Lecture 3

3. Constructing topologies

In this section we discuss several methods for constructing topologies on a given set.

Definition. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on the same space X, such that $\mathcal{T}' \subset \mathcal{T}$ (as sets), then $\mathcal{T}$ is said to be *stronger than* $\mathcal{T}'$. Equivalently, we will say that $\mathcal{T}'$ is *weaker than* $\mathcal{T}$.

Remark that this condition is equivalent to the continuity of the map

$$\mathrm{Id} : (X, \mathcal{T}) \to (X, \mathcal{T}').$$

Comment. Given a (non-empty) set X, and a collection $\mathcal{S}$ of subsets of X, one can ask the following:

> *Question 1*: *Is there a topology on X with respect to which all the sets in $\mathcal{S}$ are open?*

Of course, this question has an affirmative answer, since we can take as the topology the collection of *all* subsets of X. Therefore the above question is more meaningful if stated as:

> *Question 2*: *Is there the weakest topology on X with respect to which all the sets in $\mathcal{S}$ are open?*

The answer to this question is again affirmative, and it is based on the following:

Remark 3.1. If X is a non-empty set, and $(\mathcal{T}_i)_{i \in I}$ is a family of topologies on X, then the intersection

$$\bigcap_{i \in I} \mathcal{T}_i$$

is again a topology on X.

In particular, if one starts with an arbitrary family $\mathcal{S}$ of subsets of X, and if we take

$$\Theta(\mathcal{S}) = \big\{ \mathcal{T} : \mathcal{T} \text{ topology on } X \text{ with } \mathcal{T} \supset \mathcal{S} \big\},$$

then the intersection

$$\textsc{top}(\mathcal{S}) = \bigcap_{\mathcal{T} \in \Theta(\mathcal{S})} \mathcal{T}$$

is the weakest (i.e. smallest) among all topologies with respect to which all sets in $\mathcal{S}$ are open.

The topology $\textsc{top}(\mathcal{S})$ defined above cane also be described constructively as follows.

Proposition 3.1. *Let $\mathcal{S}$ be a collection of subsets of X. Then the sets in $\textsc{top}(\mathcal{S})$, which are a proper subsets of X, are those which can be written a (arbitrary) unions of finite intersections of sets in $\mathcal{S}$.*

PROOF. It is useful to introduce the following notations. First we define $\mathcal{V}(\mathcal{S})$ to be the collection of all sets which are finite intersections of sets in $\mathcal{S}$. In other words,

$$B \in \mathcal{V}(\mathcal{S}) \Longleftrightarrow \exists\, D_1, \dots, D_n \in \mathcal{S} \text{ such that } D_1 \cap \dots \cap D_n = B.$$

With the above notation, what we need to prove is that for a set $A \subsetneq X$, we have

$$A \in \text{TOP}(\mathcal{S}) \Longleftrightarrow \exists\, \mathcal{V}_A \subset \mathcal{V}(\mathcal{S}) \text{ such that } A = \bigcup_{B \in \mathcal{V}_A} B.$$

The implication "$\Leftarrow$" is pretty obvious. Since $\text{TOP}(\mathcal{S})$ is a topology, and every set in $\mathcal{S}$ is open with respect to $\text{TOP}(\mathcal{S})$, it follows that every finite intersection of sets in $\mathcal{S}$ is again in $\text{TOP}(\mathcal{S})$, which means that every set in $\mathcal{V}(\mathcal{S})$ is again open with respect to $\text{TOP}(\mathcal{S})$. But then arbitrary unions of sets in $\mathcal{V}(\mathcal{S})$ are again open with respect to $\text{TOP}(\mathcal{S})$.

To prove the implication "$\Rightarrow$" we define

$$\mathcal{T}_0 = \Big\{ A \subset X \,:\, \exists\, \mathcal{V}_A \subset \mathcal{V}(\mathcal{S}) \text{ such that } A = \bigcup_{B \in \mathcal{V}_A} B \Big\},$$

and we will show that

$$\text{TOP}(\mathcal{S}) \subset \{X\} \cup \mathcal{T}_0. \tag{1}$$

By the definition of $\text{TOP}(\mathcal{S})$ it suffices to prove the following

Claim: *The collection $\mathcal{T}_1 = \{X\} \cup \mathcal{T}_0$ is a topology on X, which contains all the sets in $\mathcal{S}$.*

The fact that $\mathcal{T}_1 \supset \mathcal{S}$ is trivial.

