# Lecture 5

## 5. Topology preliminaries V: Locally compact spaces

Definition. A *locally compact space* is a Hausdorff topological space with the property

(lc) *Every point has a compact neighborhood.*

One key feature of locally compact spaces is contained in the following;

Lemma 5.1. *Let $X$ be a locally compact space, let $K$ be a compact set in $X$, and let $D$ be an open subset, with $K \subset D$. Then there exists an open set $E$ with:*

(i) *$\overline{E}$ compact;*

(ii) *$K \subset E \subset \overline{E} \subset D$.*

Proof. Let us start with the following

*Particular case*: *Assume $K$ is a singleton $K = \{x\}$.*

Start off by choosing a compact neighborhood $N$ of $x$. Using the results from section 4, when equipped with the induced topology, the set $N$ is normal. In particular, if we consider the closed sets $A = \{x\}$ and $B = N \smallsetminus D$ (which are also closed in the induced topology), it follows that there exist sets $U, V \subset N$, such that

- $U \supset \{x\}$, $V \supset B$, $U \cap V = \varnothing$;
- $U$ and $V$ are open in the induced topology on $N$.

The second property means that there exist open sets $U_0, V_0 \subset X$, such that $U = N \cap U_0$ and $V = N \cap V_0$. Let $E = \operatorname{Int}(U)$. By construction $E$ is open, and $E \ni x$. Also, since $E \subset U \subset N$, it follows that

$$\overline{E} \subset \overline{N} = N. \tag{1}$$

In particular this gives the compactness of $E$. Finally, since we obviously have

$$E \cap V_0 \subset U \cap V_0 = N \cap U_0 \cap V_0 = U \cap V = \varnothing,$$

we get $E \subset X \smallsetminus V_0$, so using the fact that $X \smallsetminus V_0$ is closed, we also get the inclusion $\overline{E} \subset X \smallsetminus V_0$. Finally, combining this with (1) and with the inclusion $N \smallsetminus D \subset V \subset V_0$, we will get

$$\overline{E} \subset N \cap (X \smallsetminus V_0) \subset N \cap (N \smallsetminus D) \subset D,$$

and we are done.

Having proven the particular case, we proceed now with the general case. For every $x \in K$ we use the particular case to find an open set $E(x)$, with $\overline{E(x)}$ compact, and such that $x \in E(x) \subset \overline{E(x)} \subset D$. Since we clearly have $K \subset \bigcup_{x \in K} E(x)$, by compactness, there exist $x_1, \dots, x_n \in K$, such that $K \subset E(x_1) \cup \dots \cup E(x_n)$. Notice that if we take $E = E(x_1) \cup \dots \cup E(x_n)$, then we clearly have

$$K \subset E \subset \overline{E} \subset \overline{E(x_1)} \cup \dots \cup \overline{E(x_n)} \subset D,$$

and we are done. $\square$

One of the most useful result in the analysis on locally compact spaces is the following.

THEOREM 5.1 (Urysohn's Lemma for locally compact spaces). *Let $X$ be a locally compact space, and let $K, F \subset X$ be two disjoint sets, with $K$ compact, and $F$ closed. Then there exists a continuous function $f : X \to [0,1]$ such that $f\big|_K = 1$ and $f\big|_F = 0$.*

PROOF. Apply Lemma 5.1 for the pair $K \subset X \smallsetminus F$ and find an open set $E$, with $\overline{E}$ compact, such that $K \subset E \subset \overline{E} \subset X \smallsetminus F$. Apply again Lemma 5.1 for the pair $K \subset E$ and find another open set $G$ with $\overline{G}$ compact, such that $K \subset G \subset \overline{G} \subset E$.

