

**Chapter I**  
**Topology Preliminaries**



# LECTURE 1

## 1. Review of basic topology concepts

In this lecture we review some basic notions from topology, the main goal being to set up the language. Except for one result (Uryson Lemma) there will be no proofs.

DEFINITIONS. A *topology* on a (non-empty) set  $X$  is a family  $\mathcal{T}$  of subsets of  $X$ , which are called *open sets*, with the following properties:

- (TOP<sub>1</sub>): *both the empty set  $\emptyset$  and the total set  $X$  are open;*
- (TOP<sub>2</sub>): *an arbitrary union of open sets is open;*
- (TOP<sub>3</sub>): *a finite intersection of open sets is open.*

In this case the system  $(X, \mathcal{T})$  is called a *topological space*.

If  $(X, \mathcal{T})$  is a topological space and  $x \in X$  is an element in  $X$ , a subset  $N \subset X$  is called a *neighborhood of  $x$*  if there exists some open set  $D$  such that  $x \in D \subset N$ .

A collection  $\mathcal{N}$  of neighborhoods of  $x$  is called a *basic system of neighborhoods of  $x$* , if for any neighborhood  $M$  of  $x$ , there exists some neighborhood  $N$  in  $\mathcal{N}$  such that  $x \in N \subset M$ .

A collection  $\mathcal{V}$  of neighborhoods of  $x$  is called a *fundamental system of neighborhoods of  $x$*  if for any neighborhood  $M$  of  $x$  there exists a finite sequence  $V_1, V_2, \dots, V_n$  of neighborhoods in  $\mathcal{V}$  such that  $x \in V_1 \cap V_2 \cap \dots \cap V_n \subset M$ .

A topology is said to have the *Hausdorff property* if:

- (H) *for any  $x, y \in X$  with  $x \neq y$ , there exist open sets  $U \ni x$  and  $V \ni y$  such that  $U \cap V = \emptyset$ .*

If  $(X, \mathcal{T})$  is a topological space, a subset  $F \subset X$  will be called *closed*, if its complement  $X \setminus F$  is open. The following properties are easily derived from the definition:

- (C<sub>1</sub>) *both the empty set  $\emptyset$  and the total set  $X$  are closed;*
- (C<sub>2</sub>) *an arbitrary intersection of closed sets is closed;*
- (C<sub>3</sub>) *a finite union of closed sets is closed.*

Using the above properties of open/closed sets, one can perform the following constructions. Let  $(X, \mathcal{T})$  be a topological space and  $A \subset X$  be an arbitrary subset. Consider the set  $\text{Int}(A)$  to be the union of all open sets  $D$  with  $D \subset A$  and consider the set  $\bar{A}$  to be the intersection of all closed sets  $F$  with  $F \supset A$ . The set  $\text{Int}(A)$  (sometimes denoted simply by  $\overset{\circ}{A}$ ) is called the *interior of  $A$* , while the set  $\bar{A}$  is called the *closure of  $A$* . The properties of these constructions are summarized in the following:

PROPOSITION 1.1. *Let  $(X, \mathcal{T})$  be a topological space, and let  $A$  be an arbitrary subset of  $X$ .*

A. (Properties of the interior)

- (i) The set  $\text{Int}(A)$  is open and  $\text{Int}(A) \subset A$ .
- (ii) If  $D$  is an open set such that  $D \subset A$ , then  $D \subset \text{Int}(A)$ .
- (iii)  $x$  belongs to  $\text{Int}(A)$  if and only if  $A$  is a neighborhood of  $x$ .
- (iv)  $A$  is open if and only if  $A = \text{Int}(A)$ .

B. (Properties of the closure)

- (i) The set  $\bar{A}$  is closed and  $\bar{A} \supset A$ .
- (ii) If  $F$  is a closed set with  $F \supset A$ , then  $F \supset \bar{A}$ .
- (iii) A point  $x$  belongs to  $\bar{A}$ , if and only if,  $A \cap N \neq \emptyset$  for any neighborhood  $N$  of  $x$ .
- (iv)  $A$  is closed if and only if  $A = \bar{A}$ .