The fact that $\varnothing, X \in \mathcal{T}_1$ is also clear.

The fact that arbitrary unions of sets in $\mathcal{T}_1$ again belong to $\mathcal{T}_1$ is again clear, by construction.

Finally, we need to show that if $A_1, A_2 \in \mathcal{T}_1$, then $A_1 \cap A_2 \in \mathcal{T}_1$. If either $A_1 = X$ or $A_2 = X$, there is nothing to prove. Assume that both A_1 and A_2 are proper subsets of X, so there are subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}(\mathcal{S})$, such that

$$A_1 = \bigcup_{B \in \mathcal{V}_1} B \text{ and } A_2 = \bigcup_{E \in \mathcal{V}_1} E.$$

Then it is clear that

$$A_1 \cap A_2 = \bigcup_{\substack{B \in \mathcal{V}_1 \\ E \in \mathcal{V}_2}} (B \cap E),$$

with all the sets $B \cap E$ in $\mathcal{V}(\mathcal{S})$, so $A_1 \cap A_2$ indeed belongs to $\mathcal{T}_1$. □

DEFINITION. Let X be a (non-empty) set, let $\mathcal{T}$ be a topology on X. A collection $\mathcal{S}$ of subsets of X, with the property that

$$\mathcal{T} = \text{TOP}(\mathcal{S}),$$

is called a *sub-base for* $\mathcal{T}$. According to the above remark, the above condition is equivalent to the fact that every open set $D \subsetneq X$ can be written as a *union of finite intersections of sets in* $\mathcal{S}$.

Convergence is characterized using sub-bases as follows;

PROPOSITION 3.2. *Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{S}$ be a sub-base for $\mathcal{T}$, let x be some point in X, and define the collection $\mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}$.*

A. *The collection* $\mathcal{S}_x \cup \{X\}$ *is a fundamental system of neighborhoods of* x.
B. *For an ultrafilter* $\mathcal{U}$ *on* X*, the following are equivalent:*
   (i) $\mathcal{U}$ *is convergent to* x*;*
   (ii) $\mathcal{U}$ *contains all the sets* $S \in \mathcal{S}_x$.
C. *For a net* $(x_\lambda)_{\lambda\in\Lambda}$ *in* X*, the following are equivalent:*
   (i) $(x_\lambda)_{\lambda\in\Lambda}$ *is convergent to* x*;*
   (ii) *for every* $S \in \mathcal{S}_x$*, there exists* $\lambda_S \in \Lambda$*, such that* $x_\lambda \in S$, $\forall\, \lambda \succ \lambda_S$.

Proof. A. We need to show that for every neighborhood N of x, there exist sets $V_1, \dots, V_n \in \mathcal{S}_x \cup \{X\}$, such that $x \in V_1 \cap \dots \cap V_n \subset N$. Start with some arbitrary neighborhood N of x, so there exists some open set D with $x \in D \subset N$. If $D = X$, there is nothing to prove, since we can take $V_1 = X$. If $D \subsetneq X$, then using the fact that D is a union of finite intersections of sets in $\mathcal{S}$, together with the fact that $x \in D$, there exist sets $V_1, \dots, V_n \in \mathcal{S}$, such that $x \in V_1 \cap \dots \cap V_n \subset D \subset N$. It is clear that all the V's are in fact in $\mathcal{S}_x$.

B. Immediate from part A, and Proposition 2.1.

C. Immediate from part A, and Proposition 2.2. □

There are instances when sub-bases have a particular feature, which enables one to describe all open sets in an easier fashion.