Let us work for the moment in the space $\overline{E}$ (equipped with the induced topology). This is a compact Hausdorff space, hence it is normal. In particular, using Urysohn Lemma (see section 1) there exists a continuous function $g; \overline{E} \to [0,1]$ such that $g\big|_K = 0$ and $g\big|_{\overline{E} \smallsetminus G} = 0$. Let us now define the function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in \overline{E} \\ 0 & \text{if } x \in X \smallsetminus \overline{E} \end{cases}$$

Notice that $f\big|_E = g\big|_E$, so $f\big|_E$ is continuous. If we take the open set $A = X \smallsetminus \overline{G}$, then it is also clear that $f\big|_A = 0$. So now we have two open sets $E$ and $A$, with $A \cup E = X$, and $f\big|_A$ and $f\big|_E$ both continuous. Then it is clear that $f$ is continuous. The other two properties $f\big|_K = 1$ and $f\big|_F = 0$ are obvious. $\square$

We now discuss an important notion which makes the linkage between locally compact spaces and compact spaces

DEFINITION. Let $X$ be a locally compact space. By a *compactification of* $X$ one means a pair $(\theta, T)$ consisting of a compact Hausdorff space $T$, and of a continuous map $\theta : X \to T$, with the following properties

- (i) $\theta(X)$ is a dense open subset of $T$;
- (ii) when we equip $\theta(X)$ with the induced topology, the map $\theta : X \to \theta(X)$ is a homeomorphism.

Notice that, when $X$ is already compact, any compactification $(\theta, T)$ of $X$ is necessarily made up of a compact space $T$, and a homeomorphism $\theta : X \to T$.

EXAMPLES 5.1. A. Take $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, with the "usual" topology, in which a set $D \subset [-\infty, \infty]$ is open if $D = D_0 \cup D_1 \cup D_2$, where $D_0$ is open in $\mathbb{R}$ and

$$D_1, D_2 \in \big\{\varnothing\big\} \cup \big\{(a, \infty] \,:\, a \in \mathbb{R}\big\} \cup \big\{[-\infty, a) \,:\, a \in \mathbb{R}\big\}.$$

Then $[-\infty, \infty]$ is a compactification of $\mathbb{R}$

B. (Alexandrov compactification) Suppose $X$ is a locally compact space, which is not compact. We form a disjoint union with a singleton $X^\alpha = X \sqcup \{\infty\}$, and we equip the space $X^\alpha$ with the topology in which a subset $D \subset X^\alpha$ is declared to be open, if either $D$ is an open subset of $X$, or there exists some compact subset $K \subset X$, such that $D = (X \smallsetminus K) \sqcup \{\infty\}$. Define the inclsuion map $\iota : X \hookrightarrow X^\alpha$. Then $(\iota, X^\alpha)$ is a compactification of $X$, which is called the Alexandrov compactification. The fact that $\iota(X)$ is open in $X^\alpha$, and $\iota : X \to \iota(X)$ is a homeomorphism, is clear. The density of $\iota(X)$ in $X^\alpha$ is also clear, since every open set $D \subset X^\alpha$, with $D \ni \infty$, is of the form $(X \smallsetminus K) \sqcup \{\infty\}$, for some compact set $K \subset X$, and then we have $D \cap \iota(X) = \iota(X \smallsetminus K)$, which is non-empty, because $X$ is not compact.

Remark that, if $X$ is already compact, we can still define the topological space $X^\alpha = X \sqcup \{\infty\}$, but this time the singleto set $\{\infty\}$ will be also be open. Although $\iota(X)$ will still be open in $X^\alpha$, it will not be dense in $X^\alpha$.

*Exercise 1*$^\diamondsuit$. Let $K$ be some compact Hausdorff space, and let $p \in K$ be some point with the property that the set $X = K \smallsetminus \{p\}$ is not compact. Equip $X$ with the induced topology.

(i) Show that $X$ is locally compact (but non-compact).

(ii) If we denote by $X^\alpha$ the Alexandrov compactification of $X$, then the map $\Psi : X^\alpha \to K$, defined by

$$\Psi(x) = \begin{cases} x & \text{if } x \in X \\ p & \text{if } x = \infty \end{cases}$$

is a homeomorphism.

One should regard the Alexandrov compactification as a minimal one. More precisely, one has the following.

Proposition 5.1. *Suppose $X$ is a locally compact space which is non-compact. Let $(\theta, T)$ be a compactification of $X$, and let $X^\alpha = X \sqcup \{\infty\}$ be the Alexandrov compactification. Then there exists a unique continuous map $\Psi : T \to X^\alpha$, such that $(\Psi \circ \theta)(x) = x$, $\forall\, x \in X$. Moreover, the map $\Psi$ has the property that $\Psi(y) = \infty$, $\forall\, y \in T \smallsetminus \theta(X)$.*

Proof. The uniqueness part is pretty obvious, since $\theta(X)$ is dense in $T$. For the existence, we use the map $\theta : X \to T$ to identify $X$ with an open dense subset of $T$, and we define $\Psi : T \to X^\alpha$ by

$$\Psi(x) = \begin{cases} x & \text{if } x \in X \\ \infty & \text{if } x \in T \smallsetminus X \end{cases}$$

so that all we have to prove is the fact that $\Psi$ is continuous. Start with some open set $D$ in $X^\alpha$, and let us prove that $\Psi^{-1}$ is open in $T$. There are two cases.