C. (Relationship between interior and closure)  $\text{Int}(X \setminus A) = X \setminus \bar{A}$  and  $\overline{X \setminus A} = X \setminus \text{Int}(A)$ .

*Exercise 1*  $\diamond$ . Let  $X$  be some non-empty set, and let  $\mathcal{P}(X)$  denote the collection of all subsets of  $X$ . Let  $\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  be a map with the following properties:

- (0)  $\text{cl}(\emptyset) = \emptyset$ ;
- (1)  $\text{cl}(A) \supset A, \forall A \in \mathcal{P}(X)$ ;
- (2)  $\text{cl} \circ \text{cl} = \text{cl}$ ;
- (3)  $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B), \forall A, B \in \mathcal{P}(X)$ .

Prove there exists a unique topology  $\mathcal{T}$  on  $X$ , with respect to which we have the equality  $\text{cl}(A) = \bar{A}, \forall A \in \mathcal{P}(X)$ .

HINTS: Prove first the implication  $A \subset B \Rightarrow \text{cl}(A) \subset \text{cl}(B)$ . Define then the collection

$$\mathcal{T} = \{A \subset X : \text{cl}(X \setminus A) = X \setminus A\}.$$

A map  $\text{cl}$  with the above properties is called a *closure operator*.

DEFINITION. Suppose  $(X, \mathcal{T})$  is a topological space. Assume  $A \subset X$  is a subset of  $X$ . On  $A$  we can introduce a natural topology, sometimes denoted by  $\mathcal{T}|_A$  which consists of all subsets of  $A$  of the form  $A \cap U$  with  $U$  open set in  $X$ . This topology is called the *relative* (or *induced*) *topology*.

REMARK 1.1. If  $A$  is already open in the topology  $\mathcal{T}$ , then a subset  $V \subset A$  is open in the induced topology if and only if  $V$  is open in the topology  $\mathcal{T}$  (this follows from the fact that the intersection of any two open sets in  $\mathcal{T}$  is again an open set in  $\mathcal{T}$ ).

DEFINITION. Suppose  $(X, \mathcal{T})$  and  $(Y, \mathcal{S})$  are topological spaces and  $x$  is an element in  $X$ . A map  $f : X \rightarrow Y$  is said to be *continuous at  $x$* , if for any neighborhood  $N$  of  $f(x)$  in the topology  $\mathcal{S}$  (on  $Y$ ), the set

$$f^{-1}(N) = \{x \in X \mid f(x) \in N\}$$

is a neighborhood of  $x$  in the topology  $\mathcal{T}$  (on  $X$ ).

If  $f$  is continuous at every point in  $X$ , then  $f$  is said to be *continuous*.

Continuity is “well behaved” with respect to compositions:

PROPOSITION 1.2. Suppose  $(X, \mathcal{T})$ ,  $(Y, \mathcal{S})$ , and  $(Z, \mathcal{Z})$  are topological spaces, and  $X \xrightarrow{f} Y \xrightarrow{g} Z$  are two functions.

- (i) If  $f$  is continuous at a point  $x \in X$ , and if  $g$  is continuous at  $f(x)$ , then  $g \circ f$  is continuous at  $x$ .
- (ii) If  $f$  and  $g$  are (globally) continuous, then so is  $g \circ f$ .

The identity map on a topological space is always continuous.

In terms of open/closed sets, the characterization of continuity is given by the following.

PROPOSITION 1.3. *If  $(X, \mathcal{T})$ ,  $(Y, \mathcal{S})$  are topological spaces and  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$  is a map, then the following are equivalent:*

- (i)  *$f$  is continuous.*
- (ii) *Whenever  $U \subset Y$  is an open set, it follows that  $f^{-1}(U)$  is also an open set (in  $X$ ).*
- (iii) *Whenever  $F \subset Y$  is a closed set, it follows that  $f^{-1}(F)$  is also a closed set (in  $X$ ).*

We conclude this section with a useful technical result.

