Lecture 6

6. Metric spaces

In this section we review the basic facts about metric spaces.

Definitions. A *metric* on a non-empty set X is a map

$$d : X \times X \to [0, \infty)$$

with the following properties:

(i) If $x, y \in X$ are points with $d(x, y) = 0$, then $x = y$;
(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

A *metric space* is a pair (X, d), where X is a set, and d is a metric on X.

Notations. If (X, d) is a metric space, then for any point $x \in X$ and any $r > 0$, we define the open and closed balls:

$$\mathcal{B}_r(x) = \{y \in X \,:\, d(x, y) < r\},$$
$$\overline{\mathcal{B}}_r(x) = \{y \in X \,:\, d(x, y) \leq r\}.$$

Definition. Suppose (X, d) is a metric space. Then X carries a natural toplogy constructed as follows. We say that a set $D \subset X$ is *open*, if it has the property:

- *for every $x \in D$, there exists some $r_x > 0$, such that $\mathcal{B}_{r_x}(x) \subset D$.*

One can prove that the collection

$$\mathcal{T}_d = \{D \subset X \,:\, D \text{ open}\}$$

is indeed a *topology*, i.e. we have

- $\varnothing$ and X are open;
- if $(D_i)_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} D_i$ is again open;
- if D_1 and D_2 are open, then $D_1 \cap D_2$ is again open.

The topology thus constructed is called the *metric topology*.

Remark 6.1. Let (X, d) be a metric space. Then for every $p \in X$, and for every $r > 0$, the set $\mathcal{B}_r(p)$ is open, and the set $\overline{B}_r(p)$ is closed.

If we start with some $x \in \mathcal{B}_r(p)$, an if we define $r_x = r - d(x, p)$, then for every $y \in \mathcal{B}_{r_x}(x)$ we will have

$$d(y, p) \leq d(y, x) + d(x, p) < r_x + d(x, p) = r,$$

so y belongs to $\mathcal{B}_r(p)$. This means that $\mathcal{B}_{r_x}(x) \subset \mathcal{B}_r(p)$. Since this is true for all $x \in \mathcal{B}_r(p)$, it follows that $\mathcal{B}_r(p)$ is indeed open.

To prove that $\overline{B}_r(p)$ is closed, we need to show that its complement

$$X \smallsetminus \overline{B}_r(p) = \{x \in X \,:\, d(x,p) > r\}$$

is open. If we start with some $x \in X \smallsetminus \overline{\mathcal{B}}_r(p)$, an if we define $\rho_x = d(p,x) - r$, then for every $y \in \mathcal{B}_{\rho_x}(x)$ we will have

$$d(y,p) \geq d(p,x) - d(y,x) > d(p,x) - \rho_x = r,$$

so y belongs to $X \smallsetminus \overline{\mathcal{B}}_r(p)$. This means that $\mathcal{B}_{\rho_x}(x) \subset X \smallsetminus \overline{\mathcal{B}}_r(p)$. Since this is true for all $x \in X \smallsetminus \overline{\mathcal{B}}_r(p)$, it follows that $X \smallsetminus \overline{\mathcal{B}_r(p)}$ is indeed open.

REMARK 6.2. The metric toplogy on a metric space (X,d) is Hausdorff. Indeed, if we start with two points $x,y \in X$, with $x \neq y$, then if we choose r to be a real number, with

$$0 < r < \frac{d(x,y)}{2},$$

then we have $\mathcal{B}_r(x) \cap \mathcal{B}_r(y) = \varnothing$. (Otherwise, if we have a point $z \in \mathcal{B}_r(x) \cap \mathcal{B}_r(y)$, we would have $2r < d(x,y) \leq d(x,z) + d(y,z) < 2r$, which is impossible.)

REMARK 6.3. Let (X,d) be a metric space, and let M be a subset of X. Then $d\big|_{M\times M}$ is a metric on M, and the metric topology on M defined by this metric is precisely the *induced toplogy* from X. This means that a set $A \subset M$ is open in M if and only if there exists some open set $D \subset X$ with $A = M \cap D$.

The metric space framework is particularly convenient because one can use *convergence.*

DEFINITION. Let (X,d) be a metric space. For a point $x \in X$, we say that a sequence $(x_n)_{n\geq 1} \subset X$ is is *convergent to* x, if $\lim_{n\to\infty} d(x_n,x) = 0$.

REMARK 6.4. Let (X,d) is a metric space, and if the sequence $(x_n)_{n\geq 1} \subset X$ is convergent to some point $x \in X$, then

$$\lim_{n\to\infty} d(x_n,y) = d(x,y), \ \ \forall\, y \in X. \tag{1}$$

This is an immediate consequence of the inequalities

$$d(x,y) - d(x_n,x) \leq d(x_n,y) \leq d(x,y) + d(x_n,x).$$

Among other things, the equality (1) gives the fact that $(x_n)_{n\geq 1}$ cannot be convergent to any other point $y \neq x$. Therefore, if $(x_n)_{n\geq 1}$ is convergent to *some* x, then x is uniquely determined, and will be denoted by $\lim_{n\to\infty} x_n$.

Convergence is useful for characterizing closure.

PROPOSITION 6.1. *Let (X,d) be a metric space, and let $A \subset X$ be a non-empty subset. For a point $x \in X$, the following are equivalent:*

(i) *x belongs to the closure $\overline{A}$ of A;*
(ii) *there exists some sequence $(x_n)_{n\geq 1} \subset A$, with $\lim_{n\to\infty} x_n = x$.*

PROOF. *(i)* $\Rightarrow$ *(ii)*. Assume $x \in \overline{A}$. This means that

(∗) *For every open set $D \subset X$ with $D \ni x$, the intersection $D \cap A$ is non-empty.*

We use this property for the open sets $\mathcal{B}_{1/n}(x)$, $n = 1, 2, \dots$. So, for every integer $n \geq 1$, we can find a point $x_n \in \mathcal{B}_{1/n}(x) \cap A$. This way we have built a sequence $(x_n)_{n\geq 1} \subset A$, such that

$$d(x_n, x) < \frac{1}{n}, \ \ \forall\, n \geq 1.$$

It is clear that this gives $x = \lim_{n\to\infty} x_n$.

(ii) $\Rightarrow$ *(i)*. Assume x satisfies property (ii). Fix $(x_n)_{n\geq 1} \subset A$ to be a sequence with $\lim_{n\to\infty} x_n = x$. We need to prove property $(*)$. Start with some arbitrary open set $D \subset X$, with $x \in D$. Let $\varepsilon > 0$ be chosen such that $\mathcal{B}_\varepsilon(x) \subset D$. Since $\lim_{n\to\infty} d(x_n, x) = 0$, there exists some n_ε such that $d(x_{n_\varepsilon}, x) < \varepsilon$. It is now clear that

$$x_{n_\varepsilon} \in \mathcal{B}_\varepsilon(x) \cap A \subset D \cap A,$$

so the intersection $D \cap A$ is indeed non-empty. □

Continuity can be characterized using convergence, as follows.