*Case I*: $D \subset X$.

This case is trivial, $\Phi^{-1}(D) = D$, and $D$ is open in $X$, hence also open in $T$.

*Case II*: $D \not\subset X$.

In this case $D \ni \infty$, so there exists some compact set $K$ in $X$, such that $D = X^\alpha \smallsetminus K = (X \smallsetminus K) \cup \{\infty\}$. We then have

$$\Psi^{-1}(D) = \Psi^{-1}(X \smallsetminus K) \cup \Psi^{-1}(\{\infty\}) = (X \smallsetminus K) \cup (T \smallsetminus X) = T \smallsetminus K.$$

Since $K$ is compact in $X$, it will be compact in $T$ as well. In particular, $K$ is closed in $T$, hence the set $\Psi^{-1}(D) = T \smallsetminus K$ is indeed open in $T$. □

It turns out that there exists another compactification which is described below, which can be regarded as the largest.

Theorem 5.2 (Stone-Čech). *Let $X$ be a locally compact space. Consider the set*

$$F = \{f : X \to [0,1] \,:\, f \text{ continuous}\},$$

*and consider the product space*

$$T = \prod_{f \in F} [0,1],$$

*equipped with the product topology, and define the map* $\theta : X \to T$ *by*

$$\theta(x) = \big(f(x)\big)_{f \in F}, \ \ \forall\, x \in X.$$

*Equip the closure* $\overline{\theta(X)}$ *with the topology induced from* $T$*. Then the pair* $(\theta, \overline{\theta(X)})$ *is a compactification of* $X$*.*

Proof. For every $f \in F$, let us denote by $\pi_f : T \to [0,1]$ the coordinate map. Remark that $\theta : X \to T$ is continuous. This is immediate from the definition of the product topology, since the continuity of $\theta$ is equivalent to the continuity of all compositions $\pi_f \circ \theta$, $f \in F$. The fact that these compositions are continuous is however trivial, since we have $\pi_f \circ \theta = f$, $\forall\, f \in F$.

Denote for simplicity $\overline{\theta(X)}$ by $B$. By Tihonov's Theorem, the space $T$ is compact (and obviously Hausdorff), so the set $B$ is compact as well, being a closed subset of $T$. By construction, $\theta(X)$ is dense in $B$, and $\theta$ is continuous.

At this point, it is interesting to point out the following property

> *Claim 1***:** *For every* $f \in F$*, there exists a unique continuous map* $\tilde{f} : B \to [0,1]$*, such that* $\tilde{f} \circ \theta = f$*.*

The uniqueness is trivial, since $\theta(X)$ is dense in $B$. The existence is also trivial, because we can take $\tilde{f} = \pi_f\big|_B$.

We can show now that $\theta$ is injective. If $x, y \in X$ are such that $x \neq y$, then using Urysohn Lemma we can find $f \in F$, such that $f(x) \neq f(y)$. The function $\tilde{f}$ given by Claim 1, clearly satisfies

$$\tilde{f}\big(\theta(x)\big) = f(x) \neq f(y) = \tilde{f}\big(\theta(y)\big),$$

which forces $\theta(x) \neq \theta(y)$.

In order to show that $\theta(X)$ is open in $B$, we need some preparations. For every compact subset $K \subset X$, we define

$$F_K = \big\{ f : X \to [0,1] \, : \, f \text{ continuous}, \ f\big|_{X \smallsetminus K} = 0 \big\}.$$

On key observation is the following.