THEOREM 1.1 (Urysohn's Lemma). *Let  $(X, \mathcal{T})$  be a topological Hausdorff space with the following property:*

- (N) *For any two disjoint closed sets  $A, B \subset X$ , there exist two disjoint open sets  $U, V \subset X$ , such that  $U \supset A$  and  $V \supset B$ .*

*Then for any two disjoint closed sets  $A, B \subset X$ , there exists a continuous function  $f : X \rightarrow [0, 1]$  such that  $f|_A = 0$  and  $f|_B = 1$ .*

PROOF. We begin with a refinement of property (N):

- (N') *For any disjoint closed sets  $A, B \subset X$ , there exist two open sets  $U, W \subset X$ , such that  $A \subset U$ ,  $\bar{U} \subset W$ , and  $\bar{W} \cap B = \emptyset$ .*

To prove (N'), we first apply (N) to find two disjoint open sets  $W, Z \subset X$  such that

$$(1) \quad W \supset A \text{ and } Z \supset B.$$

Next we apply again (N) to the pair of closed sets  $A$  and  $X \setminus W$ , and find two disjoint open sets  $U, V \subset X$  such that

$$(2) \quad U \supset A \text{ and } V \supset X \setminus W.$$

On the one hand, using the fact that  $U \cap V = \emptyset$  and the fact that  $V$  is open, we get the inclusion  $\bar{U} \subset X \setminus V$ . Using (2) this gives

$$\bar{U} \subset X \setminus V \subset W.$$

On the other hand, using the fact that  $W \cap Z = \emptyset$  and the fact that  $Z$  is open, we get  $\bar{W} \subset X \setminus Z$ . But using (1) this will give

$$\bar{W} \subset X \setminus Z \subset X \setminus B,$$

and we are done.

To prove the Theorem, start with two disjoint closed sets  $A, B \subset X$ . For every integer  $n \geq 0$  we define the set  $\mathcal{D}_n = \{\frac{k}{2^n} : k \in \mathbb{Z}, 0 \leq k \leq 2^n\}$ , and we consider

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n.$$

(Notice that  $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ , for all  $n \geq 0$ .)

We are going to construct a family  $(V_t)_{t \in \mathcal{D}}$  of open sets in  $X$  with the following properties

- (i)  $V_0 \supset A$  and  $\bar{V}_1 \cap B = \emptyset$ ;
- (ii)  $\bar{V}_t \subset V_s$ , for all  $t, s \in \mathcal{D}$  with  $t < s$ .

Let us start by constructing  $V_0$  and  $V_1$ . We use property (N') to find open sets  $U, W \subset X$ , with

$$A \subset U \subset \bar{U} \subset W \text{ and } \bar{W} \cap B = \emptyset,$$

and we simply take  $V_0 = U$  and  $V_1 = W$ .

The construction of the family  $(V_t)_{t \in \mathcal{D}}$  is carried on recursively. Assume, for some integer  $n \geq 0$ , we have constructed the sets  $(V_t)_{t \in \mathcal{D}_n}$  with property (i) and (ii) (satisfied for  $t, s \in \mathcal{D}_n$ ), and let us construct the next block of sets  $(V_t)_{t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n}$ . We start off by observing that for every  $t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ , then the numbers

$$t^\pm = t \pm \frac{1}{2^{n+1}}$$

belong to  $\mathcal{D}_n$ . Apply (N') to the pair of disjoint closed sets  $\bar{V}_{t^-}$  and  $X \setminus V_{t^+}$  to find two open sets  $U, W \subset X$  such that

$$\bar{V}_{t^-} \subset U \subset \bar{U} \subset W \text{ and } \bar{W} \cap X \setminus V_{t^+} = \emptyset.$$

