\varepsilon > 0$ there exists some finite set $F_{\varepsilon} \subset I$ such that*

$$\left| s - \sum_{i \in F} \alpha(i) \right| < \varepsilon, \text{ for all finite sets } F \text{ with } F_{\varepsilon} \subset F \subset I.$$

If such an s exists, then it is unique, and it is denoted by $\sum_{i \in I} \alpha(i)$. In the case when I is *finite*, every map $\alpha : I \rightarrow \mathbb{K}$ is summable, and the above notation agrees with the usual notation for the sum.

Exercise 6. Assume $\alpha : I \rightarrow \mathbb{K}$ is summable. Prove that, for every $\lambda \in \mathbb{K}$, the map $\lambda\alpha : I \rightarrow \mathbb{K}$ is summable, and

$$\sum_{i \in I} \lambda\alpha(i) = \lambda \sum_{i \in I} \alpha(i).$$

If $\beta : I \rightarrow \mathbb{K}$ is another summable map, prove that $\alpha + \beta : I \rightarrow \mathbb{K}$ is summable, and

$$\sum_{i \in I} [\alpha(i) + \beta(i)] = \left[\sum_{i \in I} \alpha(i) \right] + \left[\sum_{i \in I} \beta(i) \right].$$

The following result characterizes summability for non-negative terms

LEMMA 2.1. *Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , let I be a non-empty set, and let $\alpha : I \rightarrow [0, \infty)$. The following are equivalent:*

- (i) α is summable;
- (ii) $\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} < \infty$.

Moreover, in this case we have

$$\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} = \sum_{i \in I} \alpha(i).$$

PROOF. We denote the quantity $\sup \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\}$ simply by t .

(i) \Rightarrow (ii). Assume α is summable, and denote $\sum_{i \in I} \alpha(i)$ simply by s . Choose, for each $\varepsilon > 0$ a finite set $F_\varepsilon \subset I$ such that

$$\left| s - \sum_{i \in F} \alpha(i) \right| < \varepsilon, \text{ for all finite subsets } F \subset I \text{ with } F \supset F_\varepsilon.$$

Claim: For any finite set $G \subset I$, and any $\varepsilon > 0$, one has the inequality

$$\sum_{i \in G} \alpha(i) < s + \varepsilon.$$

Indeed, if we take the finite set $G \cup F_\varepsilon$, then using the fact that all α 's are non-negative, we get

$$\sum_{i \in G} \alpha(i) \leq \sum_{i \in G \cup F_\varepsilon} \alpha(i) < s + \varepsilon.$$

Using the Claim, which holds for any $\varepsilon > 0$, we immediately get

$$\sum_{i \in G} \alpha(i) \leq s, \text{ for all finite subsets } G \subset I,$$

so taking supremum yields $t \leq s$, in particular $t < \infty$.

(ii) \Rightarrow (i). Assume condition (ii) is true. We are going to show that α is summable, by proving that the number t satisfies the definition of summability. Consider the set

$$S = \left\{ \sum_{i \in F} \alpha(i) : F \text{ finite subset of } I \right\},$$

so that $\sup S = t < \infty$. Start with some $\varepsilon > 0$. Since $t - \varepsilon$ is no longer an upper bound for S , there exists some finite set $F_\varepsilon \subset I$, such that $\sum_{i \in F_\varepsilon} \alpha(i) > t - \varepsilon$. Notice that, for any finite set $F \subset I$ with $F \supset F_\varepsilon$, we have

$$t - \varepsilon < \sum_{i \in F_\varepsilon} \alpha(i) \leq \sum_{i \in F} \alpha(i) \leq t,$$

so we immediately get

$$\left| t - \sum_{i \in F} \alpha(i) \right| < \varepsilon.$$

□

Exercise 7. Let $\alpha : I \rightarrow [0, \infty)$ be summable. Prove that every map $\beta : I \rightarrow [0, \infty)$ with

$$\beta(j) \leq \alpha(j), \quad \forall j \in I,$$

is summable, and $\sum_{j \in I} \beta(j) \leq \sum_{j \in I} \alpha(j)$.

REMARK 2.1. It is obvious that the above result has a version for non-positive maps as well. More explicitly, for a map $\alpha : I \rightarrow (-\infty, 0]$ the following are equivalent:

- (i) α is summable;
- (ii) $\inf \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} > -\infty$.

Moreover, in this case we have

$$\inf \left\{ \sum_{i \in F} \alpha(i) : F \subset I, \text{ finite} \right\} = \sum_{i \in I} \alpha(i).$$

LEMMA 2.2. Let I be a non-empty set. For a function $\alpha : I \rightarrow \mathbb{C}$, the following are equivalent:

- (i) α is summable;
- (ii) both functions $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable.

Moreover, in this case we have the equality

$$\sum_{j \in I} \alpha(j) = \sum_{j \in I} \operatorname{Re} \alpha(j) + i \sum_{j \in I} \operatorname{Im} \alpha(j).$$

PROOF. (i) \Rightarrow (ii). Assume α is summable. Denote the sum $\sum_{j \in I} \alpha(j)$ simply by s . For every $\varepsilon > 0$ choose a finite set $F_\varepsilon \subset I$ such that

$$\left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \quad \text{for all finite sets } F \subset I \text{ with } F \supset F_\varepsilon.$$

Using the inequality

$$\max \{ |\operatorname{Re} z|, |\operatorname{Im} z| \} \leq |z|, \quad \forall z \in \mathbb{C},$$

we immediately get the inequalities

$$\begin{aligned} \left| \operatorname{Re} s - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| &= \left| \operatorname{Re} \left[s - \sum_{j \in F} \alpha(j) \right] \right| \leq \left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \\ \left| \operatorname{Im} s - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| &= \left| \operatorname{Im} \left[s - \sum_{j \in F} \alpha(j) \right] \right| \leq \left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \end{aligned}$$

for all finite sets $F \subset I$ with $F \supset F_\varepsilon$,

so $\operatorname{Re} \alpha$ and $\operatorname{Im} \alpha$ are indeed summable and moreover, we have

$$\sum_{j \in I} \operatorname{Re} \alpha(j) = \operatorname{Re} s \quad \text{and} \quad \sum_{j \in I} \operatorname{Im} \alpha(j) = \operatorname{Im} s.$$