> *Claim 2***:** *If* $K \subset X$ *is compact, and if* $f \in F_K$*, then the continuous function* $\tilde{f} : B \to [0,1]$*, given by Claim 1, has the property* $\tilde{f}\big|_{B \smallsetminus \theta(K)} = 0$*.*

We start with some $\alpha \in B \smallsetminus \theta(K)$, and we use Urysohn Lemma to find some continuous function $\phi : B \to [0,1]$ such that $\phi(\alpha) = 1$ and $\phi\big|_{\theta(K)} = 0$. Consider the function $\psi = \phi \cdot \tilde{f}$. Notice that $(\phi \circ \theta)\big|_K = 0$, which combined with the fact that $f\big|_{X \smallsetminus K} = 0$, gives

$$\psi \circ \theta = (\phi \circ \theta) \cdot (\tilde{f} \circ \theta) = (\phi \circ \theta) \cdot f = 0,$$

so using Claim 1 (the uniqueness part), we have $\psi = 0$. In particular, since $\phi(\alpha) = 1$, this forces $\tilde{f}(\alpha) = 0$, thus proving the Claim.

We define now the collection

$$F_c = \bigcup_{\substack{K \subset X \\ K \text{ compact}}} F_K.$$

Define the set

$$S = \bigcap_{f \in F_c} \pi_f^{-1}\big(\{0\}\big).$$

By the definition of the product topology, it follows that $S$ is closed in $T$. The fact that $\theta(X)$ is open in $B$, is then a consequence of the following fact.

*Claim 3***:** *One has the equality* $\theta(X) = B \smallsetminus S$.

Start first with some point $x \in X$, and let us show that $\theta(x) \notin S$. Choose some open set $D \subset X$, with $\bar{D}$ compact, such that $D \ni x$, and apply Urysohn Lemma to find some continuous map $f : X \to [0,1]$ such that $f(x) = 1$ and $f\big|_{X \smallsetminus D} = 0$. It is clear that $f \in F_{\bar{D}} \subset F_c$, but $\pi_f\big(\theta(x)\big) = f(x) = 1 \neq 0$, which means that $\theta(x) \notin \pi_f^{-1}\big(\{0\}\big)$, hence $\theta(x) \notin S$. Conversely, let us start with some point $\alpha = (\alpha_f)_{f \in F} \in B \smallsetminus S$, and let us prove that $\alpha \in \theta(X)$. Since $\alpha \notin S$, there exists some $f \in F_c$, such that $\pi_f(\alpha) > 0$. Since $f \in F_c$, there exists some compact subset $K \subset X$, such that $f\big|_{X \smallsetminus K} = 0$. Using Claim 2, we know that $\tilde{f}\big|_{B \smallsetminus \theta(K)} = 0$. Since $\tilde{f}(\alpha) = \pi_f(\alpha) \neq 0$, this forces $\alpha \in \theta(K) \subset \theta(X)$.

To finish the proof of the Theorem, all we need to prove now is the fact that $\theta : X \to \theta(X)$ is a homeomorphism, which amounts to proving that, whenever $D \subset X$ is open, it follows that $\theta(D)$ is open in $B$. Fix an open subset $D \subset X$. In order to show that $\theta(D)$ is open in $B$, we need to show that $\theta(D)$ is a neighborhood for each of its points. Fix some point $\alpha \in \theta(D)$, i.e. $\alpha = \theta(x)$, for some $x \in D$. Choose some compact subset $K \subset D$, such that $x \in \text{Int}(K)$, and apply Urysohn Lemma to find a function $f \in F_K$, with $f(x) = 1$. Consider the continuous function $\tilde{f} : B \to [0,1]$ given by Claim 1, and apply Claim 2 to conclude that $\tilde{f}\big|_{B \smallsetminus \theta(K)} = 0$. In particular the open set

$$N = \tilde{f}^{-1}\big((1/2, \infty)\big) \subset B$$

is contained in $\theta(K) \subset \theta(D)$. Since $\tilde{f}(\alpha) = f(x) = 1$, we clearly have $x \in N$. □

Definition. The compactification $(\theta, \overline{\theta(X)})$, constructed in the above Theorem, is called the *Stone-Cech compactification of* $X$. The space $\overline{\theta(X)}$ will be denoted by $X^\beta$. Using the map $\theta$, we shall identify from now on $X$ with a dense open subset of $X^\beta$. Remark that if $X$ is compact, then $X^\beta = X$.