Proposition 3.3. *Let* $(X, \mathcal{T})$ *be a topological space. Suppose* $\mathcal{V}$ *is a colletion of subsets of* X*. The following are equivalent:*

(i) $\mathcal{V}$ *is a sub-base for* $\mathcal{T}$*, and*

$$\forall\, U, V \in \mathcal{V} \text{ and } x \in U \cap V,, \ \exists\, W \in \mathcal{V} \text{ with } x \in W \subset U \cap V. \tag{2}$$

(ii) *Every open set* $A \subsetneq X$ *is a union of sets in* $\mathcal{V}$.

Proof. $(i) \Rightarrow (ii)$. From property (i), it follows that *every finite intersection of sets in* $\mathcal{V}$ *is a union of sets in* $\mathcal{V}$. Then the desired implication is immeadiate from the previous result.

$(ii) \Rightarrow (i)$. Assume (ii) and start with two sets $U, V \in \mathcal{V}$, and an element $x \in U \cap V$. Since $U \cap V$ is open, by (ii) either we have $U \cap V = X$, in which case we get $U = V = X$, and we take $W = X$, or $U \cap V \subsetneq X$, in which case $U \cap V$ is a union of sets in $\mathcal{V}$, so in particular there exists $W \in \mathcal{V}$ with $x \in W \subset U \cap V$. □

Definition. If $(X, \mathcal{T})$ is a topological space, a collection $\mathcal{V}$ which satisfies the above equivalent conditions, is called a *base for* $\mathcal{T}$.

The following is a useful technical result.

Lemma 3.1. *Let* $(Y, \mathcal{T})$ *be a topological space, let* X *be some (non-empty) set, and let* $f : X \to Y$ *be a function. Then the collection*

$$f^*(\mathcal{T}) = \{f^{-1}(D) \,:\, D \in \mathcal{T}\}$$

is a topology on X*. Moreover,* $f^*(\mathcal{T})$ *is the weakest topology on* X*, with respect to which the map* f *is continuous.*

Proof. Clearly $\varnothing = f^{-1}(\varnothing)$ and $X = f^{-1}(Y)$ both belong to $f^*(\mathcal{T})$. If $(A_i)_{i\in I}$ is a family of sets in $f^*(\mathcal{T})$, say $A_i = f^{-1}(D_i)$, for some $D_i \in \mathcal{T}$, for all $i \in I$, then the equality

$$\bigcup_{i\in I} A_i = \bigcup_{i\in I} f^{-1}(D_i) = f\big(\bigcup_{i\in I} D_i\big)$$

clearly shows that $\bigcup_{i\in I} A_i$ again belongs to $f^*(\mathcal{T})$. Likewise, if $A_1, A_2 \in f^*(\mathcal{T})$, say $A_1 = f^{-1}(D_1)$ and $A_2 = f^{-1}(D_2)$ for some $D_1, D_2 \in \mathcal{T}$, then the equality

$$A_1 \cap A_2 = f^{-1}(D_1) \cap f^{-1}(D_2) = f^{-1}(D_1 \cap D_2)$$

proves that $A_1 \cap A_2$ again belongs to $f^*(\mathcal{T})$.

Having proven that $f^*(\mathcal{T})$ is a topology on X, let us prove now the second statement. The fact that f is continuous with respect to $f^*(\mathcal{T})$ is clear by construction. If $\mathcal{T}'$ is another topology which still makes f continuous, then this will force all the sets of the form $f^{-1}(D)$, $D \in \mathcal{T}$ to belong to $\mathcal{T}'$, which means that $f^*(\mathcal{T}) \subset \mathcal{T}'$. □

Remark 3.2. Using the above notations, if $\mathcal{V}$ is a (sub)base for $\mathcal{T}$, then

$$f^*(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$$

is a (sub)base for $f^*(\mathcal{T})$. This is pretty obvious since the correspondence

$$\{\text{subsets of } Y\} \ni D \longmapsto f^{-1}(D)$$

is compatible with the operation of intersection and union (of arbitrary families).