(ii) \Rightarrow (i). Assume $\operatorname{Re} \alpha$ and $\operatorname{Im} \alpha$ are both summable. Denote $\sum_{j \in I} \operatorname{Re} \alpha(j)$ by u and denote $\sum_{j \in I} \operatorname{Im} \alpha(j)$ by v . Fix some $\varepsilon > 0$. Choose finite sets $E_\varepsilon, G_\varepsilon \subset I$

such that

$$\left| u - \sum_{j \in E} \operatorname{Re} \alpha(j) \right| < \frac{\varepsilon}{2}, \text{ for all finite sets } E \subset I \text{ with } E \supset E_\varepsilon,$$

$$\left| v - \sum_{j \in G} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2}, \text{ for all finite sets } G \subset I \text{ with } G \supset G_\varepsilon.$$

Put $F_\varepsilon = E_\varepsilon \cup G_\varepsilon$. Suppose $F \subset I$ is a finite set with $F \supset F_\varepsilon$. Using the inclusions $F \supset E_\varepsilon$ and $F \supset G_\varepsilon$, we then get

$$\left| u - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| < \frac{\varepsilon}{2} \text{ and } \left| v - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2},$$

so we get

$$\begin{aligned} \left| [u + iv] - \sum_{j \in F} \alpha(j) \right| &= \left| [u - \sum_{j \in F} \operatorname{Re} \alpha(j)] + i[v - \sum_{j \in F} \operatorname{Im} \alpha(j)] \right| \leq \\ &\leq \left| u - \sum_{j \in F} \operatorname{Re} \alpha(j) \right| + \left| v - \sum_{j \in F} \operatorname{Im} \alpha(j) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that α is indeed summable, and $\sum_{j \in I} \alpha(j) = u + iv$. \square

Exercise 8. Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. Suppose one has two non-empty sets I_1, I_2 with $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Suppose $\alpha : I \rightarrow \mathbb{K}$ has the property that both $\alpha|_{I_1} : I_1 \rightarrow \mathbb{K}$ and $\alpha|_{I_2} : I_2 \rightarrow \mathbb{K}$ are summable. Prove that α is summable, and

$$\sum_{j \in I} \alpha(j) = \sum_{j \in I_1} \alpha(j) + \sum_{j \in I_2} \alpha(j).$$

PROPOSITION 2.1. *Let I be a non-empty set, let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} . For a map $\alpha : I \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) α is summable;
- (ii) $|\alpha|$ is summable.

Moreover, in this case one has the inequality

$$(1) \quad \left| \sum_{j \in I} \alpha(j) \right| \leq \sum_{j \in I} |\alpha(j)|.$$

PROOF. (i) \Rightarrow (ii). Assume α is summable. We divide the proof in two cases: Case $\mathbb{K} = \mathbb{R}$. Define the sets

$$\begin{aligned} I^+ &= \{j \in I : \alpha(j) > 0\}, \\ I^- &= \{j \in I : \alpha(j) < 0\}, \\ I^0 &= \{j \in I : \alpha(j) = 0\}. \end{aligned}$$

More generally, for any subset $F \subset I$ we define $F^\pm = F \cap I^\pm$ and $F^0 = F \cap I^0$.

Claim: Both maps $\alpha|_{I^+} : I^+ \rightarrow \mathbb{R}$ and $\alpha|_{I^-} : I^- \rightarrow \mathbb{R}$ are summable. Moreover, one has the equality

$$(2) \quad \sum_{j \in I} \alpha(j) = \sum_{j \in I^+} \alpha(j) + \sum_{j \in I^-} \alpha(j).$$

Denote the sum $\sum_{j \in I} \alpha(j)$ simply by s . Start by choosing some finite set $F \subset I$ such that

$$\left| s - \sum_{j \in G} \alpha(j) \right| < 1, \text{ for all finite sets } G \subset I \text{ with } G \supset F.$$

Let $E \subset I^+$ be a finite subset. Then the set $\tilde{E} = E \cup F$ will be a finite subset of I with $\tilde{E} \supset F$, so we will have

$$\left| s - \sum_{j \in \tilde{E}} \alpha(j) \right| < 1,$$

so we get

$$\begin{aligned} \sum_{j \in E} \alpha(j) &\leq \sum_{j \in E \cup F^+} \alpha(j) = \left[\sum_{j \in E \cup F^+} \alpha(j) + \sum_{j \in F^0 \cup F^-} \alpha(j) \right] - \left[\sum_{j \in F^0 \cup F^-} \alpha(j) \right] = \\ &= \left[\sum_{j \in \tilde{E}} \alpha(j) \right] - \left[\sum_{j \in F^-} \alpha(j) \right] < s + 1 - \left[\sum_{j \in F^-} \alpha(j) \right]. \end{aligned}$$

In particular this gives

$$\sup \left\{ \sum_{j \in E} \alpha(j) : E \subset I^+, \text{ finite} \right\} \leq s + 1 - \left[\sum_{j \in F^-} \alpha(j) \right],$$

so by Lemma ??, the map $\alpha|_{I^+} : I^+ \rightarrow [0, \infty)$ is indeed summable. The fact that the map $\alpha|_{I^-} : I^- \rightarrow (-\infty, 0]$ is summable is proven the exact same way. The equality (2) follows from Exercise ??

Having proven the Claim, we notice now that the map $-\alpha|_{I^-} : I^- \rightarrow [0, \infty)$ is also summable. Using Exercise ??, it is clear then that the map $|\alpha| : I \rightarrow [0, \infty)$ is summable, simply because all the three maps $|\alpha|_{I^+} = \alpha|_{I^+}$, $|\alpha|_{I^-} = -\alpha|_{I^-}$, and $|\alpha|_{I^0} = 0$ are all summable.

Case $\mathbb{K} = \mathbb{C}$. By Lemma ?? we know that the maps $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable. In particular, using the real case, we get the fact that the maps $|\operatorname{Re} \alpha|, |\operatorname{Im} \alpha| : I \rightarrow [0, \infty)$ are summable. Using the obvious inequality

$$|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|, \quad \forall z \in \mathbb{C},$$

we get

$$\sum_{j \in F} |\alpha(j)| \leq \sum_{j \in F} |\operatorname{Re} \alpha(j)| + \sum_{j \in F} |\operatorname{Im} \alpha(j)| \leq \sum_{j \in I} |\operatorname{Re} \alpha(j)| + \sum_{j \in I} |\operatorname{Im} \alpha(j)|,$$

for every finite subset $F \subset I$. Then we get

$$\sup \left\{ \sum_{j \in F} |\alpha(j)| : F \subset I, \text{ finite} \right\} \leq \sum_{j \in I} |\operatorname{Re} \alpha(j)| + \sum_{j \in I} |\operatorname{Im} \alpha(j)| < \infty,$$

so $|\alpha| : I \rightarrow [0, \infty)$ is indeed summable.