Comment. The Stone-Cech compactification is inherently "Zorn Lemma type" construction. For example, if $X$ is a locally compact space, then every ultrafilter on $X$ gives rise to a point in $X^\beta$, constructed as follows. If $\theta : X \to X^\beta$ denotes the inclusion map, then for every ultrafilter $\mathcal{U}$ on $X$, we consider the ultrafilter $\theta_*\mathcal{U}$ on $X^\beta$, and by compactness this ultrafilter converges to some (unique) point in $X^\beta$. This way one gets a correspondence

$$\mathbf{lim}_X : \big\{\mathcal{U} \subset \mathcal{P}(X) \,:\, \mathcal{U} \text{ ultrafilter on } X\big\} \to X^\beta.$$

The next two exercises discuss the features of this map.

*Exercise 2.* Let $X$ be a locally compact space.

A. Prove that, for an ultrafilter $\mathcal{U}$ on $X$, the condition $\mathbf{lim}_X\mathcal{U} \in X$ is equivalent to the condition that $\mathcal{U}$ contains a compact subset of $X$.
B. Prove that, for two ultrafilters $\mathcal{U}_1, \mathcal{U}_2$, the condition $\mathbf{lim}_X(\mathcal{U}_1) \neq \mathbf{lim}_X(\mathcal{U}_2)$ is equivalent to the existence of two sets $A_1 \in \mathcal{U}_1$ and $A_2 \in \mathcal{U}_2$, that are "separated by a continuous function," that is, for which there exists a continuous function $f : X \to \mathbb{R}$, and numbers $\alpha_1 < \alpha_2$, such that $f(A_1) \subset (-\infty, \alpha_1]$ and $f(A_2) \subset [\alpha_2, \infty)$.
C. Prove that the correspondence $\mathbf{lim}_X$ is surjective.

*Exercise 3.* Suppose a set $X$ is equipped with the *discrete* topology. Prove that the correspondence $\mathbf{lim}_X$ is bijective.

The Stone-Cech compactification is functorial, in the following sense.

Proposition 5.2. *If $X$ and $Y$ are locally compact spaces, and if $\Phi : X \to Y$ is a continuous map, then there exists a unique continuous map $\Phi^\beta : X^\beta \to Y^\beta$, such that $\Phi^\beta\big|_X = \Phi$.*

Proof. We use the notations from Theorem 5.2. Define

$$F = \{f : X \to [0,1] \,:\, f \text{ continuous}\} \text{ and } G = \{g : Y \to [0,1] \,:\, g \text{ continuous}\},$$

the product spaces

$$T_X = \prod_{f\in F}[0,1] \text{ and } T_Y = \prod_{g\in G}[0,1],$$

as well as the maps $\theta_X : X \to T_X$ and $\theta_Y : Y \to T_Y$, defined by

$$\theta_X(x) = \big(f(x)\big)_{f\in F}, \ \ \forall\, x \in X;$$
$$\theta_Y(y) = \big(g(y)\big)_{g\in G}, \ \ \forall\, y \in Y.$$

With these notations, we have $X^\beta = \overline{\theta_X(X)} \subset T_X$ and $Y^\beta = \overline{\theta_Y(Y)} \subset T_Y$. Using the fact that we have a correspondence $G \ni g \longmapsto g\circ\Phi \in F$, we define the map

$$\Psi : T_X \ni (\alpha_f)_{f\in F} \longmapsto (\alpha_{g\circ\Phi})_{g\in G} \in T_Y.$$

Remark that $\Psi$ is continuous. This fact is pretty obvious, because when we compose with corrdinate projections $\pi_g : T_Y \to [0,1]$, $g \in G$, we have $\pi_g \circ \Psi = \pi_{g\circ\Phi}$ where $\pi_{g\circ\Phi} : T_X \to [0,1]$ is the coordinate projection, which is automatically continuous. Remark that if we start with some point $x \in X$, then

$$\Psi\big(\theta_X(x)\big) = \big((g\circ\Phi)(x)\big)_{g\in G} = \theta_Y\big(\Phi(x)\big), \tag{2}$$

which means that we have the equality $\Psi\circ\theta_X = \theta_Y\circ\Phi$. Remark first that, since $Y^\beta$ is closed, it follows that $\Psi^{-1}(Y^\beta)$ is closed in $T_X$. Second, using (2), we clearly have the inclusion $\theta_X(X) \subset \Psi^{-1}\big(\theta_Y(Y)\big) \subset \Psi^{-1}(Y^\beta)$, so using the fact that $\Psi^{-1}(Y^\beta)$ is closed, we get the inclusion