Proposition 6.2. *Let X and Y be metric spaces, and let $f : X \to Y$ be a function. For a point $p \in X$, the following are equivalent:*

- (i) *f is continuous at p;*
- (ii) *for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that*

$$d\big(f(x), f(p)\big) < \varepsilon, \text{ for all } x \in X \text{ with } d(x, p) < \delta_\varepsilon.$$

- (iii) *if $(x_n)_{n\geq 1} \subset X$ is a sequence with $\lim_{n\to\infty} x_n = p$, then $\lim_{n\to\infty} f(x_n) = f(p)$.*

Proof. *(i)* $\Rightarrow$ *(ii)*. The condition that f is continuoous at p means

- $(*)$ *for every open set $D \subset Y$, with $D \ni f(p)$, there exists some open set $E \subset X$, with $p \in E \subset f^{-1}(D)$.*

Assume f is continuous at p. For every $\varepsilon > 0$, we consider the open ball $\mathcal{B}^Y_\varepsilon\big(f(p)\big)$. Using $(*)$, there exists some open set $E \subset X$, with $E \ni p$, and $f(E) \subset \mathcal{B}^Y_\varepsilon\big(f(p)\big)$. In particular, there exists $\delta > 0$, such that $\mathcal{B}^X_\delta(p) \subset E$, so now we have

$$f\big(\mathcal{B}^X_\delta(p)\big) \subset \mathcal{B}^Y_\varepsilon\big(f(p)\big),$$

which clearly gives (ii).

(ii) $\Rightarrow$ *(iii)*. Assume f satisfies (ii), and start with some sequence $(x_n)_{n\geq 1} \subset X$, which converges to p. For every $\varepsilon > 0$, we choose $\delta_\varepsilon > 0$ as in (ii), and using the fact that $\lim_{n\to\infty} x_n = p$, we can also choose some N_ε such that

$$d(x_n, p) < \delta_\varepsilon, \ \ \forall\, n \geq N_\varepsilon.$$

Using (ii) this will give

$$d\big(f(x_n), f(p)\big) < \varepsilon, \ \ \forall\, n \geq N_\varepsilon.$$

In other words, we get the fact that

$$\lim_{n\to\infty} \big(f(x_n), f(p)\big) = 0,$$

which means that we indeed have $\lim_{n\to\infty} f(x_n) = f(p)$.

(iii) $\Rightarrow$ *(i)*. Assume f satisfies (iii), but f is not continuous at p. By $(*)$ this means that there exists some open set $D_0 \subset Y$ with $D_0 \ni f(p)$, such that

- $(*')$ *for every open set $E \subset X$ with $E \ni p$, we have $f(E) \not\subset D_0$.*

It is clear that any other open set D, with $f(p) \in D \subset D_0$, will again satisfy property $(*')$. Fix then some $r > 0$, such that $\mathcal{B}_r^Y\big(f(p)\big) \subset D_0$. Using condition $(*')$ it follows that for every integer $n \geq 1$, we have

$$f\big(\mathcal{B}^X_{1/n}(p)\big) \not\subset \mathcal{B}_r^Y\big(f(p)\big).$$

This means that, for every integer $n \geq 1$, we can find a point $x_n \in X$ such that

$$d(x_n, p) < \frac{1}{n} \text{ and } d\big(f(x_n), f(p)\big) \geq r.$$

It is then clear that the sequence $(x_n)_{n\geq 1} \subset X$ is convergent to p, but the sequence $\big(f(x_n)\big)_{n\geq 1} \subset Y$ is not convergent to $f(p)$. This will contradict (iii). □

Convergence can also be used for characterizing compactness.

Theorem 6.1. *Let (X, d) be a metric space. The following are equivalent:*

(i) *X is compact in the metric topology;*
(ii) *every sequence has a convergent subsequence.*

Proof. *(i)* ⇒ *(ii)*. Assume X is compact. Start with an arbitrary sequence $(x_n)_{n\geq 1} \subset X$. For every $n \geq 1$, we define the closed set

$$T_n = \overline{\{x_k : k > n\}}.$$

It is obvious that the family of closed sets $(T_n)_{n\geq 1}$ has the *finite intersection property*, i.e. for every finite set F of indices, we have

$$\bigcap_{n\in F} T_n \neq \varnothing.$$

(This follows from the fact that the T_n's form a decreasing sequence of sets.) By compactness, it follows that

$$\bigcap_{n\geq 1} T_n \neq \varnothing.$$

Take a point $x \in \bigcap_{n\geq 1} T_n$. The key feature of x is the given by the following:

Claim 1: *For every $\varepsilon > 0$ and every integer $\ell \geq 1$, there exists some integer $N(\varepsilon, \ell) > \ell$ such that $d(x_{N(\varepsilon,\ell)}, x) < \varepsilon$.*

This is a consequence of the fact that, for every $\ell \geq 1$, the point x belongs to the closure $\overline{\{x_N : N > \ell\}}$, so for every $\varepsilon > 0$ we have

$$\mathcal{B}_\varepsilon(x) \cap \{x_N : N > \ell\} \neq \varnothing.$$

Using Claim 1, we define a sequence $(k_n)_{n\geq 0}$ of integers, recursively by

$$k_n = N(\tfrac{1}{n}, k_{n-1}), \ \ \forall\, n \geq 1.$$

(The initial term k_0 is chosen arbitrarily.) We have, by construction, $k_0 < k_1 < k_2 < \dots$, and

$$d(x_{k_n}, x) < \frac{1}{n}, \ \ \forall\, n \geq 1,$$

so $(x_{k_n})_{n\geq 1}$ is indeed a subsequence of $(x_k)_{k\geq 1}$, which is convergent (to x).

(ii) ⇒ *(i)*. Assume (ii). Before we start proving that X is compact, We shall need some preparations.

Claim 2: *For every $r > 0$ there exists a finite set $F \subset X$, such that*

$$X = \bigcup_{x\in F} \mathcal{B}_r(x).$$

We prove this by contradiction. Assume there exists some $r > 0$, such that

$$\bigcup_{x\in F} \mathcal{B}_r(x) \subsetneq X,$$

for every finite set $F \subset X$. In particular, there exists a sequence $(x_n)_{n\geq 1}$ such that

$$x_{n+1} \in X \smallsetminus \big[\mathcal{B}_r(x_1) \cup \cdots \cup \mathcal{B}_r(x_n)\big], \ \ \forall\, n \geq 1.$$

This will force

$$d(x_m, x_n) \geq r, \ \ \forall\, m > n \geq 1.$$

Notice that every subsequence $(x_{k_n})_{n\geq 1}$ will satisfy the same property

$$d(x_{k_m}, x_{k_n}) \geq r, \ \ \forall\, m > n \geq 1.$$

This proves that *no subsequence of* $(x_n)_{n\geq 1}$ *is Cauchy*, so *no subsequence of* $(x_n)_{n\geq 1}$ *can be convergent*, thus contradicting (ii).

Having proven Claim 2, we choose, for every integer $n \geq 1$, finite set F_n such that

$$X = \bigcup_{x\in F_n} \mathcal{B}_{\frac{1}{n}}(x).$$

Claim 3: *The collection* $\mathcal{W} = \big\{\mathcal{B}_{\frac{1}{n}}(x) \ : \ n \in \mathbb{N},\, x \in F_n\big\}$ *is a base for the metric topology.*

What we need to show is that every open set is a union of sets in $\mathcal{W}$. Fix an open set D and a point $p \in D$. Choose $r > 0$, such that $\mathcal{B}_r(p) \subset D$. Choose then some integer $n \geq 1$, such that $\frac{1}{n} < \frac{r}{2}$, and choose some point $x \in F_n$, such that $p \in \mathcal{B}_{\frac{1}{n}}(x)$. Notice that, for every $y \in \mathcal{B}_{\frac{1}{n}}(x)$, we have

$$d(y,p) \leq d(y,x) + d(x,p) < \frac{1}{n} + \frac{1}{n} \leq r,$$

which proves that $y \in \mathcal{B}_r(p)$. Therefore we have

$$p \in \mathcal{B}_{\frac{1}{n}}(x) \subset \mathcal{B}_r(p) \subset D.$$

Since $p \in D$ is arbitrary, this proves that D is a union of sets in $\mathcal{W}$.

