Lecture 4

4. Compactness

Definition. Let X be a topological space X. A subset $K \subset X$ is said to be *compact set in* X, if it has the *finite open cover* property:

(f.o.c) *Whenever* $\{D_i\}_{i\in I}$ *is a collection of open sets such that* $K \subset \bigcup_{i\in I} D_i$*, there exists a finite sub-collection* $D_{i_1}, \dots, D_{i_n}$ *such that*

$$K \subset D_{i_1} \cup \cdots \cup D_{i_n}.$$

An equivalent description is the *finite intersection* property:

(f.i.p.) *If* $\{F_i\}_{i\in I}$ *is is a collection of closed sets such that for any finite sub-collection* $F_{i_1}, \dots, F_{i_n}$ *we have* $K \cap F_{i_1} \cap \dots F_{i_n} \neq \varnothing$*, it follows that*

$$K \cap \Big(\bigcap_{i\in I} F_i \Big) \neq \varnothing.$$

A topological space $(X, \mathcal{T})$ is called compact if X itself is a compact set.

Remark 4.1. Suppose $(X, \mathcal{T})$ is a topological space, and K is a subset of X. Equip K with the induced topology $\mathcal{T}\big|_K$. Then it is straightforward from the definition that the following are equivalent:

- *K is compact, as a subset in $(X, \mathcal{T})$;*
- *$(K, \mathcal{T}\big|_K)$ is a compact space, that is, K is compact as a subset in $(K, \mathcal{T}\big|_K)$.*

The following three results give methods of constructing compact sets.

Proposition 4.1. *A finite union of compact sets is compact.*

Proof. Immediate from the definition. □

Proposition 4.2. *Suppose $(X, \mathcal{T})$ is a topological space and $K \subset X$ is a compact set. Then for every closed set $F \subset X$, the intersection $F \cap K$ is again compact.*

Proof. Immediate, using the finite intersection property. □

Proposition 4.3. *Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are topological spaces, $f : X \to Y$ is a continuous map, and $K \subset X$ is a compact set. Then $f(K)$ is compact.*

Proof. Immediate from the definition. □

Besides the two equivalent conditions (f.o.c) and (f.i.p.), there are some other useful characterizations of compactness, listed in the following.

Theorem 4.1. *Let $(X, \mathcal{T})$ be a topological space. The following are equivalent:*

(i) *X is compact.*

(ii) (Alexander sub-base Theorem) *There exists a sub-base $\mathcal{S}$ with the finite open cover property:*

(s) *For any collection $\{S_i \mid i \in I\} \subset \mathcal{S}$ with $X = \bigcup_{i\in I} S_i$, there exists a finite sub-collection $\{S_{i_1}, S_{i_2}, \dots, S_{i_n}\}$ (for some finite sequence of indices $i_1, i_2, \dots, i_n \in I$) such that $X = S_{i_1} \cup S_{i_2} \cup \dots \cup S_{i_n}$.*

(iii) *Every ultrafilter in X is convergent.*

(iv) *Every net in X has a convergent subnet.*

Proof. *(i)* $\Rightarrow$ *(ii)*. This is obvious. (In fact *any* sub-base has the open cover property.)

(ii) $\Rightarrow$ *(iii)*. Let $\mathcal{U}$ be an ultrafilter on X. Assume $\mathcal{U}$ is not convergent to any point $x \in X$. By Proposition 3.2 it follows that, for each $x \in X$, one can find a set $S_x \in \mathcal{S}$ with $S_x \ni x$, but such that $S_x \notin \mathcal{U}$. Using property (s), one can find a finite collection of points $x_1, \dots, x_n \in X$, such that

$$S_{x_1} \cup \dots \cup S_{x_n} = X. \tag{1}$$

Since $S_{x_p} \notin \mathcal{U}$, it means that $X \smallsetminus S_{x_p}$ belongs to $\mathcal{U}$, for every $p = 1, \dots, n$. Then, using (1), we get

$$\mathcal{U} \ni (X \smallsetminus S_{x_1}) \cap \dots \cap (X \smallsetminus S_{x_n}) = \varnothing,$$

which is impossible.