Remark 3.3. As a consequence of the above remark, we see that (sub)bases can be useful for verifying continuity. More specifically, if $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ are topological spaces, and $\mathcal{V}$ is a sub-base for $\mathcal{T}'$, then a function $f : X \to X'$ is continuous, if and only if $f^{-1}(V)$ is open, for all $V \in \mathcal{V}$.

The construction outlined in Lemma 3.1 can be generalized as follows.

Proposition 3.4. *Let X be a set, and let $\Phi = (f_i, Y_i)_{i\in I}$ be a family consisting of maps $f_i : X \to Y_i$, where Y_i is a topological space, for all $i \in I$. Then there is a unique toplogy $\mathcal{T}^\Phi$ on X, with the following properties*

(i) *Each of the maps $f_i : X \to Y_i$, $i \in I$ is continuous with respect to $\mathcal{T}^\Phi$.*
(ii) *Given a topological space $(Z, \mathcal{S})$, and a map $g : Z \to X$, such that the composition $f_i \circ g : Z \to Y_i$ is continuous, for every $i \in I$, it follows that g is continuous as a map $(Z, \mathcal{S}) \to (X, \mathcal{T}^\Phi)$.*

Proof. For every $i \in I$ we define

$$\mathcal{D}_i = \{f_i^{-1}(D) : D \text{ open subset of } Y_i\},$$

and we form the collection

$$\mathcal{D} = \bigcup_{i\in I} \mathcal{D}_i.$$

Take $\mathcal{T}^\Phi = \text{TOP}(\mathcal{D})$ Property (i) follows from the simple observation that, by construction, every set in $\mathcal{D}$ is open.

To prove property (ii) start with a topological space $(Z, \mathcal{S})$, and a map $g : Z \to X$, such that the composition $f_i \circ g : Z \to X_i$ is continuous, for every $i \in I$, and let us prove that g is continuous. By Remark 3.3 it suffices to prove that $g^{-1}(A)$ is open (in Z) for every $A \in \mathcal{D}$. By the definition of $\mathcal{D}$ this is equivalent to proving the fact that, for each $i \in I$, and each open set $D \subset Y_i$, the set $g^{-1}\big(f_i^{-1}(D)\big)$ is open. But this is obvious, since we have

$$g^{-1}\big(f_i^{-1}(D)\big) = (f_i \circ g)^{-1}(D),$$

and $f_i \circ g : Z \to Y_i$ is continuous.

To prove the uniqueness, let $\mathcal{T}$ be another topology on X with properties (i) and (ii). Consider the map $h = \text{Id} : (X, \mathcal{T}) \to (X, \mathcal{T}^\Phi)$. Using property (i) for $\mathcal{T}$, combined with property (ii) for $\mathcal{T}^\Phi$, it follows that h is continuous, which means that $\mathcal{T}^\Phi \subset \mathcal{T}$. Reversing the roles, and arguing exactly the same way, we also get the other inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. □

REMARK 3.4. Using the above setting, assume that for each $i \in I$ a sub-base $\mathcal{S}_i$ for the topology of Y_i is given. Consider the sets $f_i^*\mathcal{S}_i = \{f_i^{-1}(S) : S \in \mathcal{S}_i\}$. Then the collection

$$\mathcal{S} = \bigcup_{i\in I} f_i^*\mathcal{S}_i$$

is a sub-base for the topology $\mathcal{T}^\Phi$.

To prove this, we take $\mathcal{T} = \text{TOP}(\mathcal{S})$, so that we obviously have the inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. In order to prove the equality $\mathcal{T} = \mathcal{T}^\Phi$, all we have to prove are (use the notations from the proof of the above Proposition) the inclusions

$$\mathcal{D}_i \subset \mathcal{T}, \ \forall\, i \in I.$$

By construction however, we have $\mathcal{D}_i = f_i^*\mathcal{T}_i$, and since $\mathcal{S}_i$ is a sub-base for $\mathcal{T}_i$, it follows that $f_i^*\mathcal{S}_i$ is a sub-base for $\mathcal{D}_i$, which means that we have

$$\mathcal{D}_i = \text{TOP}(f_i^*\mathcal{S}_i) \subset \text{TOP}(\mathcal{S}) = \mathcal{T}.$$

COMMENT. Using the notations above, it is immediate that the topology $\mathcal{T}^\Phi$ can also be described as *the weakest topology on X, with respect to which all the maps $f_i : X \to Y_i$, $i \in I$, are continuous.* In the light of this remark, we will call the topology $\mathcal{T}^\Phi$ the *weak topology defined by* Φ.