We now begin proving that X is compact. Start with a collection $(D_i)_{i\in I}$ of open sets, with $\bigcup_{i\in I} D_i = X$. We need to find a finite set of indices $I_0 \subset I$, such that $\bigcup_{i\in I_0} D_i = X$. First we show that:

Claim 4: *There exists a countable set of indices* $I_1 \subset I$, *such that*

$$\bigcup_{i\in I_1} D_i = X.$$

The key fact is that the base $\mathcal{W}$ is *countable*. Let us enumerate the base $\mathcal{W}$ as a sequence

$$\mathcal{W} = \{W_m \ : \ m \in \mathbb{N}\}.$$

For each $i \in I$, we define the set

$$M_i = \{m \geq 1 \ : \ W_m \subset D_i\}.$$

By Claim 3, we know that for every $x \in D_i$ there exists some $m \in M_i$ such that $x \in W_m \subset D_i$. In particular this proves the equality

$$D_i = \bigcup_{m\in M_i} W_m, \ \ \forall\, i \in I.$$

Consider then the union $M = \bigcup_{i\in I} M_i$, which is countable, being a subset of the integers. We clearly have

$$\bigcup_{m\in M} W_m = \bigcup_{i\in I}\big(\bigcup_{m\in M_i} W_m\big) = \bigcup_{i\in I} D_i = X.$$

For every $m \in M$ we choose an $i_m \in I$, such that $m \in M_{i_m}$. If we take

$$I_1 = \{i_m : m \in M\},$$

then I_1 is obviously countable, and since we clearly have $W_m \subset D_{i_m}$, we get

$$X = \bigcup_{m\in M} W_m \subset \bigcup_{m\in M} D_{i_m} = \bigcup_{i\in I_1} D_i,$$

so the Claim is proven.

Let us list the countable set I_1 as

$$I_1 = \{i_k : k \geq 1\}.$$

(Of course, if I_1 is already finite, there is nothing to prove. So we will assume that I_1 is infinite.) In order to finish the proof, we must find some k, such that $D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k} = X$. Assume no such k can be found, which means that

$$D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k} \subsetneq X,\ \ \forall\, k \geq 1.$$

In other words, if we define for each $k \geq 1$, the close set

$$A_k = X \smallsetminus (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k}),$$

we have

$$A_k \neq \varnothing,\ \ \forall\, k \geq 1.$$

For each $k \geq 1$ we choose a point $x_k \in A_k$. This way we have constructed a sequence $(x_k)_{k\geq 1} \subset X$, so using property (i) we can find a convergent subsequence. This means that we have a sequence of integers

$$1 \leq k_1 < k_2 < \dots$$

and a point $x \in X$, such that $\lim_{n\to\infty} x_{k_n} = x$. Notice that, since

$$k_n \geq n,\ \ \forall\, n \geq 1,$$

and since the sequence $(A_k)_{k\geq 1}$ is decreasing, we get the fact that, for each $m \geq 1$, we have

$$x_{k_n} \in A_m,\ \ \forall\, n \geq m.$$

Since A_m is closed, this forces $x \in A_m$, for all $m \geq 1$. But this is clearly impossible, since

$$\bigcap_{m\geq 1} A_m = X \smallsetminus \big(\bigcup_{m\geq 1} (D_{i_1} \cup \cdots \cup D_{i_m})\big) = X \smallsetminus \big(\bigcup_{i\in I_1} D_i\big) = \varnothing.$$

□

Corollary 6.1 (of the proof). *Evry compact metric space is second countable, which means that there exists a sequence $(W_m)_{m\geq 1}$ of open sets, with the property*

(b) *for every open set D, there exists a subset $M \subset \mathbb{N}$ such that*

$$D = \bigcup_{m\in M} W_m.$$

Proof. Use (i) and the steps in the proof of $(i) \Rightarrow (ii)$, up to the proof of Claim 3. □

COROLLARY 6.2. *Let (X,d) be a metric space. For a subset $K \subset X$ the following are equivalent:*

(i) *every sequence in K has a subsequence which is convergent to some point in K;*

(ii) *K is compact in X.*

PROOF. *(i)* $\Rightarrow$ *(ii)*. By the above Theorem, we know that when we equip K with the metric $d\big|_{K\times K}$, then K is compact. This means that K is compact in the induced topology, which means exactly that K is compact in X.

(ii) $\Rightarrow$ *(i)*. Argue as above. If K is compact in X, then K is compact when equipped with the induced toplogy, which means that $(K, d\big|_{K\times K})$ is compact. $\square$

COROLLARY 6.3. *Let X and Y be metric spaces, and let $f : X \to Y$ be a continuous map. If X is compact, then f is uniformly continuous, that is,*

- *for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$, such that*

$$d\big(f(x), f(x')\big) < \varepsilon, \text{ for all } x, x' \in X \text{ with } d(x,x') < \delta_\varepsilon.$$

PROOF. Suppose f is not uniformly continuos, so there exists some $\varepsilon_0 > 0$, with the property that for any $\delta > 0$ there exists $x, x' \in X$, with $d(x.x') < \delta$, but $d\big(f(x), f(x')\big) \geq \varepsilon_0$. In particular, one can construct two sequences $(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ with

$$d(x_n, x'_n) < \frac{1}{n} \text{ and } d\big(f(x_n), f(x'_n)\big) \geq \varepsilon_0, \ \ \forall\, n \geq 1. \tag{2}$$

Using compactness, we can find a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ which converges to some point p. On the one hand, we have

$$d(p, x'_{n_k}) \leq d(p, x_{n_k}) + d(x_{n_k}, x'_{n_k}) < d(p, x_{n_k}) + \frac{1}{n_k}, \ \ \forall\, k \geq 1,$$

which proves that

$$\lim_{k\to\infty} x'_{n_k} = p. \tag{3}$$

On the other hand, using (2) we also have

$$\varepsilon_0 \leq d\big(f(x_{n_k}), f(x'_{n_k})\big) \leq d\big(f(p), f(x_{n_k})\big) + d\big(f(p), f(x'_{n_k})\big),$$

which leads to a contradiction, because the equalities

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} x'_{n_k} = p,$$

together with the continuity of f, will force

$$\lim_{k\to\infty} d\big(f(p), f(x_{n_k}) = \lim_{k\to\infty} d\big(f(p), f(x'_{n_k})\big) = 0.$$

$\square$

REMARK 6.5. Let X be a metric space. Then any compact subset $K \subset X$ is *closed* (this is a consequence of the fact that X is Hausdorff) and *bounded*, in the sense that for every $p \in X$ we have

$$\sup_{x\in K} d(x,p) < \infty.$$

This is a consequence of the continuity (see ??) of the map

$$K \ni x \longmapsto d(x,p) \in [0,\infty).$$

In general however the converse is not true, i.e. there are metric spaces in which closed bounded sets may fail to be compact.