Having proven the implication (i) \Rightarrow (ii), let us prove the inequality (1). If s denotes the sum $\sum_{j \in I} \alpha(j)$, then for every $\varepsilon > 0$ there exists $F_\varepsilon \subset I$ finite such that

$$\left| s - \sum_{j \in F} \alpha(j) \right| < \varepsilon, \text{ for all finite sets } F \subset I \text{ with } F \supset F_\varepsilon.$$

In particular, we get

$$|s| \leq \varepsilon + \left| \sum_{j \in F_\varepsilon} \alpha(j) \right| \leq \varepsilon + \sum_{j \in F_\varepsilon} |\alpha(j)| \leq \varepsilon + \sum_{j \in I} |\alpha(j)|.$$

Since this inequality holds for all $\varepsilon > 0$, we then get

$$|s| \leq \sum_{j \in F_\varepsilon} |\alpha(j)|.$$

(ii) \Rightarrow (i). Assume now $|\alpha| : I \rightarrow [0, \infty)$ is summable.

Case $\mathbb{K} = \mathbb{R}$. It is obvious that $|\alpha|_J : J \rightarrow [0, \infty)$ is summable, for any subset $J \subset I$. In particular, using the notations from the proof of (i) \Rightarrow (ii), it follows that $\alpha|_{I^+} = |\alpha|_{I^+}$, $\alpha|_{I^-} = -|\alpha|_{I^-}$, and $\alpha|_{I^0} = 0$ are all summable. Then the summability of α follows from Exercise ??.

Case $\mathbb{K} = \mathbb{C}$. Using the inequality

$$\max \{ |\operatorname{Re} z|, |\operatorname{Im} z| \} \leq |z|, \quad \forall z \in \mathbb{C},$$

combined with Exercise ??, it follows that both maps $|\operatorname{Re} z|, |\operatorname{Im} z| : I \rightarrow [0, \infty)$ are summable. Using the real case it then follows that both maps $\operatorname{Re} \alpha, \operatorname{Im} \alpha : I \rightarrow \mathbb{R}$ are summable. Then the summability of α follows from Lemma ??. \square

The following result shows that summability is essentially the same as the summability of series.

PROPOSITION 2.2. *Suppose $\alpha : I \rightarrow \mathbb{K}$ is summable. Then the support set*

$$[[\alpha]] = \{j \in I : \alpha(j) \neq 0\}$$

is at most countable.

PROOF. For every integer $n \geq 1$, we define the set $J_n = \{j \in I : |\alpha(j)| \geq \frac{1}{n}\}$. Since $|\alpha|$ is summable, the sets $J_n, n \geq 1$ are all finite. The desired result then follows from the obvious equality $[[\alpha]] = \bigcup_{n=1}^{\infty} J_n$. \square

We are now ready to discuss our next class of examples.

EXAMPLE 2.4. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , let I be a non-empty set, and let $p \in [1, \infty)$ be a real number. We define

$$\ell_{\mathbb{K}}^p(I) = \{ \alpha : I \rightarrow \mathbb{K} : |\alpha|^p : I \rightarrow [0, \infty) \text{ summable} \}.$$

For $\alpha \in \ell_{\mathbb{K}}^p(I)$ we define

$$\|\alpha\|_p = \left[\sum_{j \in I} |\alpha(j)|^p \right]^{\frac{1}{p}}.$$

When $\mathbb{K} = \mathbb{C}$, the space $\ell_{\mathbb{C}}^{\infty}(I)$ is simply denoted by $\ell^{\infty}(I)$. When $I = \mathbb{N}$ - the set of natural numbers - instead of $\ell_{\mathbb{R}}^{\infty}(\mathbb{N})$ we simply write $\ell_{\mathbb{R}}^{\infty}$, and instead of $\ell^{\infty}(\mathbb{N})$ we simply write ℓ^{∞} .

In order to show that the ℓ^p spaces ($1 \leq p < \infty$) are normed vector spaces, we will need several preliminary results. The first result we are going to need is the (classical) Hölder inequality.

Exercise 9. Let $q > 1$ and let $u, v \geq 0$. Define the function $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(t) = ut + v(1 - t^q)^{\frac{1}{q}}, \quad t \in [0, 1].$$

Prove that

$$\max_{t \in [0, 1]} f(t) = (u^p + v^p)^{\frac{1}{p}},$$

where $p = \frac{q}{q-1}$. Prove that, unless $u = v = 0$, there exists a unique $s \in [0, 1]$ such that

$$f(s) = \max_{t \in [0, 1]} f(t).$$

HINT: Analyze the derivative: $f'(t) = u - v \left(\frac{t^q}{1 - t^q} \right)^{\frac{1}{p}}$, $t \in (0, 1)$.

LEMMA 2.3 (Hölder's inequality). Let $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be non-negative numbers. Let $p, q > 1$ be real number with the property $\frac{1}{p} + \frac{1}{q} = 1$. Then:

$$(3) \quad \sum_{j=1}^n a_j b_j \leq \left(\sum_{j=1}^n a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^n b_j^q \right)^{\frac{1}{q}}.$$

Moreover, one has equality only when the sequences (a_1^p, \dots, a_n^p) and (b_1^q, \dots, b_n^q) are proportional.

PROOF. The proof will be carried on by induction on n . The case $n = 1$ is trivial.

Case $n = 2$.