Convergence can be nicely characterized:

PROPOSITION 3.5. *Let X be a set, let $\Phi = (f_i, Y_i)_{i\in I}$ be a family consisting of maps $f_i : X \to Y_i$, where Y_i is a topological space, for all $i \in I$, and let x be some point in X.*

A. *For an ultrafilter $\mathcal{U}$ on X, the following are equivalent:*
   (i) *$\mathcal{U}$ is convergent to x, with respect to the topology $\mathcal{T}^\Phi$;*
   (ii) *for every $i \in I$, the ultrafilter $f_{i*}(\mathcal{U})$ is convergent to $f_i(x)$.*

B. *For a net $(x_\lambda)_{\lambda\in\Lambda}$ in X, the following are equivalent:*
   (i) *$(x_\lambda)_{\lambda\in\Lambda}$ is convergent to x, with respect to the topology $\mathcal{T}^\Phi$;*
   (ii) *for every $i \in I$, the net $\big(f_i(x_\lambda)\big)_{\lambda\in\Lambda}$ is convergent to $f_i(x)$.*

PROOF. The implications $(i) \Rightarrow (ii)$ in both A and B are clear, since all maps $f_i : (X, \mathcal{T}^\Phi) \to (Y_i, \mathcal{T}_i)$, $i \in I$, are continuous.

To prove the implications $(ii) \Rightarrow (i)$ we consider the collection

$$\mathcal{S} = \bigcup_{i\in I} \{ f_i^{-1}(D) : D \subset Y_i \text{ open} \},$$

which is a sub-base for the topology $\mathcal{T}^\Phi$, and we define $\mathcal{S}_x = \{S \in \mathcal{S} : S \ni x\}$. It is pretty clear that we have

$$\mathcal{S}_x = \bigcup_{i\in I} \{ f_i^{-1}(D) : D \subset Y_i \text{ open, with } f_i(x) \in D \} \tag{3}$$

A. $(ii) \Rightarrow (i)$. Suppose $\mathcal{U}$ satisfies (ii). Then for every $i \in I$, the ultrafilter $f_{i*}(\mathcal{U})$ contains all the open sets $D \subset Y_i$ with $D \ni f_i(x)$. In other words, for every $i \in I$ and every open set $D \subset Y_i$ with $f_i(x) \in D$, we have $f_i^{-1}(D) \in \mathcal{U}$. By (3) this means precisely that we have the inclusion $\mathcal{U} \supset \mathcal{S}_x$. Then the fact that $\mathcal{U}$ converges to x follows from Proposition 3.2.

B. $(ii) \Rightarrow (i)$. Suppose the net $(x_\lambda)_{\lambda\in\Lambda}$ satisfies (ii). In order to prove that $(x_\lambda)_{\lambda\in\Lambda}$ is convergent to x, we are going to use Proposition 3.2. Using (3), it suffices to show that for every $i \in I$ and every open set $D \subset Y_i$ with $D \ni f_i(x)$, there exists some $\lambda_{i,D} \in \Lambda$, such that

$$x_\lambda \in f_i^{-1}(D), \ \forall \lambda \succ \lambda_{i,D}.$$

This is however clear, since the condition $x_\lambda \in f_i^{-1}(D)$ is equivalent to $f_i(x_\lambda) \in D$, and we know that by (ii) the net $(f_i(x_\lambda))_{\lambda\in\Lambda}$ is convergent to $f_i(x) \in D$. □

Example 3.1. (The product topology) Supoose we have a family $(X_i, \mathcal{T}_i)$, $i \in I$ of topological spaces. Consider the Cartesian product $X = \prod_{i\in I} X_i$. For each $j \in I$ we consider the projection $\pi_j : X \to X_j$. The weakest topology on X, defined by the family $\Phi = \{\pi_j\}_{j\in I}$, is called the *product topology.*