(iii) $\Rightarrow$ *(iv)*. Assume condition (iii) holds, and let $(x_\lambda)_{\lambda\in\Lambda}$ be some net in X. Denote by Φ the collection of all finite subsets of Λ (including the empty set), and define, for each $F \in \Phi$, the sets

$$\Lambda_F = \{\lambda \in \Lambda : \lambda \succ \mu, \ \forall mu \in F\},$$
$$G_F = \{x_\lambda : \lambda \in \Lambda_F\}$$

(use the convention $\Lambda_\varnothing = \Lambda$). Since we obviously have $\Lambda_{F_1} \cap \Lambda_{F_2} = \Lambda_{F_1 \cup F_2}$, $\forall F_1, F_2 \in \Phi$, we get the equalities

$$G_{F_1} \cap G_{F_2} = G_{F_1 \cup F_2}, \ \forall F_1, F_2 \in \Phi,$$

which prove the fact that the collection $\mathcal{G} = \{G_F\}_{F\in\Phi}$ is a *filter* in X. Let $\mathcal{U}$ be then some ultrafilter with $\mathcal{U} \supset \mathcal{G}$. By the hypothesis (iii), the ultrafilter $\mathcal{U}$ is convergent to some point $x \in X$, which means that $\mathcal{U}$ contains the collection $\mathcal{N}_x$ of all neighborhoods of x. In particular, we get the fact that

$$N \cap G_F \neq \varnothing, \ \forall N \in \mathcal{N}_x, \ F \in \Phi. \tag{2}$$

We are now in position to define a subnet of $(x_\lambda)_{\lambda\in\Lambda}$, which will prove to be convergent to x. Consider the set

$$\Sigma = \{(N, \lambda) \in \mathcal{N}_x \times \Lambda : x_\lambda \in N\},$$

equipped with the ordering

$$(N_1, \lambda_1) \succ (N_2, \lambda_2) \Longleftrightarrow \begin{cases} N_1 \subset N_2 \\ \lambda_1 \succ \lambda_2 \end{cases}$$

Let us remark that Σ is a directed set. Indeed, if we start with two elements $\sigma_1 = (N_1, \lambda_1)$ and $\sigma_2 = (N_2, \lambda_2)$ in Σ, then using (2) with $N = N_1 \cap N_2$, and $F = \{\lambda_1, \lambda_2\}$, we get the existence of some $\lambda \in \Lambda$ with $\lambda \succ \lambda_1$, $\lambda \succ \lambda_2$, and such that $x_\lambda \in N$. This means that the pair $\sigma = (N, \lambda)$ belongs to Σ, and it also satisfies $\sigma \succ \sigma_1$, $\sigma \succ \sigma_2$.

Consider also the map

$$\phi : \Sigma \ni (N, \lambda) \longmapsto \lambda \in \Lambda.$$

Again using (2), it follows that ϕ is a directed map.

Define then the net $(y_\sigma)_{\sigma\in\Sigma}$ by $y_\sigma = x_{\phi(\sigma)}$, $\forall\, \sigma \in \Sigma$. By construction, $(y_\sigma)_{\sigma\in\Sigma}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$. We show now that $(y_\sigma)_{\sigma\in\Sigma}$ is convergent to x. Start with some arbitrary neighborhood N of x. Use (2) with $F = \varnothing$, to find some $\lambda_N \in \Lambda$, such that $x_{\lambda_N} \in N$. Put $\sigma_N = (N, \lambda_N)$. If $\sigma = (V, \lambda) \in \Sigma$ is such that $\sigma \succ \sigma_N$, then in particular we have $x_\lambda \in M \subset N$, i.e. $y_\sigma \in N$. In other words we have

$$y_\sigma \in N, \ \ \forall\, \sigma \succ \sigma_N,$$

thus proving that $(y_\sigma)_{\sigma\in\Sigma}$ is indeed convergent to x.