Assume $(b_1, b_2) \neq (0, 0)$. (Otherwise everything is trivial). Define the number

$$r = \frac{b_1}{(b_1^q + b_2^q)^{1/q}}.$$

Notice that $r \in [0, 1]$, and we have

$$\frac{b_2}{(b_1^q + b_2^q)^{1/q}} = (1 - r^q)^{1/q}.$$

Notice also that, upon dividing by $(b_1^q + b_2^q)^{1/q}$, the desired inequality

$$(4) \quad a_1 b_1 + a_2 b_2 \leq (a_1^p + a_2^p)^{\frac{1}{p}} (b_1^q + b_2^q)^{\frac{1}{q}}$$

reads

$$a_1 r + a_2 (1 - r^q)^{1/q} \leq (a_1^p + a_2^p)^{1/p},$$

and it follows immediately from the exercise, applied to the function

$$f(t) = a_1 t + a_2 (1 - t^q)^{1/q}, \quad t \in [0, 1].$$

Let us examine when equality holds. If $a_1 = a_2 = 0$, the equality obviously holds, and in this case (a_1, a_2) is clearly proportional to (b_1, b_2) . Assume $(a_1, a_2) \neq (0, 0)$. Put

$$s = \frac{a_1^{p/q}}{(a_1^p + a_2^p)^{1/q}},$$

and notice that

$$(1 - s^q)^{1/q} = \left(1 - \frac{a_1^p}{a_1^p + a_2^p} \right)^{1/q} = \frac{a_2^{p/q}}{(a_1^p + a_2^p)^{1/q}},$$

so we have

$$f(s) = \frac{a_1^{1+\frac{p}{q}} + a_2^{1+\frac{p}{q}}}{(a_1^p + a_2^p)^{\frac{1}{q}}} = \frac{a_1^p + a_2^p}{(a_1^p + a_2^p)^{\frac{1}{q}}} = (a_1^p + a_2^p)^{1-\frac{1}{q}} = (a_1^p + a_2^p)^{\frac{1}{p}} = \max_{t \in [0,1]} f(t).$$

By the exercise, it follows that we have equality in (4) precisely when $r = s$, i.e.

$$\frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}} = \frac{a_1^{\frac{p}{q}}}{(a_1^p + a_2^p)^{\frac{1}{q}}},$$

or equivalently

$$\frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^p}{a_1^p + a_2^p}.$$

Obviously this forces

$$\frac{b_2^q}{b_1^q + b_2^q} = \frac{a_2^p}{a_1^p + a_2^p},$$

so indeed (a_1^p, a_2^p) and (b_1^q, b_2^q) are proportional.

Having proven the case $n = 2$, we now proceed with the proof of:

The implication: Case $n = k \Rightarrow$ Case $n = k + 1$.

Start with two sequences $(a_1, a_2, \dots, a_k, a_{k+1})$ and $(b_1, b_2, \dots, a_k, b_{k+1})$. Define the numbers

$$a = \left(\sum_{j=1}^k a_j^p \right)^{\frac{1}{p}} \text{ and } b = \left(\sum_{j=1}^k b_j^q \right)^{\frac{1}{q}}.$$

Using the assumption that the case $n = k$ holds, we have

$$(5) \quad \sum_{j=1}^{k+1} a_j b_j \leq \left(\sum_{j=1}^k a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^k b_j^q \right)^{\frac{1}{q}} + a_{k+1} b_{k+1} = ab + a_{k+1} b_{k+1}.$$

Using the case $n = 2$ we also have

$$(6) \quad ab + a_{k+1} b_{k+1} \leq (a^p + a_{k+1}^p)^{\frac{1}{p}} \cdot (b^q + b_{k+1}^q)^{\frac{1}{q}} = \left(\sum_{j=1}^{k+1} a_j^p \right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{k+1} b_j^q \right)^{\frac{1}{q}},$$

so combining with (5) we see that the desired inequality (3) holds for $n = k + 1$.

Assume now we have equality. Then we must have equality in both (5) and in (6). On the one hand, the equality in (5) forces $(a_1^p, a_2^p, \dots, a_k^p)$ and $(b_1^q, b_2^q, \dots, b_k^q)$ to be proportional (since we assume the case $n = k$). On the other hand, the equality in (6) forces (a^p, a_{k+1}^p) and (b^q, b_{k+1}^q) to be proportional (by the case $n = 2$). Since

$$a^p = \sum_{j=1}^k a_j^p \text{ and } b^q = \sum_{j=1}^k b_j^q,$$

it is clear that $(a_1^p, a_2^p, \dots, a_k^p, a_{k+1}^p)$ and $(b_1^q, b_2^q, \dots, b_k^q, b_{k+1}^q)$ are proportional. \square

DEFINITION. Two numbers $p, q \in [1, \infty)$ are said to be *Hölder conjugate*, if $\frac{1}{p} + \frac{1}{q} = 1$. Here we use the convention $\frac{1}{\infty} = 0$.

PROPOSITION 2.3. *Let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , let I be a non-empty set, and let $p, q \in [1, \infty]$ be two Hölder conjugate numbers. If $\alpha \in \ell_{\mathbb{K}}^p(I)$ and $\beta \in \ell_{\mathbb{K}}^q(I)$, then $\alpha\beta \in \ell_{\mathbb{K}}^1(I)$, and*

$$\|\alpha\beta\|_1 \leq \|\alpha\|_p \cdot \|\beta\|_q.$$

PROOF. Using Lemma ??, it suffices to prove the inequality

$$(7) \quad \sum_{j \in F} |\alpha(j)\beta(j)| \leq \|\alpha\|_p \cdot \|\beta\|_q,$$

for every finite set $F \subset I$.

Fix for the moment a finite subset $F \subset I$. Assume $p, q \in (1, \infty)$, using Hölder's inequality we have

$$(8) \quad \sum_{j \in F} |\alpha(j)\beta(j)| = \sum_{j \in F} |\alpha(j)| \cdot |\beta(j)| \leq \left[\sum_{j \in F} |\alpha(j)|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{j \in F} |\beta(j)|^q \right]^{\frac{1}{q}}.$$

Notice however that

$$\begin{aligned} \sum_{j \in F} |\alpha(j)|^p &\leq \sum_{j \in I} |\alpha(j)|^p = (\|\alpha\|_p)^p, \\ \sum_{j \in F} |\beta(j)|^q &\leq \sum_{j \in I} |\beta(j)|^q = (\|\beta\|_q)^q, \end{aligned}$$

so we get

$$\left[\sum_{j \in F} |\alpha(j)|^p \right]^{\frac{1}{p}} \leq \|\alpha\|_p \text{ and } \left[\sum_{j \in F} |\beta(j)|^q \right]^{\frac{1}{q}} \leq \|\beta\|_q,$$

so when we go back to (8) we immediately get the desired inequality (7)