A sub-base for the product topolgy can be defined as follows. For each $i \in I$, we choose a sub-base $\mathcal{S}_i$ for $\mathcal{T}_i$ (for instance we can take $\mathcal{S}_i = \mathcal{T}_i$), and we take

$$\mathcal{S} = \bigcup_{i\in I} \pi_i^* \mathcal{S}_i = \bigcup_{i\in I} \{\pi_i^{-1}(D) \, : \, D \in \mathcal{S}_i\}.$$

Then $\mathcal{S}$ is a sub-base for the product topology.

For a point $x = (x_i)_{i\in I} \in X$, and an ultrafilter $\mathcal{U}$ on X, the condition $\mathcal{U} \to x$ is equivalent to the fact that $\pi_{i*}(\mathcal{U}) \to x_i$, $\forall i \in I$.

For a point $x = (x_i)_{i\in I} \in X$, and a net $(x^\lambda)_{\lambda\in\Lambda}$ in X, given in coordinates as $x^\lambda = (x_i^\lambda)_{i\in I}$, $\lambda \in \Lambda$, the condition $(x^\lambda)_{\lambda\in\Lambda}$ is convergent to x is equivalent to the fact, for each $i \in I$, the net $(x_i^\lambda)_{\lambda\in\Lambda}$ is convergent to x_i.

Another method of constructing topologies is based on the following "dual" version of Lemma 3.1.

Lemma 3.2. *Let $(Y, \mathcal{T})$ be a topological space, let X be some (non-empty) set, and let $f : Y \to X$ be a function. Then the collection*

$$f_*(\mathcal{T}) = \{D \subset X \, : \, f^{-1}(D) \in \mathcal{T}\}$$

is a topology on X. Moreover, $f_*(\mathcal{T})$ is the strongest topology on X, with respect to which the map f is continuous.

Proof. Since $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(X) = Y$, it follows that $\varnothing$ and X both belong to $f_*(\mathcal{T})$. If $(A_i)_{i\in I}$ is a family of sets in $f_*(\mathcal{T})$, then the sets $f^{-1}(A_i)$, $i \in I$ belong to $\mathcal{T}$. In particular the set

$$f^{-1}\big(\bigcup_{i\in I} A_i\big) = \bigcup_{i\in I} f^{-1}(A_i)$$

will again belong to $\mathcal{T}$, which means that $\bigcup_{i\in I} A_i$ belongs to $f_*(\mathcal{T})$. Likewise, if $A_1, A_2 \in f_*(\mathcal{T})$, then the sets $f^{-1}(A_1)$ and $f^{-1}(A_2)$ both belong to $\mathcal{T}$. The intersection

$$f^{-1}(A_1 \cap A_2) = f^{-1}(A_1) \cap f^{-1}(A_2)$$

will then belong to $\mathcal{T}$, which means that $A_1 \cap A_2$ again belongs to $f_*(\mathcal{T})$.

Having proven that $f_*(\mathcal{T})$ is a topology on X, let us prove now the second statement. The fact that f is continuous with respect to $f_*(\mathcal{T})$ is clear by construction. If $\mathcal{T}'$ is another topology which still makes f continuous, then this will force all the sets of the form $f^{-1}(A)$, $A \in \mathcal{T}'$ to belong to $\mathcal{T}$, which means that A will in fact belong to $f_*(\mathcal{T})$. In other words, we have the inclusion $\mathcal{T}' \subset f_*(\mathcal{T})$. □

A generalization of the above construction is given in the following.