$(iv) \Rightarrow (i)$. Assume (iv), and let us prove that X has the finite intersection property (F.I.P.). Let $\{F_i\}_{i\in I}$ is is a collection of closed sets such that for any finite sub-collection $F_{i_1}, \dots, F_{i_n}$ we have $F_{i_1} \cap \dots F_{i_n} \neq \varnothing$, and let us prove that $\bigcap_{i\in I} F_i \neq \varnothing$. Denote by Ω the collection of all finite non-empty subsets of I, and define, for each $J \in \Omega$, the non-empty closed set $F_J = \bigcap_{i\in J} F_i$. With this notation, what we have to prove is the fact that $\bigcap_{J\in\Omega} F_J \neq \varnothing$. The advantage here is the fact that Ω is directed (by inclusion). Since $F_J \neq \varnothing$, for each $J \in \Omega$, we can choose an element $x_J \in F_J$. This way (use the Axiom of Choice) we can construct a net $(x_J)_{J\in\Omega}$. Using (iv) there exists a subnet $(y_\sigma)_{\sigma\in\Sigma} \underset{\phi}{\subset} (x_J)_{J\in\Omega}$ which is convergent to some point $x \in X$. In order to finish the proof, it suffices to show that $x \in F_J$, $\forall\, J \in \Omega$. As above, denote by $\mathcal{N}_x$ the collection of all neighborhoods of x. Since each F_J is closed, all we need to prove is the fact that

$$N \cap F_J \neq \varnothing, \ \ \forall\, N \in \mathcal{N}_x,\ J \in \Omega.$$

Fix $N \in \mathcal{N}_x$ and $J \in \Omega$. On the one hand, since $\phi : \Sigma \to \Omega$ is a directed map, there exists $\sigma_1 \in \Sigma$, such that $\phi(\sigma) \succ J$, $\forall\, \sigma \succ \sigma_1$. On the other hand, since $(y_\sigma)_{\sigma\in\Sigma}$ is convergent to x, there exists some $\sigma_2 \in \Sigma$, such that $y_\sigma \in N$, $\forall\, \sigma \succ \sigma_2$. If we choose $\sigma \in \Sigma$ such that $\sigma \succ \sigma_1$ and $\sigma \succ \sigma_2$, then on the one hand we have $y_\sigma = x_{\phi(\sigma)} \in F_{\phi(\sigma)} \subset F_J$ (here we use the fact that, for $J, J_1 \in \Omega$, one has $J_1 \succ J \Rightarrow F_{J_1} \subset F_J$), and on the other hand we also have $y_\sigma \in N$. This means precisely that $y_\sigma \in N \cap F_J$. ☐

An interesting application of the above result is the following:

THEOREM 4.2 (Tihonov). *Suppose one has a familiy $(X_i, \mathcal{T}_i)_{i\in I}$ of compact topological spaces. Then the product space $\prod_{i\in I} X_i$ is compact in the product topology.*

PROOF. We are going to use the ultrafilter characterization (iii) from the preceding Theorem. Let $\mathcal{U}$ be an ultrafilter on $X = \prod_{i\in I} X_i$. Denote by $\pi_i : X \to X_i$, $i \in I$ the coordinate maps. Since each X_i is compact, it follows that, for every $i \in I$, the ultrafilter $\pi_{i*}(\mathcal{U})$ (in X_i) is convergent to some point $x_i \in X_i$. If we form the element $x = (x_i)_{i\in I} \in X$, this means that $\pi_{i*}(\mathcal{U})$ is convergent to $\pi_i(x)$, for every $i \in I$. Then, by the ultrafilter characterization of the product topology (see section 3) it follows that $\mathcal{U}$ is convergent to x. ☐

