4. The weak dual topology

In this section we examine the topological duals of normed vector spaces. Besides the norm topology, there is another natural topology which is constructed as follows.

Definition. Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}(=\mathbb{R}, \mathbb{C})$. For every $x \in \mathcal{X}$, let $\epsilon_x : \mathcal{X}^* \to \mathbb{K}$ be the linear map defined by

$$\epsilon_x(\phi) = \phi(x), \ \ \forall\, \phi \in \mathcal{X}^*.$$

We equipp the vector space $\mathcal{X}^*$ with the weak topology defined by the family $\Xi = (\epsilon_x)_{x\in\mathcal{X}}$. This topology is called the *weak dual topology*, which is denoted by w^*. Recall (see I.3) that this topology is characterized by the following property

(w^*) *Given a topological space T, a map $f : T \to \mathcal{X}^*$ is continuous with respect to the w^* topology, if and only if $\epsilon_x \circ f : T \to \mathbb{K}$ is continuous, for each $x \in \mathcal{X}$.*

Remark that all the maps $\epsilon_x : \mathcal{X}^* \to \mathbb{K}$, $x \in \mathcal{X}$ are already continuous with respect to the *norm* topology. This gives the fact that

- *the w^* topology on $\mathcal{X}^*$ is weaker than the norm topology.*

The weak topology can also be characterized in terms of convergent nets. More explicitly

- *a net $(\phi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{X}^*$ converges weakly to some $\phi \in \mathcal{X}^*$, if and only if*

$$\lim_{\lambda\in\Lambda} \phi_\lambda(x) = \phi(x), \ \ \forall\, x \in \mathcal{X}.$$

In this case we use the notation $\phi = w^*\text{-}\lim_{\lambda\in\Lambda} \phi_\lambda$.

Remark 4.1. The w^* topology is Hausdorff. Indeed, if $\phi, \psi \in \mathcal{X}^*$ are such that $\phi \neq \psi$, then there exists some $x \in \mathcal{X}$ such that

$$\epsilon_x(\phi) = \phi(x) \neq \psi(x) = \epsilon_x(\psi).$$

Proposition 4.1. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For every $\varepsilon > 0$, $\phi \in \mathcal{X}^*$, and $x \in \mathcal{X}$, define the set*

$$W(\phi; x, \varepsilon) = \big\{\psi \in \mathcal{X}^* \,:\, |\psi(x) - \phi(x)| < \varepsilon.$$

Then the collection

$$\mathcal{W} = \big\{W(\phi; x, \varepsilon) \,:\, \varepsilon > 0,\, \phi \in \mathcal{X}^*,\, x \in \mathcal{X}\big\}$$

is a subbase for the w^* topology. More precisely, given $\phi \in \mathcal{X}^*$, a set $N \subset \mathcal{X}^*$ is a neighborhood of ϕ with respect to the w^* topology, if and only if, there exist $\varepsilon > 0$ and $x_1, \dots, x_n \in \mathcal{X}$, such that

$$N \supset W(\phi; \varepsilon, x_1) \cap \dots \cap W(\phi; \varepsilon, x_n).$$

Proof. It is clearly sufficient to prove the second assertion, because it would imply the fact that any w^* open set is a union of finite intersections of sets in $\mathcal{W}$.

If we define the collection

$$\mathcal{S} = \big\{\epsilon_x^{-1}(D) \,:\, x \in \mathcal{X},\ D \subset \mathbb{K} \text{ open } \big\},$$

then we know that $\mathcal{S}$ is a subbase for the w^* topology.

Fix $\phi \in \mathcal{X}^*$. Start with some w^* neighborhood N of ϕ, so there exists some w^* open set E with $\phi \in E \subset N$. Using the fact that $\mathcal{S}$ is a subbase for the w^* topology, there exist open sets $D_1, \dots, D_n \subset \mathbb{K}$, and points $x_1, \dots, x_n$, such that

$$\phi \in \bigcap_{k=1}^{n} \epsilon_{x_k}^{-1}(D_k) \subset E.$$

Fix for the moment $k \in \{1, \dots, n\}$. The fact that $\phi \in \epsilon_{x_k}^{-1}(D_k)$ means that $\phi(x_k) \in D_k$. Since D_k is open in $\mathbb{K}$, there exists some $\varepsilon_k > 0$, such that