In the case when $p = 1$, we immediately have

$$\begin{aligned} \sum_{j \in F} |\alpha(j)\beta(j)| &\leq \left[\sum_{j \in F} |\alpha(j)| \right] \cdot \left[\max_{j \in F} |\beta(j)| \right] \leq \\ &\leq \left[\sum_{j \in I} |\alpha(j)| \right] \cdot \left[\sup_{j \in I} |\beta(j)| \right] = \|\alpha\|_1 \cdot \|\beta\|_\infty. \end{aligned}$$

The case $p = \infty$ is proven in the exact same way. \square

REMARK 2.2. Suppose $p, q \in [1, \infty]$ are Hölder conjugate numbers. For any $\alpha \in \ell_{\mathbb{K}}^p(I)$ and $\beta \in \ell_{\mathbb{K}}^q(I)$, the map $\alpha\beta$ is summable (by Proposition ??). In particular, one can define the number

$$\langle \alpha, \beta \rangle = \sum_{j \in I} \alpha(j)\beta(j) \in \mathbb{K}.$$

As a consequence we get the inequality

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\|_p \cdot \|\beta\|_q, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I), \beta \in \ell_{\mathbb{K}}^q(I).$$

NOTATIONS. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , let I be a non-empty set, and let $q \in [1, \infty]$ be a real number. We define

$$\mathcal{B}_{\mathbb{K}}^q(I) = \{ \alpha \in \text{fin}\mathbb{K}(I) : \|\alpha\|_q \leq 1 \}.$$

(remark that $\text{fin}_{\mathbb{K}}(I) \subset \ell_{\mathbb{K}}^q(I)$, for all $q \in [1, \infty]$.)

THEOREM 2.1 (Dual definition of ℓ^p spaces). *Let $p, q \in (1, \infty)$ be Hölder conjugate numbers, let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. For a function $\alpha : I \rightarrow \mathbb{K}$, the following are equivalent:*

- (i) $\alpha \in \ell_{\mathbb{K}}^p(I)$;
- (ii) $\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle| < \infty$.

Moreover, one has the equality

$$(9) \quad \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle| = \|\alpha\|_p, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I).$$

PROOF. It will be convenient to introduce several notations. Given a function $\alpha : I \rightarrow \mathbb{K}$, and a finite set $F \subset I$, we define the function $\beta_{\alpha}^F : I \rightarrow \mathbb{K}$, as follows:

$$\beta_{\alpha}^F(i) = \begin{cases} \frac{|\alpha(i)|^{1+\frac{p}{q}}}{\alpha(i) \cdot (\sum_{j \in F} |\alpha(j)|^p)^{1/q}} & \text{if } i \in F \text{ and } \alpha(i) \neq 0 \\ 0 & \text{if } i \notin F \text{ or } \alpha(i) = 0 \end{cases}$$

Notice that $[[\beta_{\alpha}^F]] \subset F$, and unless β_{α}^F is identically zero, we have

$$\sum_{i \in [[\beta_{\alpha}^F]]} |\beta_{\alpha}^F(i)|^q = 1.$$

So in any case we have $\beta_{\alpha}^F \in \mathcal{B}_{\mathbb{K}}^q(I)$. Notice also that, unless β_{α}^F is identically zero, we have

$$(10) \quad \begin{aligned} \langle \alpha, \beta_{\alpha}^F \rangle &= \sum_{i \in F} \alpha(i) \beta_{\alpha}^F(i) = \frac{\sum_{i \in F} |\alpha(i)|^{1+\frac{p}{q}}}{(\sum_{j \in F} |\alpha(j)|^p)^{1/q}} = \frac{\sum_{i \in F} |\alpha(i)|^p}{(\sum_{j \in F} |\alpha(j)|^p)^{1/q}} = \\ &= \left(\sum_{i \in F} |\alpha(i)|^p \right)^{1-\frac{1}{q}} = \left(\sum_{i \in F} |\alpha(i)|^p \right)^{1/p}. \end{aligned}$$

It is clear that the equality (10) actually holds even when β_{α}^F is identically zero.

To make the exposition a bit clearer, we denote the quantity $\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle|$

simply by $|||\alpha|||$.

We now proceed with the proof of the Theorem.

(i) \Rightarrow (ii). Assume $\alpha \in \ell_{\mathbb{K}}^p(I)$. In order to prove (ii) it suffices to prove the inequality

$$(11) \quad |||\alpha||| \leq \|\alpha\|_p.$$

Start with some arbitrary $\beta \in \mathcal{B}_{\mathbb{K}}^q(I)$. Using Hölder inequality we have

$$\begin{aligned} |\langle \alpha, \beta \rangle| &= \left| \sum_{j \in [[\beta]]} \alpha(j) \beta(j) \right| \leq \sum_{j \in [[\beta]]} |\alpha(j)| \cdot |\beta(j)| \leq \\ &\leq \left(\sum_{j \in [[\beta]]} |\alpha(j)|^p \right)^{1/p} \cdot \left(\sum_{j \in [[\beta]]} |\beta(j)|^q \right)^{1/q} \leq \left(\sup_{\substack{F \subset I \\ \text{finite}}} \left[\sum_{i \in F} |\alpha(i)|^p \right] \right)^{1/p} = \|\alpha\|_p. \end{aligned}$$

Since this inequality holds for all $\beta \in \mathcal{B}_{\mathbb{K}}^q(I)$, the inequality (11) follows.

(ii) \Rightarrow (i). Assume now $|||\alpha||| < \infty$. In order to prove condition (i) it suffices to prove that

$$(12) \quad \sum_{i \in F} |\alpha(i)|^p \leq |||\alpha|||^p, \text{ for every finite subset } F \subset I.$$

By (10) we know that for every finite subset $F \subset I$ we have

$$(13) \quad \sum_{i \in F} |\alpha(i)|^p \leq |||\alpha|||^p = \langle \alpha, \beta_{\alpha}^F \rangle^p.$$

In particular we get the fact that $\langle \alpha, \beta_\alpha^F \rangle = |\langle \alpha, \beta_\alpha^F \rangle|$, and the fact that β_α^F belongs to $\mathcal{B}_{\mathbb{K}}^q(I)$, combined with (13) will give

$$\sum_{i \in F} |\alpha(i)|^p = |\langle \alpha, \beta_\alpha^F \rangle|^p \leq \left(\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle| \right)^p = \|\alpha\|^p.$$

Having proven the equivalence (i) \Leftrightarrow (ii), let us now observe that (9) is an immediate consequence of (11) and (12). \square

Exercise 10. Prove that Theorem 9.1 holds also in the cases $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$.