Exercise 1. Equip $\mathbb{R}$ with the metric

$$d(x,y) = \frac{|x-y|}{1+|x-y|}, \ \ \forall\, x,y \in \mathbb{R}.$$

Prove that d is indeed a metric on $\mathbb{R}$, and the metric topology on $\mathbb{R}$ defined by d is the usual topology. Prove that $\mathbb{R}$ is bounded with respect to this metric.

Exercise 2. Start with a metric space X, and let $(x_n)_{n\geq 1} \subset X$ be a sequence which is convergent to some point x. Prove that the set

$$K = \{x\} \cup \{x_n \,:\, n \geq 1\}$$

is compact in X.

Definition. Let (X,d) be e metric space. For a point $x \in X$ and a non-empty subset $A \subset X$, one defines the *distance from x to A* as the number

$$d(x,A) = \inf \big\{ d(x,a) \,:\, a \in A \big\}.$$

Exercise 3. Let (X,d) be a metric space, and let A be a non-empty subset of X.

(i) For a point $x \in X$, prove that the equality $d(x,A) = 0$ is equivalent to the fact that $x \in \overline{A}$.
(ii) Prove the inequality

$$\big|d(x,A) - d(y,A)\big| \leq d(x,y), \ \ \forall\, x,y \in X.$$

Using (ii) conclude that the map

$$X \ni x \longmapsto d(x,A) \in [0,\infty)$$

is continuous.

Proposition 6.3. *Let (X,d) be a metric space. When equipped with the metric topology, X is normal.*

Proof. Let A and B be closed subsets of X with $A \cap B = \varnothing$. We need to find open sets $U, V \subset X$, with $U \supset A$, $V \supset B$, and $U \cap V = \varnothing$. We are going to use a converse of Urysohn Lemma. More explicitly, let us define the function $f : X \to [0,1]$ by

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}, \ \ x \in X.$$

Notice that by Exercise 3, both the numerator and denominator are continuous, and the denominator never vanishes. So f is indeed continuous. It is obvious that $f\big|_A = 0$ and $f\big|_B = 1$, so if we take the open sets $U = f^{-1}\big((-\infty, \frac{1}{2})\big)$ and $V = f^{-1}\big((\frac{1}{2}, \infty)\big)$, we clearly get the desired result. □

We continue now with a discussion on completeness.

Definitions. Let (X,d) be a metric space. A sequence $(x_n)_{n\geq 1} \subset X$ is said to be a *Cauchy sequence*, if it has the following property.

(C) *For every $\varepsilon > 0$, there exists some integer $N_\varepsilon \geq 1$ such that*

$$d(x_m, x_n) < \varepsilon, \ \ \forall\, m,n \geq N_\varepsilon.$$

The metric space (X, d) is said to be *complete*, if every Cauchy sequence is convergent.

The following result summarizes some equivalent characterizations of completeness.

PROPOSITION 6.4. *Let (X, d) be a metric space. The following are equivalent.*

(i) *(X, d) is complete.*

(ii) *Every sequence $(x_n)_{n\geq 1} \subset X$, with*

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty, \tag{4}$$

is convergent.

(iii) *Every Cauchy sequence has a convergent subsequence.*

PROOF. *(i)* $\Rightarrow$ *(ii)*. Assume X is complete. Let $(x_n)_{n\geq 1} \subset X$ be a sequence with property (4). To prove (ii) it suffices to show that $(x_n)_{n\geq 1}$ is Cauchy. For every $N \geq 1$ we define

$$R_N = \sum_{n=N}^{\infty} d(x_{n+1}, x_n).$$

Using (4) we get $\lim_{N\to\infty} R_N = 0$, so for every $\varepsilon > 0$ there exists some $N(\varepsilon)$ with $R_{N(\varepsilon)} < \varepsilon$. Notice also that the sequence $(R_N)_{N\geq 1}$ is decreasing. If $m > n \geq N(\varepsilon)$, then

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} d(x_{k+1}, x_k) = R_n \leq R_{N(\varepsilon)} < \varepsilon,$$

so $(x_n)_{n\geq 1}$ is indeed Cauchy.

(ii) $\Rightarrow$ *(iii)*. Start with some Cauchy sequence $(y_k)_{k\geq 1}$. For every $n \geq 1$ choose an integer $N(n) \geq 1$ such that

$$d(x_k, x_\ell) < \frac{1}{2^n}, \ \ \forall\, k, \ell \geq N(n). \tag{5}$$

Start with some arbitrary $k_1 \geq N(1)$ and define recursively an entire sequence $(k_n)_{n\geq 1}$ of integers, by

$$k_{n+1} = \max\{k_n + 1, N(n+1), \ \ n \geq 1.$$

Clearly we have $k_1 < k_2 < \dots$, and since we have

$$k_{n+1} > k_n \geq N(n), \ \ \forall\, n \geq 1,$$

using (5), we get

$$d(y_{k_{n+1}}, y_{k_n}) < \frac{1}{2^n}, \ \ \forall\, n \geq 1.$$

So if we define the subsequence $x_n = y_{k_n}$, $n \geq 1$, we will have

$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so the subsequence $(x_n)_{n\geq 1}$ satisfies condition (4). By (ii) the subsequence $(x_n)_{n\geq 1}$ is convergent.

(iii) ⇒ *(i)*. Assume condition (iii) holds. Start with some Cauchy sequence $(x_n)_{n\geq 1}$. For every integer $n \geq 1$ we put

$$S_n = \sup_{\ell,m\geq n} d(x_\ell, x_m).$$

Since $(x_n)_{n\geq 1}$ is Cauchy, we have

$$\lim_{n\to\infty} S_n = 0. \tag{6}$$

Using the assumption, we can find a subsequence $(x_{k_n})_{n\geq 1}$ (defined by an increasing sequence of integers $1 \leq k_1 < k_2 < \dots$) which is convergent to some point x. We are going to prove that the entire sequence $(x_n)_{n\geq 1}$ is convergent to x. Fix for the moment $n \geq 1$. For every $m \geq n$, we have $k_m \geq m \geq n$, so we have

$$S_n \geq d(x_n, x_{k_m}), \ \ \forall\, m \geq n. \tag{7}$$

By Remark 3.4, we also know that

$$\lim_{m\to\infty} d(x_n, x_{k_m}) = d(x_n, x),$$

so if we take $lim_{m\to\infty}$ in (7) we will get

$$d(x_n, x) \leq S_n.$$

Since this estimate holds for arbitrary $n \geq 1$, using (6) we immediately get the fact that $(x_n)_{n\geq 1}$ is indeed convergent to x. □

Proposition 6.5. *Suppose (X, d) is a complete metric space, and Y is a subset of X. The following are equivalent:*

(i) *Y is complete, when equipped with the metric from X;*
(ii) *Y is closed in X, in the metric topology.*

Proof. *(i)* ⇒ *(ii)*. Assume Y is complete, and let us prove that Y is closed. Start with a point $x \in \overline{Y}$. Then there exists a sequence $(y_n)_{n\geq 1} \subset Y$ with $\lim_{n\to\infty} y_n = x$. Notice that $(y_n)_{n\geq 1}$ is Cauchy in Y, so by assumption, $(y_n)_{n\geq 1}$ is convergent to som point in Y. This will then clearly force $x \in Y$.

(ii) ⇒ *(i)*. Assume Y is closed, and let us prove that Y is complete. Start with a Cauchy sequence $(y_n)_{n\geq 1} \subset Y$. Since X is complete, the sequence $(y_n)_{n\geq 1}$ is convergent to some point $x \in X$. Since Y is closed, this forces $x \in Y$. □

Remark 6.6. Using Theorem 6.1, we immediately see that a metric space, which is *compact* in the metric topology, is automatically complete.