PROPOSITION 3.6. *Let X be a set, and let $\Phi = (f_i, Y_i)_{i\in I}$ be a family consisting of maps $f_i : Y_i \to X$, where Y_i is a topological space, for all $i \in I$. Then there is a unique toplogy $\mathcal{T}_\Phi$ on X, with the following properties*

- (i) *Each of the maps $f_i : Y_i \to X$, $i \in I$ is continuous with respect to $\mathcal{T}_\Phi$.*
- (ii) *Given a topological space $(Z, \mathcal{S})$, and a map $g : X \to Z$, such that the composition $g \circ f_i : Y_i \to Z$ is continuous, for every $i \in I$, it follows that g is continuous as a map $(X, \mathcal{T}_\Phi) \to (Z, \mathcal{S})$.*

PROOF. For each $i \in I$, let $\mathcal{T}_i$ denote the topology on Y_i. We define

$$\mathcal{T}_\Phi = \bigcap_{i\in I} f_{i*}(\mathcal{T}_i).$$

Property (i) is obvious by construction.

To prove property (ii), start with some topological space $(Z, \mathcal{S})$ and a map $g : X \to Z$ such that $g \circ f_i : Y_i \to Z$ is continuous, for all $i \in I$. Start with some open set $D \subset Z$, and let us prove that the set $A = g^{-1}(D)$ is open in X, i.e. $A \in \mathcal{T}_\Phi$. Notice that, for each $i \in I$, one has

$$f_i^{-1}(A) = f_i^{-1}\big(g^{-1}(D)\big) = (g \circ f_i)^{-1}(D),$$

so using the continuity of $g \circ f_i$ we get the fact that $f_i^{-1}(A)$ is open in Y_i, which means that $A \in f_{i*}(\mathcal{T}_i)$. Since this is true for all $i \in I$, we then get $A \in \mathcal{T}_\Phi$.

To prove uniqueness, let $\mathcal{T}$ be another topology on X with properties (i) and (ii). Consider the map $h = \mathrm{Id} : (X, \mathcal{T}) \to (X, \mathcal{T}^\Phi)$. Using property (i) for $\mathcal{T}_\Phi$, combined with property (ii) for $\mathcal{T}$, it follows that h is continuous, which means that $\mathcal{T}^\Phi \subset \mathcal{T}$. Reversing the roles, and arguing exactly the same way, we also get the other inclusion $\mathcal{T} \subset \mathcal{T}^\Phi$. □

COMMENT. Using the notations above, it is immediate that the topology $\mathcal{T}_\Phi$ can also be described as *the strongest topology on X, with respect to which all the maps $f_i : Y_i \to X$, $i \in I$, are continuous*. In the light of this remark, we will call the topology $\mathcal{T}_\Phi$ the *strong topology defined by* Φ.

EXAMPLE 3.2. (The disjoint union topology) Supoose we have a family $(X_i, \mathcal{T}_i)$, $i \in I$ of topological spaces. Consider the disjoint union[1] $X = \bigsqcup_{i\in I} X_i$. For each $i \in I$ we consider the inclusion $\epsilon_i : X_i \to X$. The strongest topology on X, defined by the family $\Phi = \{\epsilon_i\}_{i\in I}$, is called the *disjoint union topology*.

If we think each X_i as a subset of X, then X_i is open in X, for all $i \in I$. Moreover, a set $D \subset X$ is open, if and only if $D \cap X_i$ is open (in X_i), for all $i \in I$. For a point $x \in X$, there exists a unique $i(x) \in I$, with $x \in X_{i(x)}$. With this notation, an ultrafilter $\mathcal{U}$ on X is convergent to x, if and only if $X_{i(x)} \in \mathcal{U}$, and the collection

$$\mathcal{U}\big|_{X_{i(x)}} = \{U \cap X_{i(x)} : U \in \mathcal{U}\}$$

is an ultrafilter on $X_{i(x)}$, which converges to x.

A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is convergent to x, if and only if there exists some $\lambda_x \in \Lambda$ such that $x_\lambda \in X_{i(x)}$, $\forall\, \lambda \succ \lambda_x$, and the net $(x_\lambda)_{\lambda\succ\lambda_x}$ is convergent, in $X_{i(x)}$, to x.

[1] Formally one uses the sets $Z = \bigcup_{i\in I} X_i$, and $Y = I \times Z$, and one realizes the disjoint union as $X = \bigcup_{i\in I} [\{i\} \times X_i]$.