Notice that the equality  $\bar{W} \cap (X \setminus V_{t^+}) = \emptyset$ , coupled with the inclusion  $\bar{U} \subset W$ , gives  $\bar{U} \cap (X \setminus V_{t^+}) = \emptyset$ , so we get  $\bar{U} \subset V_{t^+}$ . We can then define  $V_t = U$ , and we will obviously have the inclusions

$$(3) \quad \bar{V}_{t^-} \subset V_t \subset \bar{V}_t \subset V_{t^+}.$$

Now the extended family  $(V_t)_{t \in \mathcal{D}_{n+1}}$  will also satisfy property (ii), since for  $t, s \in \mathcal{D}_{n+1}$  with  $t < s$ , one of the following will hold:

- either  $t, s \in \mathcal{D}_n$ , or
- $t \in \mathcal{D}_n$ ,  $s \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ , and  $t \leq s^-$ , or
- $t \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ ,  $s \in \mathcal{D}_n$ , and  $t^+ \leq s$ , or
- $t, s \in \mathcal{D}_{n+1} \setminus \mathcal{D}_n$ , and  $t^+ \leq s^-$ .

(In either case, one uses (3) combined with the inductive hypothesis.)

Having constructed the family  $(V_t)_{t \in \mathcal{D}}$ , with properties (i) and (ii), we define the functions  $f : X \rightarrow [0, 1]$  by

$$f(x) = \begin{cases} \inf\{t \in \mathcal{D} : x \in V_t\}, & \text{if } x \in V_1 \\ 1, & \text{if } x \notin V_1 \end{cases}$$

*Claim 1: The function  $f$  is equivalently defined by*

$$(4) \quad f(x) = \begin{cases} 0, & \text{if } x \in \bar{V}_0 \\ \sup\{t \in \mathcal{D} : x \notin \bar{V}_t\}, & \text{if } x \notin \bar{V}_0 \end{cases}$$

Let us denote by  $g : X \rightarrow [0, 1]$  be the function defined by formula (4). Fix some point  $x \in X$ . We break the proof in several cases

**CASE I:**  $x \in \bar{V}_0$ .

In particular, using (ii) we get  $x \in V_t$ , for all  $t \in \mathcal{D}$ , with  $t > 0$ , and since  $x \in V_1$ , we have

$$f(x) = \inf\{t \in \mathcal{D} : x \in V_t\} = \inf\{t \in \mathcal{D} : t > 0\} = 0 = g(x).$$

**CASE II:**  $x \notin V_1$ .

Using (ii) we have  $x \notin \bar{V}_t$ , for all  $t \in \mathcal{D}$ , with  $t < 1$ , and since  $x \notin \bar{V}_0$ , we have

$$g(x) = \sup\{t \in \mathcal{D} : x \notin \bar{V}_t\} = \sup\{t \in \mathcal{D} : t < 1\} = 1 = f(x).$$

**CASE III:**  $x \in V_1 \setminus \bar{V}_0$ .

By the definition of  $f(x)$  we know:

- (5)  $x \notin V_t, \forall t \in \mathcal{D}, \text{ with } t < f(x).$   
 (6)  $\forall \varepsilon > 0, \exists s_\varepsilon \in \mathcal{D}, \text{ with } f(x) \leq s_\varepsilon < f(x) + \varepsilon, \text{ such that } x \in V_{s_\varepsilon}.$

By the definition of  $g(x)$  we know:

- (7)  $x \in \bar{V}_t, \forall t \in \mathcal{D}, \text{ with } t > g(x);$   
 (8)  $\forall \varepsilon > 0, \exists r_\varepsilon \in \mathcal{D}, \text{ with } g(x) \geq r_\varepsilon > g(x) - \varepsilon, \text{ such that } x \notin \bar{V}_{r_\varepsilon}.$

Using (6) and (8) we see that we must have

- (9)  $s_\varepsilon \geq r_\varepsilon, \forall \varepsilon > 0.$

Indeed, if there exists some  $\varepsilon > 0$  for which we have  $s_\varepsilon < r_\varepsilon$ , then using (6) we would have

$$x \in V_{s_\varepsilon} \subset \bar{V}_{s_\varepsilon} \subset V_{r_\varepsilon} \subset \bar{V}_{r_\varepsilon},$$

which contradicts (8).