COROLLARY 2.1. *Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , let I be a non-empty set, and let $p \geq 1$.*

- (i) *When equipped with point-wise addition and scalar multiplication, the set $\ell_{\mathbb{K}}^p(I)$ is a \mathbb{K} -vector space.*
- (ii) *The map*

$$\ell_{\mathbb{K}}^p(I) \ni \alpha \longmapsto \|\alpha\|_p \in [0, \infty)$$

is a norm.

PROOF. Let q be the Hölder conjugate of p . If $\alpha \in \ell_{\mathbb{K}}^p(I)$, and $\lambda \in \mathbb{K}$, then

$$\langle \lambda\alpha, \beta \rangle = \lambda \langle \alpha, \beta \rangle, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I),$$

so we get

$$\sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \lambda\alpha, \beta \rangle| = |\lambda| \cdot \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha, \beta \rangle|,$$

which gives the fact that $\lambda\alpha \in \ell_{\mathbb{K}}^p(I)$, as well as the equality $\|\lambda\alpha\|_p = |\lambda| \cdot \|\alpha\|_p$.

If $\alpha_1, \alpha_2 \in \ell_{\mathbb{K}}^p(I)$, then

$$\langle \alpha_1 + \alpha_2, \beta \rangle = \langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I),$$

so we get

$$\begin{aligned} \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha_1 + \alpha_2, \beta \rangle| &= \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha_1, \beta \rangle + \langle \alpha_2, \beta \rangle| \leq \\ &\leq \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} (|\langle \alpha_1, \beta \rangle| + |\langle \alpha_2, \beta \rangle|) \leq \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha_1, \beta \rangle| + \sup_{\beta \in \mathcal{B}_{\mathbb{K}}^q(I)} |\langle \alpha_2, \beta \rangle|, \end{aligned}$$

which gives the fact that $\alpha_1 + \alpha_2 \in \ell_{\mathbb{K}}^p(I)$, as well as the inequality

$$\|\alpha_1 + \alpha_2\|_p \leq \|\alpha_1\|_p + \|\alpha_2\|_p.$$

The implication $\|\alpha\|_p = 0 \Rightarrow \alpha = 0$ is obvious. \square

Exercise 11. Let $p \geq 1$ be a real number, let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let I be a non-empty set. Prove that $\text{fin}_{\mathbb{K}}(I)$ is a *dense* linear subspace in $\ell_{\mathbb{K}}^p(I)$.

REMARK 2.3. Let $p, q \in [1, \infty]$ be Hölder conjugate. Then the map

$$\ell_{\mathbb{K}}^p(I) \times \ell_{\mathbb{K}}^q(I) \ni (\alpha, \beta) \longmapsto \langle \alpha, \beta \rangle \in \mathbb{K}$$

is *bilinear*, in the sense that for any $\gamma \in \ell_{\mathbb{K}}^p(I)$ and any $\eta \in \ell_{\mathbb{K}}^q(I)$, the maps

$$\ell_{\mathbb{K}}^p(I) \ni \alpha \longmapsto \langle \alpha, \eta \rangle \in \mathbb{K},$$

$$\ell_{\mathbb{K}}^q(I) \ni \beta \longmapsto \langle \gamma, \beta \rangle \in \mathbb{K}$$

are linear. These facts follow immediately from Exercise ??

We now examine linear continuous maps between normed spaces.

PROPOSITION 2.4. *Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , let \mathcal{X} and \mathcal{Y} be normed \mathbb{K} -vector spaces, and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a \mathbb{K} -linear map. The following are equivalent:*

- (i) *T is continuous;*
- (ii) $\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} < \infty$;
- (iii) $\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \} < \infty$;
- (iv) *T is continuous at 0.*

PROOF. (i) \Rightarrow (ii). Assume T is continuous, but

$$\sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} < \infty,$$

which means there exists some sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ such that

- (a) $\|x_n\| \leq 1, \forall n \geq 1$;
- (b) $\lim_{n \rightarrow \infty} \|Tx_n\| = \infty$.

Put

$$z_n = \|Tx_n\|^{-1} x_n, \quad \forall n \geq 1.$$

On the one hand, we have

$$\|z_n\| = \frac{\|x_n\|}{\|Tx_n\|} \leq \frac{1}{\|Tx_n\|}, \quad \forall n \geq 1,$$

which gives $\lim_{n \rightarrow \infty} \|z_n\| = 0$, i.e. $\lim_{n \rightarrow \infty} z_n = 0$. Since T is assumed to be continuous, we will get

$$(14) \quad \lim_{n \rightarrow \infty} Tz_n = T0 = 0.$$

On the other hand, since T is linear, we have $Tz_n = \|Tx_n\|^{-1} Tx_n$, so in particular we get

$$\|Tz_n\| = 1, \quad \forall n \geq 1,$$

which clearly contradicts (14).

(ii) \Rightarrow (iii). This is obvious, since the supremum in (iii) is taken over a subset of the set used in (ii).

(iii) \Rightarrow (iv). Let $(x_n)_{n \geq 1} \subset \mathcal{X}$ be a sequence with $\lim_{n \rightarrow \infty} x_n = 0$. For each $n \geq 1$, define

$$u_n = \begin{cases} \|x_n\|^{-1} x_n, & \text{if } x_n \neq 0 \\ \text{any vector of norm 1,} & \text{if } x_n = 0 \end{cases}$$

so that we have

$$\|u_n\| = 1 \text{ and } x_n = \|x_n\| u_n, \quad \forall n \geq 1.$$

Since T is linear, we have

$$(15) \quad Tx_n = \|x_n\| Tu_n, \quad \forall n \geq 1.$$

If we define $M = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \}$, then $\|Tu_n\| \leq M, \forall n \geq 1$, so (15) will give

$$\|Tx_n\| \leq M \cdot \|x_n\|, \quad \forall n \geq 1,$$

and the condition $\lim_{n \rightarrow \infty} x_n = 0$ will force $\lim_{n \rightarrow \infty} Tx_n = 0$.