$$X^\beta = \overline{\theta_X(X)} \subset \Psi^{-1}(Y^\beta).$$

In other words, we get now a continuous map $\Phi^\beta = \Psi\big|_{X^\beta} : X^\beta \to Y^\beta$, which clearly satisfies $\Phi^\beta\circ\theta_X = \theta_Y\circ\Phi$, which using our conventions means that $\Phi^\beta\big|_X = \Phi$. The uniqueness is obvious, by the density of $X$ in $X^\beta$. ☐

Remark 5.1. Suppose $X$ is a locally compact space which is not compact, and $Y$ is a compact Hausdorff space. By the above result, combined with the identification $Y^\beta \simeq Y$, we see that any continuous map $\Phi : X \to Y$ has a unique extension to a continuous map $\Phi^\beta : X^\beta \to Y$. In particular, if one takes $(\theta, T)$ to be a compactification of $X$, then $\theta : X \to T$ extends to a unique continuous map $\theta^\beta : X^\beta \to T$. This explains why the Stone-Cech compactification is sometimes referred to as the "largest" compactification. In particular, if we take $(\iota, X^\alpha)$ to be the Alexandrov compactification, we have a continuous map $\iota^\beta : X^\beta \to X^\alpha$, which is given by $\iota^\beta(x) = \infty, \ \forall\, x \in X^\beta \smallsetminus X$.

*Exercise 4.* Let $X$ be a locally compact space, let $X^\beta$ denote its Stone-Cech compactification, and let $(\theta, T)$ be an arbitrary compactification of $X$. Denote by $\theta^\beta : X^\beta \to T$ the map described in the above remark. Prove that for a topological space $Y$ and a map $f : T \to Y$, the following are equivalent

- (i) $f$ is continuous;
- (ii) the composition $f \circ \theta^\beta : X^\beta \to Y$ is continuous.

This explains how the topology of $T$ can be reconstructed using the map $\theta^\beta$. More precisely, the topology of $T$ is the *strong topology defined by* $\theta^\beta$ (see Lemma 3.2)

*Exercise 5.* The Alexandrov compactification is not functorial. In other words, given locally compact spaces $X$ and $Y$, and a continuous map $f : X \to Y$, in general there does not exist a continuous map $f^\alpha : X^\alpha \to Y^\alpha$, with $f^\alpha\big|_X = f$. Give an example of such a situation.

Hint: Consider $X = Y = \mathbb{N}$, equipped with the discrete topology, and define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$$

It turns out that one can define a certain type of continuous maps, with respect to which the Alexandrov compactification is functorial.

Definition. Let $X$, $Y$ be locally compact spaces, and let $\Phi : X \to Y$ be a continuous map. We say that $\Phi$ is *proper*, if it satisfies the condition

$$K \subset Y, \text{ compact } \Rightarrow \Phi^{-1}(K) \text{ compact in } X.$$

*Exercise 6*$^\diamondsuit$ (Functoriality of Alexandrov compactification). Let $X$ and $Y$ be a locally compact spaces, which are non-compact, and let $X^\alpha$ and $Y^\alpha$ denote their respective Alexandrov compactifications. For a continuous map $\Phi : X \to Y$, prove that the following are equivalent:

- (i) $\Phi$ is proper;
- (ii) the map $\Phi^\alpha : X^\alpha \to Y^\alpha$ defined by $\Phi^\alpha\big|_X = \Phi$ and $\Phi^\alpha(\infty) = \infty$ is continuous.

The following is an interesting property of proper maps, which will be exploited later, is the following.