COMMENT. Another interesting application of Theorem 4.1 is the following construction. Suppose $(X, \mathcal{T})$ is a compact Hausdorff space, and $(x_i)_{i\in I} \subset X$ is an arbitray family of elements. (Here I is an arbitrary set.) Suppose $\mathcal{U}$ is an ultrafilter *on* I. If we regard the family $(x_i)_{i\in I}$ simply as a function $f : I \to X$, then we can construct the ultrafilter $f_*(\mathcal{U})$ on X. More explicitly

$$f_*(\mathcal{U}) = \big\{W \subset X : \text{ the set } \{i \in I : x_i \in W\} \text{ belongs to } \mathcal{U}\big\}.$$

Since X is compact Hausdorff, the ultrafilter $f_*(\mathcal{U})$ is convergent to some unique point $x \in X$. This point is denoted by $\lim_{\mathcal{U}} x_i$.

We conclude this section with some results on compactness in Hausdorff spaces.

Proposition 4.4. *Suppose* $(X, \mathcal{T})$ *is toplological Hausdorff space.*

(i) *Any compact set* $K \subset X$ *is closed.*
(ii) *If* K *is a compact set, then a subset* $F \subset K$ *is compact, if and only if* F *is closed (in* X*).*

Proof. (i) The key step is contained in the following

Claim: *For every* $x \in X \smallsetminus K$*, there exists some open set* D_x *with* $x \in D_x \subset X \smallsetminus K$*.*

Fix $x \in X \smallsetminus K$. For every $y \in K$, using the Hausdorff property, we can find two open sets U_y and V_y with $U_y \ni x$, $V_y \ni y$, and $U_y \cap V_y = \varnothing$. Since we obviously have $K \subset \bigcup_{y\in K} V_y$, by compactness, there exist points $y_1, \dots, y_n \in K$, such that $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. The claim immediately follows if we then define $D_x = U_{y_1} \cap \dots \cap U_{y_n}$.

Using the Claim we now see that we can write the complement of K as a union of open sets:

$$X \smallsetminus K = \bigcup_{x\in X\smallsetminus K} D_x,$$

so $X \smallsetminus K$ is open, which means that K is indeed closed. (ii). If F is closed, then F is compact by Proposition 4.2. Conversely, if F is compact, then by (i) F is closed. □

Proposition 4.5. *Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. Let $A, B \subset X$ be two closed sets with $A \cap B = \varnothing$. We need to find two open sets $U, V \subset X$, with $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$. We start with the following

Particular case: *Assume* B *is a singleton,* $B = \{b\}$*.*

The proof follows line by line the first part of the proof of part (i) from Proposition 4.4. For every $a \in A$ we find open sets U_a and V_a, such that $U_a \ni a$, $V_a \ni b$, and $U_a \cap V_a = \varnothing$. Using Proposition 4.4 we know that A is compact, and since we clearly have $A \subset \bigcup_{a\in A} U_a$, there exist $a_1, \dots, a_n \in A$, such that $U_{a_1} \cup \dots \cup U_{a_n} \supset A$. Then we are done by taking $U = U_{a_1} \cup \dots \cup U_{a_n}$ and $V = V_{a_1} \cap \dots \cap V_{a_n}$.

Having proven the above particular case, we proceed now with the general case. For every $b \in B$, we use the particular case to find two open sets U_b and V_b, with $U_b \supset A$, $V_b \ni b$, and $U_b \cap V_b = \varnothing$. Arguing as above, the set B is compact, and we have $B \subset \bigcup_{b\in B} V_b$, so there exist $b_1, \dots, b_n \in B$, such that $V_{b_1} \cup \dots \cup V_{b_n} \supset B$. Then we are done by taking $U = U_{b_1} \cap \dots \cap U_{b_n}$ and $V = V_{b_1} \cup \dots \cup V_{b_n}$. □