(iv) \Rightarrow (i). Assume T is continuous at 0, and let us prove that T is continuous at any point. Start with some arbitrary $x \in \mathcal{X}$ and an arbitrary sequence $(x_n)_{n \geq 1} \subset \mathcal{X}$ with $\lim_{n \rightarrow \infty} x_n = x$. Put $z_n = x_n - x$, so that $\lim_{n \rightarrow \infty} z_n = 0$. Then we will have $\lim_{n \rightarrow \infty} Tz_n = 0$, which (use the linearity of T) means that

$$0 = \lim_{n \rightarrow \infty} \|Tz_n\| = \lim_{n \rightarrow \infty} \|Tx_n - Tx\|,$$

thus proving that $\lim_{n \rightarrow \infty} Tx_n = Tx$. \square

REMARK 2.4. Using the notations above, the quantities in (ii) and (iii) are in fact equal. Indeed, if we define

$$M_1 = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \},$$

$$M_2 = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \},$$

then as observed during the proof, we have $M_2 \leq M_1$. Conversely, if we start with some arbitrary $x \in \mathcal{X}$ with $\|x\| \leq 1$, then we can always write $x = \|x\|u$, for some $u \in \mathcal{X}$ with $\|u\| = 1$. In particular we will get

$$\|Tx\| = \|x\| \cdot \|Tu\| \leq \|x\| \cdot M_2 \leq M_2.$$

Taking supremum in the above inequality, over all $x \in \mathcal{X}$ with $\|x\| \leq 1$, will then give the inequality $M_1 \leq M_2$.

NOTATIONS. Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , and let \mathcal{X} and \mathcal{Y} be normed \mathbb{K} -vector spaces. We define

$$\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \{ T : \mathcal{X} \rightarrow \mathcal{Y} : T \text{ } \mathbb{K}\text{-linear and continuous} \}.$$

For $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we define (see the above remark)

$$\|T\| = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| \leq 1 \} = \sup \{ \|Tx\| : x \in \mathcal{X}, \|x\| = 1 \}$$

When $\mathcal{Y} = \mathbb{K}$ (equipped with the absolute value as the norm), the space $\mathcal{L}(\mathcal{X}, \mathbb{K})$ will be denoted simply by \mathcal{X}^* , and will be called the *topological dual of \mathcal{X}* .

PROPOSITION 2.5. *Let \mathbb{K} be either \mathbb{R} or \mathbb{C} , and let \mathcal{X} and \mathcal{Y} be normed \mathbb{K} -vector spaces.*

- (i) *The space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a \mathbb{K} -vector space.*
- (ii) *For $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ we have*

$$(16) \quad \|T\| = \min \{ C \geq 0 : \|Tx\| \leq C\|x\|, \forall x \in \mathcal{X} \}.$$

In particular one has

$$(17) \quad \|Tx\| \leq \|T\| \cdot \|x\|, \forall x \in \mathcal{X}.$$

- (iii) *The map $\mathcal{L}(\mathcal{X}, \mathcal{Y}) \ni T \mapsto \|T\| \in [0, \infty)$ is a norm.*

PROOF. The fact that $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a vector space is clear.

(ii). Assume $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. We begin by proving (17). Start with some arbitrary $x \in \mathcal{X}$, and write it as $x = \|x\|u$, for some $u \in \mathcal{X}$ with $\|u\| = 1$. Then by definition we have $\|Tu\| \leq \|T\|$, and by linearity we have

$$\|Tx\| = \|x\| \cdot \|Tu\| \leq \|x\| \cdot \|T\|.$$

To prove the equality (16) let us define the set

$$\mathcal{C}_T = \{ C \geq 0 : \|Tx\| \leq C\|x\|, \forall x \in \mathcal{X} \}.$$

On the one hand, by (17) we know that $\|T\| \in \mathcal{C}_T$. On the other hand, if we take an arbitrary $C \in \mathcal{C}_T$, then for every $u \in \mathcal{X}$ with $\|u\| = 1$, we will have

$$\|Tu\| \leq C\|u\| = C,$$

so taking supremum, over all u with $\|u\| = 1$, will immediately give $\|T\| \leq C$. Since we now have

$$\|T\| \leq C, \forall C \in \mathcal{C}_T,$$

we clearly get $\|T\| = \min \mathcal{C}_T$.

(iii). Let $T, S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Using (17), we have

$$\|(T + S)x\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|) \cdot \|x\|, \quad \forall x \in \mathcal{X}.$$

Then using (16) we get

$$\|T + S\| \leq \|T\| + \|S\|.$$

If $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\lambda \in \mathbb{K}$, then the equality

$$\|(\lambda T)x\| = |\lambda| \cdot \|Tx\|, \quad x \in \mathcal{X}$$

will immediately give $\|\lambda T\| = |\lambda| \cdot \|T\|$.

Finally if $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has $\|T\| = 0$, then using (17) one immediately gets $T = 0$. \square

NOTATION. Let I be a non-empty set, let \mathbb{K} be one of the fields \mathbb{R} or \mathbb{C} , and let $p \in [1, \infty]$. Let q be the Hölder conjugate of p . For every element $\alpha \in \ell_{\mathbb{K}}^p(I)$ we define the map $\theta_{\alpha} : \ell_{\mathbb{K}}^q(I) \rightarrow \mathbb{K}$ by

$$\theta_{\alpha}(\beta) = \langle \alpha, \beta \rangle = \sum_{i \in I} \alpha(i)\beta(i), \quad \beta \in \ell_{\mathbb{K}}^q(I).$$

We know that θ_{α} is linear, and by Remark 9.2, we have

$$|\theta_{\alpha}(\beta)| \leq \|\alpha\|_p \cdot \|\beta\|_q, \quad \forall \beta \in \ell_{\mathbb{K}}^q(I),$$

so θ_{α} is continuous, and we have the inequality

$$(18) \quad \|\theta_{\alpha}\| \leq \|\alpha\|_p.$$

PROPOSITION 2.6. *Using the above notations, but assuming $p \in (1, \infty]$, the map*

$$\Theta : \ell_{\mathbb{K}}^p(I) \ni \alpha \longmapsto \theta_{\alpha} \in (\ell_{\mathbb{K}}^q(I))^*$$

is a linear isomorphism of \mathbb{K} -vector spaces. Moreover, Θ is isometric, in the sense that

$$(19) \quad \|\Theta\alpha\| = \|\alpha\|_p, \quad \forall \alpha \in \ell_{\mathbb{K}}^p(I).$$