The next result identifies those complete metric spaces that are compact. In order to formulate it, we need the following:

Definition. Let (X, d) be a metric space, and let $\varepsilon > 0$. A subset $A \subset X$ is said to be *ε-rare*, if

$$d(a, b) \geq \varepsilon, \text{ for all } a, b \in A \text{ with } a \neq b.$$

Proposition 6.6. *Let (X, d) be a complete metric space. The following are equivalent:*

(i) *X is compact in the metric topology;*
(ii) *for each $\varepsilon > 0$, all ε-rare subsets of X are finite;*
(iii) *for any $\varepsilon > 0$, there exist finitely many points $p_1, p_2, \dots, p_n \in X$, such that*

$$X = \mathcal{B}_\varepsilon(p_1) \cup \mathcal{B}_\varepsilon(p_2) \cup \dots \cup \mathcal{B}_\varepsilon(p_n).$$

PROOF. $(i) \Rightarrow (ii)$. Assume X is compact. We prove (ii) by contradiction. Assume there exists some $\varepsilon > 0$ and an infinite ε-rare set $A \subset X$. It then follows that there exists a sequence $(a_n)_{n\geq 1} \subset A$, such that

$$d(a_m, a_n) \geq \varepsilon, \ \forall\, m > n \geq 1.$$

It is clear that *no subsequence of* $(a_n)_{n\geq 1}$ *is Cauchy*, which means that $(a_n)_{n\geq 1}$ *does not have any convergent subsequence*, thus contradicting the fact that X is compact.

$(ii) \Rightarrow (iii)$. Assume property (ii) and let us prove (iii) by contradiction. Assume there exists some $\varepsilon > 0$, such that, for every finite set $F \subset X$, one has a strict inclusion

$$\bigcup_{x\in F} \mathcal{B}_\varepsilon(x) \subsetneq X.$$

Start with some arbitrary point $a_1 \in X$, and construct recursively a seqeuence $(a_n)_{n\geq 1} \subset X$, by choosing

$$a_{n+1} \in X \smallsetminus \big[\mathcal{B}_\varepsilon(a_1) \cup \dots \cup \mathcal{B}_\varepsilon(a_n)\big], \ \forall\, n \geq 1.$$

This will then force

$$d(a_m, a_n) \geq \varepsilon, \ \forall\, m > n \geq 1,$$

so $A = \{a_n : n \in \mathbb{N}\}$ will be an infinite ε-rare set, thus contradicting (ii).

$(iii) \Rightarrow (i)$. Assume property (iii), and let us prove that X is compact. We are going to use Theorem 6.1. Start with an arbitrary sequence $(x_n)_{n\geq 1} \subset X$, and let us construct a convergent subsequence.

> *Claim*: *There exists a sequence $(p_n)_{n\geq 1} \subset X$, such that for every integer $k \geq 1$, the set*
>
> $$M_k = \big\{n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^{k} \mathcal{B}_{\frac{1}{\ell}}(p_\ell)\big\}$$
>
> *is infinite.*

The sequence $(p_n)_{n\geq 1}$ is constructed recursively. To start, we use (ii) to find a finite set $F_1 \subset X$, such that

$$X = \bigcup_{p\in F_1} \mathcal{B}_1(p).$$

If we define, for each $p \in F_1$, the set

$$S_1(p) = \{n \in \mathbb{N} : x_n \in \mathcal{B}_1(p)\},$$

then we clearly have

$$\bigcup_{p\in F_1} S_1(p) = \mathbb{N},$$

so in particular one of the sets $S_1(p)$, $p \in F_1$, is infinite.

Suppose now we have constructed points $p_1, p_2, \dots, p_{m-1}$, such that, for every $k \in \{1, \dots, m-1\}$, the set

$$M_k = \big\{n \in \mathbb{N} : x_n \in \bigcap_{\ell=1}^{k} \mathcal{B}_{\frac{1}{\ell}}(p_\ell)\big\}$$

is infinite, and let us indicate how the next term p_m is to be constructed. Start with a finite set $F_m \subset X$, such that

$$X = \bigcup_{p \in F_m} \mathcal{B}_{\frac{1}{m}}(p),$$

and define, for each $p \in F_m$, the set

$$S_m(p) = \big\{ n \in M_{m-1} \,:\, x_n \in \mathcal{B}_{\frac{1}{m}}(p) \big\}.$$

It is clear that

$$M_{m-1} = \bigcup_{p \in F_m} S_m(p),$$

and since M_{m-1} is infinite, it follows that one of the sets $S_m(p)$, $p \in F_m$ is infinite. We then choose $p_m \in F_m$ to be one point for which $S_m(p_m)$ is infinite.

Having proven the Claim, let us us construct a sequence of integers $1 \leq n_1 < n_2 < \dots$ as follows. Start with some arbitrary $n_1 \in M_1$. Once $n_1 < n_2 < \dots < n_k$ have been constructed, we choose the integer $n_{k+1} \in M_{k+1}$, such that $n_{k+1} > n_k$. (It is here that we use the fact that M_{k+1} is *infinite*.) By construction, we have $n_k \in M_k$, $\forall\, k \geq 1$.

Suppose $k \geq \ell \geq 1$. Then by construction we have $n_k \in M_k \subset M_\ell$ and $n_\ell \in M_\ell$. In particular we get

$$d(x_{n_k}, x_{n_\ell}) \leq d(x_{n_k}, p_\ell) + d(x_{n_\ell}, p_\ell) < \frac{2}{\ell}.$$

The above estimate clearely proves that the subsequence $(x_{n_k})_{k\geq 1}$ is Cauchy. Since X is complete, it follows that $(x_{n_k})_{k\geq 1}$ is convergent. □

Corollary 6.4. *Let (X, d) be a complete metric space, and let A be a subset of X. The following are equivalent:*

(i) *the closure $\overline{A}$ is compact in X;*
(ii) *for each $\varepsilon > 0$, all ε-rare subsets of A are finite.*

Proof. *(i)* ⇒ *(ii)*. This is trivial from the above result.

(ii) ⇒ *(i)*. Assume (ii), and let us prove that $\overline{A}$ is compact. Since $\overline{A}$ is complete, it suffices to prove that, for each $\varepsilon > 0$, all ε-rare subsets of $\overline{A}$ are finite. Fix $\varepsilon > 0$, and let B be an ε-rare subset of $\overline{A}$. For each $x \in B$, let us choose a point $a_x \in A$, such that $x \in \mathcal{B}_{\varepsilon/3}(a_x)$. Suppose $x, y \in B$ are such that $x \neq y$. Then

$$d(a_x, a_y) \geq d(x, y) - d(a_x, x) - d(a_y, y) > \varepsilon - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

In particular, this shows that the map

$$f : B \ni x \longmapsto a_x \in A$$

is injective, and the set $f(B)$ is an $(\varepsilon/3)$-rare subset of A. By condition (ii) this forces B to be finite. □

We continue with an important construction.