Proposition 5.3. *Let $X$, $Y$ be locally compact spaces, let $\Phi : X \to Y$ be a proper continuous map, and let $T \subset X$ be a closed subset. Then the set $\Phi(T)$ is closed in $X$.*

Proof. Start with some point $y \in \overline{\Phi(T)}$. This means that

$$D \cap \Phi(T) \neq \varnothing, \text{ for every open set } D \subset Y, \text{ with } D \ni y. \tag{3}$$

Denote by $\mathcal{V}$ the collection of all compact neighborhoods of $y$. In other words, $V \in \mathcal{V}$, if and only if $V \subset Y$ is compact, and $y \in \operatorname{Int}(V)$. For each $V \in \mathcal{V}$ we define the set $\tilde{V} = \Phi^{-1}(V) \cap T$. Since $\Phi$ is proper, all sets $\tilde{V}$, $V \in \mathcal{V}$, are compact. Notice also that, for every finite number of sets $V_1, \dots, V_n \in \mathcal{V}$, if we form the intersection $V = V_1 \cap \dots \cap V_n$, then $V \in \mathcal{V}$, and $\tilde{V} \subset \tilde{V}_j$, $\forall\, j = 1, \dots, n$. Remark now that, by (3), we have $\tilde{V} \neq \varnothing$, $\forall\, V \in \mathcal{V}$. Indeed, if we start with some $V \in \mathcal{V}$ and we choose some point $x \in T$, such that $\Phi(x) \in V$, then $x \in \tilde{V}$. Use now the finite intersection property, to get the fact that $\bigcap_{V \in \mathcal{V}} \tilde{V} \neq \varnothing$. Pick now a point $x \in \bigcap_{V \in \mathcal{V}} \tilde{V}$. This means that $x \in T$, and

$$\Phi(x) \in V, \ \ \forall\, V \in \mathcal{V}. \tag{4}$$

But now we are done, because this forces $\Phi(x) = y$. Indeed, if $\Phi(x) \neq y$, using the Hausdorff property, one could find some $V \in \mathcal{V}$ with $\Phi(x) \notin V$, thus contradicting (4). ☐

COMMENT. When one deals with various compactifications of a non-compact locally compact space, the following extension problem is often discussed.

> *Question*: *Let $(\theta, T)$ be a compactification of a locally compact space $X$, let $Y$ be some topological Hausdorff space, and let $\Phi : X \to Y$ be a continuous map. When does there exist a continuous map $\Psi : T \to Y$, such that $\Psi \circ \theta = \Phi$?*

Of course, such a map (if it exists) is unique. Obviously, by density the existence of $\Psi$ will force $\Psi(T) = \overline{\Phi(X)}$, so we see that a necessary condition is the fact that $\overline{\Phi(X)}$ is *compact*. In the case of the Stone-Cech compactification, this comdition is also sufficient, by Remark 5.1.

For the Alexandrov compactification, the answer is given by the following.

PROPOSITION 5.4. *Let $X$ be a non-compact locally compact space, let $Y$ be a topological Hausdorff space, and let $\Phi : X \to Y$ be a continuous map. The following are equivalent.*

- (i) *There exists a continuous map $\Psi : X^\alpha \to Y$ with $\Psi\big|_X = \Phi$.*
- (ii) *There exists some point $p \in Y$ such that*
  - ($*$) *for every neighborhood $V$ of $p$, there exists some compact subset $K_V \subset X$ with $\Phi(X \smallsetminus K_V) \subset V$.*

*Moreover, the map $\Psi$ in (i) is unique, the point $p$ mentioned in (ii) is also unique, and $p = \Psi(\infty)$.*

PROOF. *(ii)* $\Rightarrow$ *(i)*. Assume $\Psi$ is as in (ii), and let us prove (i). Take $p = \Psi(\infty)$. Start with some neighborhood $V$ of $p$. Since $\Psi$ is continuous at $\infty$, the set $\Psi^{-1}(V)$ is a neighborhood of $\infty$ in $X^\alpha$. In particular there exists some compact set $K \subset X$, such that $\Psi^{-1}(V) \supset (X \smallsetminus K) \cup \{\infty\}$. We then obviously have $\Phi(x) = \Psi(x) \in V$, $\forall\, x \in X \smallsetminus K$.