PROOF. We begin by proving (19). Since we have the inclusion

$$\{\beta \in \ell_{\mathbb{K}}^q(I) : \|\beta\|_q \leq 1\} \supset \mathcal{B}_{\mathbb{K}}^q(I),$$

it follows that

$$(20) \quad \|\theta_{\alpha}\| = \sup \{|\theta_{\alpha}(\beta)| : \beta \in \ell_{\mathbb{K}}^q(I), \|\beta\|_q \leq 1\} \geq \sup \{|\theta_{\alpha}(\beta)| : \beta \in \mathcal{B}_{\mathbb{K}}^q(I)\}.$$

We know however (see Theorem 9 and Exercise 7) that

$$\sup \{|\theta_{\alpha}(\beta)| : \beta \in \mathcal{B}_{\mathbb{K}}^q(I)\} = \|\alpha\|_p,$$

so using (20) we get

$$\|\theta_{\alpha}\| \geq \|\alpha\|_p.$$

Combining this with (18) yields the desired equality.

The fact that Θ is linear is pretty obvious. Notice now that since Θ is isometric, it is clear that Θ is injective, so the only thing we need to prove is the fact that Θ is surjective. Start with an arbitrary linear continuous map $\phi : \ell_{\mathbb{K}}^q(I) \rightarrow \mathbb{K}$. For every $i \in I$ we define the function $\delta^i : I \rightarrow \mathbb{K}$ by

$$\delta^i(j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

It is clear that $\delta^i \in \ell_{\mathbb{K}}^q(I)$, for all $i \in I$. (In fact $\delta^i \in \text{fin}_{\mathbb{K}}(I)$.) We define $\alpha : I \rightarrow \mathbb{K}$ by

$$\alpha(i) = \phi(\delta^i), \quad \forall i \in I.$$

Notice that, for every $\beta \in \text{fin}_{\mathbb{K}}$, we have

$$(21) \quad \sum_{i \in I} \alpha(i)\beta(i) = \sum_{i \in I} \beta(i)\phi(\delta^i) = \phi\left(\sum_{i \in [\beta]} \beta(i)\delta^i\right) = \phi(\beta),$$

where $[\beta] = \{i \in I : \beta(i) \neq 0\}$. (Since $\beta \in \text{fin}_{\mathbb{K}}(I)$, the set $[\beta]$ is finite.) Using Hölder's inequality, the above computation shows that

$$|\langle \alpha, \beta \rangle| \leq \|\phi\| \cdot \|\beta\|_q, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

By Theorem 9.1 and Exercise 7, this proves that $\alpha \in \ell_{\mathbb{K}}^p(I)$. Going back to (21) we now have

$$\theta_{\alpha}(\beta) = \phi(\beta), \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

Since both θ_{α} and ϕ are continuous, and $\text{fin}_{\mathbb{K}}(I)$ is dense in $\ell_{\mathbb{K}}^q(I)$ (by Exercise 10), it follows that $\phi = \theta_{\alpha}$. \square

REMARK 2.5. In the case $p = 1$, the map

$$\Theta : \ell_{\mathbb{K}}^1(I) \ni \alpha \longmapsto \theta_{\alpha} \in (\ell_{\mathbb{K}}^{\infty}(I))^*$$

is still isometric, but it is no longer surjective, unless I is finite. The explanation is the fact that when I is infinite, the subspace $\text{fin}_{\mathbb{K}}(I)$ is *not* dense in $\ell_{\mathbb{K}}^{\infty}(I)$. For example, if we take $1 \in \ell_{\mathbb{K}}^{\infty}(I)$ to be the constant function 1, then it is pretty obvious that

$$\|1 - \beta\| \geq 1, \quad \forall \beta \in \text{fin}_{\mathbb{K}}(I).$$

The above equality can be immediately extended to

$$(22) \quad \|\lambda 1 + \beta\| \geq |\lambda|, \quad \forall \lambda \in \mathbb{K}, \beta \in \text{fin}_{\mathbb{K}}(I).$$

If we then consider the subspace

$$\widetilde{\text{fin}}_{\mathbb{K}}(I) = \{\lambda 1 + \beta : \beta \in \text{fin}_{\mathbb{K}}(I), \lambda \in \mathbb{K}\},$$

we see that the map

$$\phi_0 : \widetilde{\text{fin}}_{\mathbb{K}}(I) \ni \lambda 1 + \beta \longmapsto \lambda \in \mathbb{K}$$

is linear, continuous, and has the property that

$$(23) \quad \phi_0|_{\text{fin}_{\mathbb{K}}(I)} = 0, \quad \phi_0(1) = 1,$$

$$(24) \quad |\phi_0(\gamma)| \leq \|\gamma\|, \quad \forall \gamma \in \widetilde{\text{fin}}_{\mathbb{K}}(I).$$

Using the Hahn-Banach Theorem, we can then extend ϕ_0 to a linear map $\phi : \ell_{\mathbb{K}}^{\infty}(I) \rightarrow \mathbb{K}$ which will still satisfy (23) and (24), in particular we have $\phi \in (\ell_{\mathbb{K}}^{\infty}(I))^*$. Notice however that if we had $\phi = \theta_{\alpha}$, for some $\alpha \in \ell_{\mathbb{K}}^1(I)$, then we must have $\alpha(i) = \phi(\delta^i) = 0$, for all $i \in I$, so this would force $\phi = 0$, which is impossible, since $\phi(1) = 1$.

Exercise 12. Use the notations above. For every $\alpha \in \ell_{\mathbb{K}}^1(I)$, define

$$\sigma_{\alpha} = \theta_{\alpha}|_{c_0^{\mathbb{K}}(I)} : c_0^{\mathbb{K}}(I) \rightarrow \mathbb{K}.$$

Prove that σ_{α} is linear and continuous. Prove that the map

$$\Sigma : \ell_{\mathbb{K}}^1(I) \ni \alpha \longmapsto \sigma_{\alpha} \in (c_0^{\mathbb{K}}(I))^*$$

is an isometric linear isomorphism of \mathbb{K} -vector spaces.