Definitions. Let (X, d) be a metric space. We define

$$\textsc{cs}(X, d) = \big\{ \boldsymbol{x} = (x_n)_{n\geq 1} \,:\, \boldsymbol{x} \text{ Cauchy sequence in } X \big\}.$$

We say that two Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{y} = (y_n)_{n\geq 1}$ in X are *equivalent*, if

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

In this case we write $\boldsymbol{x} \sim \boldsymbol{y}$. (It is fairly obvious that $\sim$ is indeed an equivalence relation.) We define the quotient space

$$\widetilde{X} = \text{CS}(X,d)/\sim .$$

For an element $\boldsymbol{x} \in \text{CS}(X,d)$, we denote its equivalence class by $\widetilde{\boldsymbol{x}}$.

Finally, for a point $x \in X$, we define $\langle x \rangle \in \widetilde{X}$, to be the equivalence class of the constant sequence x (which is obviously Cauchy).

REMARK 6.7. Let (X,d) be a metric space. If $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{y} = (y_n)_{n\geq 1}$ are Cauchy sequences in X, then the sequence of real numbers $\big(d(x_n,y_n)\big)_{n\geq 1}$ is convergent. Indeed, for any m,n we have

$$\big|d(x_m,y_m) - d(x_n,y_n)\big| \leq \big|d(x_m,y_m) - d(x_n,y_m)\big| + \big|d(x_n,y_m) - d(x_n,y_n)\big| \leq \\ \leq d(x_m,x_n) + d(y_m,y_n).$$

We can then define

$$\delta(\boldsymbol{x},\boldsymbol{y}) = \lim_{n\to\infty} d(x_n,y_n).$$

PROPOSITION 6.7. *Let (X,d) be a metric space.*

A. *The map $\delta : \text{CS}(X,d) \times \text{CS}(X,d) \to [0,\infty)$ has the following properties:*
   (i) $\delta(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{y},\boldsymbol{x})$, $\forall\, \boldsymbol{x},\boldsymbol{y} \in \text{CS}(X,d)$;
   (ii) $\delta(\boldsymbol{x},\boldsymbol{y}) \leq \delta(\boldsymbol{x},\boldsymbol{z}) = \delta(\boldsymbol{z},\boldsymbol{y})$, $\forall\, \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \text{CS}(X,d)$;
   (iii) $\delta(\boldsymbol{x},\boldsymbol{y}) = 0 \Rightarrow \boldsymbol{x} \sim \boldsymbol{y}$;
   (iv) *If $\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y},\boldsymbol{y}' \in \text{CS}(X,d)$ are such that $\boldsymbol{x} \sim \boldsymbol{x}'$ and $\boldsymbol{y} \sim \boldsymbol{y}'$, then $\delta(\boldsymbol{y},\boldsymbol{x}) = \delta(\boldsymbol{x}',\boldsymbol{y}')$.*

B. *The map $\widetilde{d} : \widetilde{X} \times \widetilde{X} \to [0,\infty)$, correctly defined by*

$$\widetilde{d}(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}) = \delta(\boldsymbol{x},\boldsymbol{y}),\ \ \forall\, \boldsymbol{x},\boldsymbol{y} \in \text{CS}(X,d),$$

   is a metric on $\widetilde{X}$.

C. *The map $X \ni x \longmapsto \langle x \rangle \in \widetilde{X}$ is isometric, in the sense that*

$$\widetilde{d}(\langle x \rangle,\, \langle y \rangle) = d(x,y),\ \ \forall\, x,y \in X.$$

PROOF. A. Properties (i), (ii) and (iii) are obvious. To prove property (iv) let $\boldsymbol{x} = (x_n)_{n\geq 1}$, $\boldsymbol{x}' = (x'_n)_{n\geq 1}$, $\boldsymbol{y} = (y_n)_{n\geq 1}$, and $\boldsymbol{y}' = (y'_n)_{n\geq 1}$. The inequality

$$d(x'_n,y'_n) \leq d(x'n,x_n) + d(x_n,y_n) + d(y_n,y'_n),$$

combined with $\lim_{n\to\infty} d(x'_n,x_n) = \lim_{n\to\infty} d(y_n,y'_n) = 0$ immediately gives

$$\delta(\boldsymbol{x}',\boldsymbol{y}') = \lim_{n\to\infty} d(x'_n,y'_n) \leq \lim_{n\to\infty} d(x_n,y_n) = \delta(\boldsymbol{x},\boldsymbol{y}).$$

By symmetry we also have $\delta(\boldsymbol{x},\boldsymbol{y}) \leq \delta(\boldsymbol{x}',\boldsymbol{y}')$, and we are done.

B. This is immediate from A.

C. Obvious, from the definition. □

PROPOSITION 6.8. *Let (X,d) be a metric space.*

(i) *For any Cauchy sequence $\boldsymbol{x} = (x_n)_{n\geq 1}$ in X, one has*

$$\lim_{n\to\infty} \langle x_n \rangle = \widetilde{\boldsymbol{x}},\ \text{in } \widetilde{X}.$$

(ii) *The metric space $(\widetilde{X},\widetilde{d})$ is complete.*

Proof. (i). For every $n \geq 1$, we have

$$\widetilde{d}\big(\langle x_n\rangle, \widetilde{\boldsymbol{x}}\big) = \lim_{m\to\infty} d(x_n, x_m). \tag{8}$$

Now if we start with some $\varepsilon > 0$, and we choose N_ε such that

$$d(x_n, x_m) < \varepsilon, \ \ \forall\, m, n \geq N_\varepsilon,$$

then (8) shows that

$$\widetilde{d}\big(\langle x_n\rangle, \widetilde{\boldsymbol{x}}\big) \leq \varepsilon, \ \ \forall\, n \geq N_\varepsilon,$$

so we indeed have

$$\lim_{n\to\infty} \widetilde{d}\big(\langle x_n\rangle, \widetilde{\boldsymbol{x}}\big) = 0.$$

(ii). Let $\big(p_k\big)_{k\geq 1}$ be a Cauchy sequence in $\widetilde{X}$. Using (i), we can choose, for each $k \geq 1$, an element $x_k \in X$, such that

$$\widetilde{d}\big(\langle x_k\rangle, p_k\big) \leq \frac{1}{2^k}.$$

*Claim 1***:** *The sequence* $\boldsymbol{x} = (x_k)_{k\geq 1}$ *is Cauchy in* X*.*

Indeed, for $k \geq \ell \geq 1$ we have

$$d(x_k, x_\ell) = \widetilde{d}\big(\langle x_k\rangle, \langle x_\ell\rangle\big) \leq \widetilde{d}\big(\langle x_k\rangle, p_k)\big) + \widetilde{d}(p_k, p_\ell) + \widetilde{d}\big(p_\ell, \langle x_\ell\rangle\big) \leq \widetilde{d}(p_k, p_\ell) + \frac{1}{2^\ell}.$$

This clearly gives

$$\lim_{n\to\infty}\left[\sup_{k,\ell\geq N} d(x_k, x_\ell)\right] \leq \lim_{n\to\infty}\left[\sup_{k,\ell\geq N} \widetilde{d}(p_k, p_\ell)\right] = 0,$$

so $\boldsymbol{x} = (x_k)_{k\geq 1}$ is indeed Cauchy.