Now the inequality (9) gives

$$f(x) + \varepsilon > g(x) - \varepsilon, \forall \varepsilon > 0,$$

so we have in fact the inequality

$$f(x) \geq g(x).$$

Suppose now this inequality is strict. Using (5) and (7) we will get

- (10)  $x \in \bar{V}_t \text{ and } x \notin V_t, \text{ for all } t \in \mathcal{D}, \text{ with } f(x) > t > g(x).$

Using the fact that  $\mathcal{D}$  is dense in  $[0, 1]$ , we could then find at least two elements  $t_1, t_2 \in \mathcal{D}$  such that

$$f(x) > t_1 > t_2 > g(x).$$

In this case (10) immediately creates a contradiction, since

$$x \in \bar{V}_{t_2} \subset V_{t_1}.$$

*Claim 2: The function  $f$  is continuous.*

Since any open set in  $\mathbb{R}$  is a union of open intervals, it suffices to prove the following two properties<sup>1</sup>

(USC):  $f^{-1}((\infty, t))$  is open for all  $t \in \mathbb{R}$ ;

(LSC):  $f^{-1}((t, \infty))$  is open for all  $t \in \mathbb{R}$ .

In order to prove property (USC) it suffices to prove the equality

$$(11) \quad f^{-1}((\infty, t)) = \bigcup_{\substack{s \in \mathcal{D} \\ s < t}} V_s.$$

Start with a point  $x \in f^{-1}((t, \infty))$ , which means that  $f(x) < t$ . Using (6), there exists some  $s \in \mathcal{D}$  with  $f(x) < s < t$ , such that  $x \in V_s$ , so  $x$  indeed belongs to the right hand side of (11). Conversely, if  $x$  belongs to the right hand side of (11), there exists some  $s < t$  such that  $x \in V_s$ . By the definition of  $f(x)$ , it follows that  $f(x) \leq s < t$ , so  $x \in f^{-1}((\infty, t))$ .

<sup>1</sup> The condition (USC) means that  $f$  is *upper semi-continuous*, while the condition (LSC) means that  $f$  is *lower semi-continuous*.

In order to prove property (LSC) it suffices to prove the equality

$$(12) \quad f^{-1}((t, \infty)) = \bigcup_{\substack{r \in \mathcal{D} \\ r > t}} (X \setminus \bar{V}_r).$$

Start with a point  $x \in f^{-1}((t, \infty))$ , which means that  $f(x) > t$ . Using (8), there exists some  $r \in \mathcal{D}$  with  $f(x) > r > t$ , such that  $x \notin \bar{V}_r$ , that is,  $x \in X \setminus \bar{V}_r$ , so  $x$  indeed belongs to the right hand side of (12). Conversely, if  $x$  belongs to the right hand side of (12), there exists some  $r > t$  such that  $x \in X \setminus \bar{V}_r$ , i.e.  $x \notin \bar{V}_r$ . By the equivalent definition of  $f(x)$  given by Claim 1, it follows that  $f(x) \geq r > t$ , so  $x \in f^{-1}((t, \infty))$ .

Having proven that  $f$  is continuous, let us finish the proof. Since  $A \subset V_0$ , by the definition of  $f$ , we get  $f|_A = 0$ . Since  $B \subset X \setminus V_1$ , again by the definition of  $f$ , we get  $f|_B = 1$ .  $\square$

DEFINITION. A Hausdorff space  $(X, \mathcal{T})$  with property (N) is called *normal*.