Lecture 7

7. Baire theorem(s)

In this section we discuss some topological phenomenon that occurs in certain topological spaces. This deals with interiors of closed sets.

Exercise 1. Let X be a topological space, and let A and B be *closed* sets with the property that $\text{int}(A \cup B) \neq \varnothing$. Prove that either $\text{Int}(A) \neq \varnothing$, or $\text{Int}(B) \neq \varnothing$.

Exercise 2. Give an example of a topological space X and of two (non-closed) sets A and B such that $\text{Int}(A \cup B) \neq \varnothing$, but $\text{Int}(A) = \text{Int}(B) = \varnothing$.

Theorem 7.1 (Baire's Theorem). *Let $(X, \mathcal{T})$ be a topological Hausdorff space, which satisfies one (or both) of the following properties:*

(A) *There exists a meatric d on X, which meakes (X, d) a complete metric space, and $\mathcal{T}$ is the metric topology.*

(B) *X is locally compact.*

Suppose one has a sequence $(F_n)_{n \geq 1}$ of closed subsets of X, such that $X = \bigcup_{n=1}^{\infty} F_n$. Then there exists some integer $n \geq 1$, such that $\text{Int}(F_n) \neq \varnothing$.

Proof. For every $n \geq 1$ we define the closed set $G_n = \bigcup_{k=1}^{n} F_k$, so that we still have $X = \bigcup_{n=1}^{\infty} G_n$, but we also have $G_1 \subset G_2 \subset \dots$. According to Exercise 1 (use an inductive argument) it suffices to show that there exists some $n \geq 1$, with $\text{Int}(G_n) \neq \varnothing$. We are going to prove this property by contradiction.

($*$) *Assume $\text{Int}(G_n) = \varnothing$, for all $n \geq 1$.*

Claim: *Under the assumption ($*$) there exists a sequence $(D_n)_{n \geq 1}$ of non-empty open sets, such that for all $n \geq 1$ we have:*

(i) *$D_n \cap G_n = \varnothing$;*

(ii) *$\overline{D}_{n+1} \subset D_n$;*

(iii) *In case* (A) *we have $\text{diam}(D_n) \leq 2^{-n}$; in case* (B) *$\overline{D}_n$ is compact.*

The sequence is constructed recursivley. To construct D_1 we use the fact that $\text{Int}(G_1) = \varnothing$ forces $X \smallsetminus G_1 \neq \varnothing$. We then choose a point $x \in X \smallsetminus G_1$. In case (A) we know that there exists $r > 0$ such that $\mathcal{B}_r(x) \subset X \smallsetminus G_1$. We put $\rho = \min\{r, \frac{1}{4}\}$ and we set $D_1 = \mathcal{B}_\rho(x)$. In the case (B) we apply Lemma 5.1 to find D_1 open with $\overline{D}_1$ compact, such that $x \in D_1 \subset \overline{D}_1 \subset X \smallsetminus G_1$.

Let us assume now that we have constructed $D_1, D_2, \dots, D_k$, such that (i) and (iii) hold for all $n \in \{1, \dots, k\}$, and such that (ii) hold for all $n \in \{1, \dots, k-1\}$, and let us indicate how the next set D_{k+1} is constructed. Using the assumption that $\text{Int}(G_{k+1}) = \varnothing$, it follows that the open set $D_k \smallsetminus G_{k+1}$ is non-empty. Choose then a point $x \in D_k \smallsetminus G_{k+1}$. In case (A) there exists some $r > 0$ such that $\mathcal{B}_r(x) \subset D_k \smallsetminus G_{k+1}$. We then put $\rho = \min\{\frac{r}{2}, \frac{1}{2^{k+2}}\}$, and we define $D_{k+1} = \mathcal{B}_\rho(x)$.

In case (B) we apply Lemma 5.1 an find an open set D_{k+1} with $\overline{D}_{k+1}$ compact, and $x \in D_{k+1} \subset \overline{D}_{k+1} \subset D_k \smallsetminus G_{k+1}$. All properties (i)-(iii) are easily verified.

Having proven the Claim, let us see now that the assumption (∗) produces a contradiction.

CASE (A): In this case we choose, for each $n \geq 1$ a point $x_n \in D_n$. Notice that, for every $m \geq n \geq 1$ we have

$$x_m, x_n \in D_n \text{ and } d(x_m, x_n) \leq \operatorname{diam}(D_n) \leq \frac{1}{2^n}.$$

In particular, this proves that the sequence $(x_n)_{n\geq 1}$ is *Cauchy*, hence convergent to some point x. Since $x_m \in D_n$, $\forall\, m \geq n \geq 1$, we see that $x \in \overline{D}_n$, for all $n \geq 1$. In other words we get

$$\bigcap_{n=1}^{\infty} \overline{D}_n \neq \varnothing. \tag{1}$$

CASE (B): In this case we also get (1), this time as a consequence of the compactness of the sets $\overline{D}_n$ (and the finite intersection property).

Let us notice now that (1) combined with (ii) will also give $\bigcap_{n=1}^{\infty} D_n \neq \varnothing$. But this is impossible, since by (i) we have

$$\bigcap_{n=1}^{\infty} D_n \subset \bigcap_{n=1}^{\infty} (X \smallsetminus G_n) = X \smallsetminus \Big(\bigcup_{n=1}^{\infty} G_n \Big) = \varnothing.$$

□