The proof of (ii) will the be finished, once we prove:

*Claim 2***:** *We have* $\lim_{n\to\infty} p_k = \widetilde{\boldsymbol{x}}$ *in* $\widetilde{X}$*.*

To see this, we observe that, for $\ell \geq k \geq 1$ we have the inequality

$$\widetilde{d}\big(p_k, \langle x_\ell\rangle\big) \leq \widetilde{d}\big(p_k, \langle x_k\rangle\big) + \widetilde{d}\big(\langle x_k\rangle, \langle x_\ell\rangle\big) \leq \frac{1}{2^k} + d(x_k, x_\ell). \tag{9}$$

If we now start with some $\varepsilon > 0$, and we choose N_ε such that

$$d(x_k, x_\ell) < \varepsilon, \ \ \forall\, k, \ell \geq N_\varepsilon,$$

then (9) gives

$$\widetilde{d}\big(p_k, \langle x_\ell\rangle\big) \leq \frac{1}{2^k} + \varepsilon, \ \ \forall\, \ell \geq k \geq N_\varepsilon.$$

If we keep $k \geq N_\varepsilon$ fixed and take $\lim_{\ell\to\infty}$, using (i) we get

$$\widetilde{d}(p_k, \widetilde{\boldsymbol{x}}) = \lim_{\ell\to\infty} \widetilde{d}(p_k, \langle x_\ell\rangle) \leq \frac{1}{2^k} + \varepsilon, \ \ \forall\, k \geq N_\varepsilon.$$

The above estimate clearly proves that

$$\lim_{k\to\infty} \widetilde{d}(p_k, \widetilde{\boldsymbol{x}}) = 0,$$

so the sequence $(p_k)_{k\geq 1}$ is convergent (to $\widetilde{\boldsymbol{x}}$). □

Definition. The metric space $(\widetilde{X}, \widetilde{d})$ is called the *completion of* (x, d).

The completion has a certain universality property. In order to formulate this property we need the following

DEFINITION. Let (X,d) and (Y,ρ) be metric spaces. A map $f : X \to Y$ is said to be a *Lipschitz function*, if there exists some constant $C \geq 0$, such that

$$\rho\big(f(x), f(x')\big) \leq C \cdot d(x,x'),\ \ \forall\, x, x' \in X.$$

Such a constant C is then called a *Lipschitz constant for f*.

PROPOSITION 6.9. *Let (X,d) be a metric space, and let $(\widetilde{X}, \widetilde{d})$ be its completion. If (Y,ρ) is a complete metric space, and $f : X \to Y$ is a Lipschitz function with Lipschitz constant $C \geq 0$, then there exists a unique continuous function $\widetilde{f} : \widetilde{X} \to Y$, such that*

$$\widetilde{f}(\langle x \rangle) = f(x),\ \ \forall\, x \in X.$$

Moreover, $\widetilde{f}$ is Lipschitz, with Lipschitz constant C.

PROOF. Start with some Cauchy sequence $\boldsymbol{x} = (x_n)_{n\geq 1}$ in X. Using the inequality

$$\rho\big(f(x_m), f(x_n)\big) \leq C \cdot d(x_m, x_n),\ \ \forall\, m, n \geq 1,$$

it is obvious that $\big(f(x_n)\big)_{n\geq 1}$ is a Cauchy sequence in Y. Since Y is complete, this sequence is convergent. Define,

$$\phi(\boldsymbol{x}) = \lim_{n\to\infty} f(x_n).$$

This way we have constructed a map $\phi : \mathrm{CS}(X,d) \to Y$.

Claim: *If $\boldsymbol{x} \sim \boldsymbol{x}'$, then $\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}')$.*

Indeed, if $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{x}' = (x'_n)_{n\geq 1}$, then the Lipschitz property will give

$$\rho\big(f(x_n), f(x'_n)\big) \leq C \cdot d(x_n, x'_n),\ \ \forall\, n \geq 1,$$

and using the fact that $\lim_{n\to\infty} d(x_n, x'_n) = 0$, we get $\lim_{n\to\infty} \rho\big(f(x_n), f(x'_n)\big) = 0$. This clearly forces

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x'_n).$$

Having proven the claim, we now see that we have a correctly define map $\widetilde{f} : \widetilde{X} \to Y$, with the property that

$$\widetilde{f}(\widetilde{\boldsymbol{x}}) = \phi(\boldsymbol{x}),\ \ \forall\, \boldsymbol{x} \in \mathrm{CS}(X,d).$$

The equality

$$\widetilde{f}(\langle x \rangle) = f(x),\ \ \forall\, x \in X$$

is trivially satisfied.

Let us check now that $\widetilde{f}$ is Lipschitz, with Lipschitz constant C. Start with two points $p, p' \in \widetilde{X}$, represented as $p = \widetilde{\boldsymbol{x}}$ and $p' = \widetilde{\boldsymbol{x}'}$, for two Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{x}' = (x'_n)_{n\geq 1}$ in X. Using the definition, we have

$$\widetilde{f}(p) = \lim_{n\to\infty} f(x_n) \text{ and } \widetilde{f}(p') = \lim_{n\to\infty} f(x'_n).$$

This will give

$$\rho\big(f(p), f(p')\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(x'_n)\big).$$

Notice however that

$$\rho\big(f(x_n), f(x'_n)\big) \leq C \cdot d(x_n, x'_n),\ \ \forall\, n \geq 1,$$

so taking the limit yields

$$\rho\big(f(p), f(p')\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(x'_n)\big) \leq C \cdot \lim_{n\to\infty} d(x_n, x'_n) = C \cdot \widetilde{d}(p, p').$$

Finally, let us show that $\widetilde{f}$ is unique. Let $F : \widetilde{X} \to Y$ be another continuous function with $F(\langle x\rangle) = f(x)$, for all $x \in X$. Start with an arbitrary point $p \in \widetilde{X}$, represented as $p = \boldsymbol{x}$, for some Cauchy sequence $\boldsymbol{x} = (x_n)_{n\geq 1}$ in X. Since $\lim_{n\to\infty}\langle x_n\rangle = p$ in $\widetilde{X}$, by continuity we have

$$F(p) = \lim_{n\to\infty} F(\langle x_n\rangle) = \lim_{n\to\infty} f(x_n) = \phi(\boldsymbol{x}) = \widetilde{f}(p).$$

□

Corollary 6.5. *Let (X, d) be a metric space, let (Y, ρ) be a complete metric space, and let $f : X \to Y$ be an isometric map, that is*

$$\rho\big(f(x), f(x')\big) = d(x, x'),\ \ \forall\, x, x' \in X.$$

Then the map $\tilde{f} : \tilde{X} \to Y$, given by the above result, is isometric and $\tilde{f}(\tilde{X}) = \overline{f(X)}$ - the closure of $f(X)$ in Y..

Proof. To show that $\tilde{f}(\tilde{X}) = \overline{f(X)}$, start with some arbitrary point $y \in \overline{f(X)}$. Then there exists a sequence $(x_n)_{n\geq 1} \subset X$, with $\lim_{n\to\infty} f(x_n) = y$. Since $\big(f(x_n)\big)_{n\geq 1}$ is Cauchy in Y, and

$$d(x_m, x_n) = \rho\big(f(x_m), f(x_n)\big),\ \ \forall\, m, n \geq 1,$$

it follows that the sequence $\boldsymbol{x} = (x_n)_{n\geq 1}$ is cauchy in X. We then have

$$y = \lim_{n\to\infty} f(x_n) = \tilde{f}(\tilde{\boldsymbol{x}}).$$

Finally, we show that $\tilde{f}$ is isometric. Start with two points $p, q \in \tilde{X}$, represented as $p = \tilde{\boldsymbol{x}}$ and $q = \tilde{\boldsymbol{z}}$, for some Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ and $\boldsymbol{z} = (z_n)_{n\geq 1}$ in X. Then by construction we have

$$\begin{aligned}\rho\big(\tilde{f}(p), \tilde{f}(q)\big) &= \lim_{n\to\infty} \rho\big(\tilde{f}(\langle x_n\rangle), \tilde{f}(\langle z_n\rangle)\big) = \lim_{n\to\infty} \rho\big(f(x_n), f(z_n)\big) = \\ &= \lim_{n\to\infty} d(x_n, z_n) = \tilde{d}(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{z}}) = \tilde{d}(p, q).\end{aligned}$$