*(i)* $\Rightarrow$ *(ii)*. Assume $p \in Y$ satisfies condition ($*$). Define the map $\Psi : X^\alpha \to Y$ by

$$\Psi(x) = \begin{cases} \Phi(x) & \text{if } x \in X \\ p & \text{if } x = \infty \end{cases}$$

and let us show that $\Psi$ is continuous. Since $\Psi\big|_X = \Phi$, and $\Phi$ is continuous, all we need to show is the fact that $\Psi$ is continuous at $\infty$. Let $V$ be some neighborhood of $p = \Psi(\infty)$, and let us show that $\Psi^{-1}(V)$ is a neighborhood of $\infty$ in $X^\alpha$. Take $D$ an open set in $Y$ with $p \in D \subset V$, and use condition ($*$) to choose some compact set $K$ in $X$, such that $\Phi(X \smallsetminus K) \subset D$, i.e. $\Phi^{-1}(D) \supset X \smallsetminus K$. We then have

$X \smallsetminus \Phi^{-1}(D) \subset K$. Since $\Phi$ is continuous, the set $\Phi^{-1}(D)$ is open in $X$, so the set $L = X \smallsetminus \Phi^{-1}(D)$ is a closed subset of $K$. In particular, $L$ is compact. It is then obvious that the set

$$\Psi^{-1}(D) = \{\infty\} \cup \Phi^{-1}(D) = \{\infty\} \cup (X \smallsetminus L)$$

is an open set in $X^\alpha$, and so $\Psi^{-1}(V) \supset \Psi^{-1}(D)$ is indeed a neighborhood of $\infty$ in $X^\alpha$. □

Definition. With the notations from the above result, a map $\Phi : X \to Y$ that satisfies condition (i), is said to *have limit p at infinity*. With this terminology, we see that a map has limit at infinity, if and only if it can be extended by continuity to a map defined on the Alexandrov compactification.

*Exercise 7.* Let $X$ be a non-compact locally compact space, let $Y$ be a topological Hausdorff space, and let $\Phi : X \to Y$ be a continuous map. Prove that the following are equivalent.

(i) $\Phi$ has limit at infinity.

(ii) There exists some point $p \in Y$ such that $\Phi(X) \cup \{p\}$ is compact, and the map $\Phi\big|_{X \smallsetminus \Phi^{-1}(\{p\})} : X \smallsetminus \Phi^{-1}(\{p\}) \to Y$ is proper.

(iii) The set $\overline{\Phi(X)}$ is compact, and if one denotes by $\Phi^\beta : X^\beta \to Y$ the unique continuous extension of $\Phi$ to the Stone-Cech compactification, then $\Phi^\beta\big|_{X^\beta \smallsetminus X}$ is constant, i.e. there exists some $p \in Y$, such that $\Phi^\beta(x) = p$, $\forall\, x \in X^\beta \smallsetminus X$.

Moreover, the points $p$ mentioned in (ii) and (iii) are unique, and they coincide with the limit in (i).

Comment. Let $X$ be a non-compact locally compact space, and let $(\theta, T)$ be a compactification of $X$. Think, for simplicity, $X$ as an open subset in $T$. Of course, since $X$ is dense in $T$, it follows that for every point $p \in T$ there exists some net $(x_\lambda)_{\lambda\in\Lambda}$ in $X$, which converges to $p$. Conversely, it is natural to pose the following

*Question: Given a net $(x_\lambda)_{\lambda\in\Lambda}$ in $X$, when does $(x_\lambda)_{\lambda\in\Lambda}$ converge to a point in $T$?*

(To make this question interesting, we exclude the case when $(x_\lambda)_{\lambda\in\Lambda}$ converges to some point in $X$.) For the Alexandrov compactification, the answer is pretty simple, since $\infty$ is the only "new" point, and by construction we see that the condition $\lim_{\lambda\in\Lambda} x_\lambda = \infty$ (in $X^\alpha$) is equivalent to the condition

- *for every compact set $K \subset X$, there exists some $\lambda_K \in \Lambda$, such that $x_\lambda \in X \smallsetminus K$, $\forall\, \lambda \succ \lambda_K$.*

For the Stone-Cech compactification, the answer is given in the following.

*Exercise 8.* Let $X$ be a non-compact locally compact space, and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in $X$. Prove that the following are equivalent:

(i) the net $(x_\lambda)_{\lambda\in\Lambda}$ is convergent to some point $p \in X^\beta$;

(ii) for every continuous function $f : X \to [0,1]$, the net $(f(x_\lambda))_{\lambda\in\Lambda}$ is convergent to some number in $[0,1]$.

Moreover, for every continuous function $f : X \to [0,1]$, one has the equality $\lim_{\lambda\in\Lambda} f(x_\lambda) = f^\beta(p)$, where $f^\beta : X^\beta \to [0,1]$ is the continuous extension of $f$.