□

Corollary 6.6. *If (X, d) is a complete metric space, and $\tilde{X}$ is its completion, then the map $\iota : X \ni x \longmapsto \langle x\rangle \in \tilde{X}$ is bijective.*

Proof. Apply the previous result to the map $\mathrm{Id} : X \to X$, to get a bijective (isometric) map $\widetilde{\mathrm{Id}} : \tilde{X} \to X$. Since the map $\widetilde{\mathrm{Id}}$ is obviously a left inverse for ι, it follows that ι itself is bijective. □

In the remainder of this section we will address the following question: *Given a topological Hausdorff space X, when does there exists a metric d on X, such that the given topology coincides with the metric topology defined by d?* A topolgical Hausdorff space with the above property is said to be *metrizable*. It is difficult to give non-trivial necessary and sufficient conditions for mtrizability. One instance in which this is possible is the compact case (see the *Urysohn Metrizability Theorem* later in these notes). Here is a useful result, which is an example of a sufficient condition for mterizabilty.

Proposition 6.10 (Metrizability of Countable Products). *Let $(X_i, d_i)_{i\in I}$ be a countable family of metric spaces. Then the product space $X = \prod_{i\in I} X_i$, equipped with the product topology, is metrizable.*

PROOF. Denote by $\mathcal{T}$ the product topology on X. What we need is a metric d on X, such that the maps

$$\text{Id} : (X, d) \to (X, \mathcal{T}) \text{ and } \text{Id} : (X, \mathcal{T}) \to (X, d)$$

are continuous. (Here the notation (X, d) signifies that X is equipped with the metric topology defined by d.) For each $i \in I$, let $\pi_i : X \to X_i$ denote the projection onto the i^{th} coordinate.

CASE I: *Assume I is finite*. In this case we define the metric d on X as follows. If $\boldsymbol{x} = (x_i)_{i\in I}$ and $\boldsymbol{y} = (y_i)_{i\in I}$ are elements in X, we put

$$d(\boldsymbol{x}, \boldsymbol{y}) = \max_{i\in I} d_i(x_i, y_i).$$

The continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the fact that all maps

$$\pi_i : (X, d) \to (X_i, d_i),\ \ i \in I$$

are continuous. This is obvious, because by construction we have

$$d_i\big(\pi_i(\boldsymbol{x}), \pi_i(\boldsymbol{y})\big) \le d(\boldsymbol{x}, \boldsymbol{y}),\ \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in X.$$

Conversely, to prove the continuity of $\text{Id} : (X, \mathcal{T}) \to (X, d)$, we are going to prove that *every d-open set is open in the product topology.* It suffices to prove this only for open balls. Fix then $\boldsymbol{x} = (x_i)_{i\in I} \in \prod_{i\in I} X_i$ and $r > 0$, and consider the open ball $\mathcal{B}_r(\boldsymbol{x})$. If we define, for each $i \in I$, the open ball $\mathcal{B}_r^{X_i}(x_i)$, then it is obvious that

$$\mathcal{B}_r(\boldsymbol{x}) = \bigcap_{i\in I} \pi_i^{-1}\big(\mathcal{B}_r^{X_i}(x_i)\big),$$

and since π_i are all continuous, this proves that $\mathcal{B}_r(\boldsymbol{x})$ is indeed open in the product toplogy.

CASE II: *Assume I is infinite.* In this case we identify $I = \mathbb{N}$. For every $n \in \mathbb{N}$ we define a new metric δ_n on X_n, as follows. If

$$\sup_{p,q\in X_n} d_n(p, q) \le 1,$$

we put $\delta_n = d_n$. Otherwise, we define

$$\delta_n(p, p) = \frac{d_n(p, q)}{1 + d_n(p, q)},\ \ \forall\, p, q \in X_n.$$

It is not hard to see that the metric topology defined by δ_n coincides with the one defined by d_n. The advantage is that δ_n takes values in $[0, 1]$. We define the metric $d : X \times X \to [0, \infty)$, as follows. If $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ and $\boldsymbol{y} = (y_n)_{n\in\mathbb{N}}$ are elements in $\prod_{n\in\mathbb{N}} X_n$, we define

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \frac{\delta_n(x_n, y_n)}{2^n}.$$

Due to the fact that δ_n takes values in $[0, 1]$, the above series is convergent, and it obviously defines a metric on X.

As above, the continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the continuity of all the maps $\pi_n : (X, d) \to (X_n, d_n)$, or equivalently for $\pi_n : (X, d) \to (X_n, \delta_n)$, $n \in \mathbb{N}$. But this is an immediate consequence of the (obvioous) inequalities

$$\delta_n\big(\pi_n(\boldsymbol{x}), \pi_n(\boldsymbol{y})\big) \le 2^n \cdot d(\boldsymbol{x}, \boldsymbol{y}),\ \ \forall\, \boldsymbol{x}, \boldsymbol{y} \in X.$$

As before, in order to prove the continuity of the other map $\text{Id} : (X, \mathcal{T}) \to (X, d)$, we start with some d-open set D, and we show that D is open in the product topology. Since D is a union of of open balls, we need to prove that for any $\boldsymbol{x} \in X$ and any $r > 0$, the open ball $\mathcal{B}_r(\boldsymbol{x})$, in (X, d), is a *neighborhood of $\boldsymbol{x}$ in the product topology.* Fix $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n$, as well as $r > 0$. Choose some integer $N \geq 1$, such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{r}{2},$$

and define, for each $k \in \{1, 2, \dots, N\}$ the set

$$D_k = \{\boldsymbol{y} = (y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \, : \, \delta_n(x_k, y_k) < \frac{r}{2}\}.$$

It is clear that D_k is *open in the product topology*, for each $k = 1, 2, \dots, N$. (This is a consequence of the fact that $D_k = \pi_k^{-1}\big(\mathcal{B}_{r/2}(x_k)\big)$, where $\mathcal{B}_{r/2}(x_k)$ is the δ_k-open ball in X_k of radius $r/2$, centered at x_k.) Then the set $D = D_1 \cap D_2 \cap \cdots \cap D_N$ is also open in the product topology. Obviously we have $\boldsymbol{x} \in D$. We now prove that $D \subset \mathcal{B}_r(\boldsymbol{x})$. Start with some arbitrary $\boldsymbol{y} \in D$, say $\boldsymbol{y} = (y_n)_{n\in\mathbb{N}}$. On the one hand, we have

$$\delta_k(x_k, y_k) < \frac{r}{2}, \ \forall\, k \in \{1, 2, \dots, N\},$$

so we get

$$\sum_{n=1}^{N} \frac{1}{2^n} \delta_n(x_n, y_n) < \frac{r}{2} \sum_{n=1}^{N} \frac{1}{2^n} < \frac{r}{2}.$$

On the other hand, since δ_n takes values in $[0, 1)$, we also have

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < \frac{r}{2},$$

so we get

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < r,$$

thus proving that $\boldsymbol{y}$ indeed belongs to $\mathcal{B}_r(\boldsymbol{x})$. $\square$