$$D_k \supset \mathcal{B}_{\varepsilon_k}\big(\phi(x_k)\big).$$

Then if we have an arbitrary $\psi \in W(\phi; \varepsilon_k, x_k)$, we will have

$$|\psi(x_k) - \phi(x_k)| < \varepsilon_k,$$

which gives $\psi \in \epsilon_{x_k}^{-1}(D_k)$. This proves that

$$W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

Notice that, if one takes $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, then we clearly have the inclusions

$$W(\phi; \varepsilon, x_k) \subset W(\phi; \varepsilon_k, x_k) \subset \epsilon_{x_k}^{-1}(D_k).$$

We then immediately get

$$W(\phi; \varepsilon, x_k) \subset \bigcap_{k=1}^{n} \epsilon_{x_k}^{-1}(D_k) \subset E \subset N,$$

and we are done. □

COROLLARY 4.1. *Let $\mathcal{X}$ be a normed vector space. Then the w^* topology on $\mathcal{X}^*$ is locally convex, i.e.*

- *for every $\phi \in \mathcal{X}^*$ and every w^*-neighborhood N of ϕ, there exists a convex w^*-open set D such that $\phi \in D \subset N$.*

PROOF. Apply the second part of the proposition, together with the obvious fact that each of the sets $W(\phi; \varepsilon, x)$ is convex and w^*-open. □

PROPOSITION 4.2. *Let $\mathcal{X}$ be a normed vector space. When equipped with the w^* topology, the space $\mathcal{X}^*$ is a topological vector space. This means that the maps*

$$\mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \phi + \psi \in \mathcal{X}^*$$
$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \lambda\phi \in \mathcal{X}^*$$

are continuous with respect to the w^* topology on the target space, and the w^* product topology on the domanin.

PROOF. According to the definition of the w^* topology, it suffices to prove that, for every $x \in \mathcal{X}$, the maps

$$\sigma_x : \mathcal{X}^* \times \mathcal{X}^* \ni (\phi, \psi) \longmapsto \gamma_x : \epsilon_x(\phi + \psi) \in \mathbb{K}$$
$$\mathbb{K} \times \mathcal{X}^* \ni (\lambda, \phi) \longmapsto \epsilon_x(\lambda\phi) \in \mathbb{K}$$

are continuous. But the continuity of σ_x and γ_x is obvious, since we have

$$\sigma_x(\phi, \psi) = \phi(x) + \phi(x) = \epsilon_x(\phi) + \epsilon_x(\psi), \ \ \forall\, (\phi, \psi) \in \mathcal{X}^* \times \mathcal{X}^*;$$
$$\gamma_x(\lambda, \phi) = \lambda\phi(x) = \lambda\epsilon_x(\phi), \ \ \forall\, (\lambda, \phi, \psi) \in \mathbb{K} \times \mathcal{X}^*.$$

□

Our next goal will be to describe the linear maps $\mathcal{X}^* \to \mathbb{K}$, which are continuous in the w^* topology.

Proposition 4.3. *Let $\mathcal{X}$ be a normed vector space over $\mathbb{K}$. For a linear map $\omega : \mathcal{X}^* \to \mathbb{K}$, the following are equivalent:*

(i) *ω is continuous with respect to the w^* topology;*
(ii) *there exists some $x \in \mathcal{X}$, such that*

$$\omega(\phi) = \phi(x), \ \forall \phi \in \mathcal{X}^*.$$

Proof. The implication $(ii) \Rightarrow (i)$ is trivial, since condition (ii) gives $\omega = \epsilon_x$

$(i) \Rightarrow (ii)$. Suppose ω is continuous. In particular, ω is continuous at 0, so if we take the set

$$D = \{\lambda \in \mathbb{K} : |\lambda| < 1\},$$

the set

$$\omega^{-1}(D) = \{\phi \in \mathcal{X}^* : |\omega(\phi)| < 1\}$$

is an open neighborhood of 0 in the w^* topology. By Proposition ?? there exist $x_1, \dots, x_n \in \mathcal{X}$, and $\varepsilon > 0$, such that

$$W(0; \varepsilon, x_1) \cap \dots \cap W(0; \varepsilon, x_n) \subset D. \tag{1}$$

Claim 1: *One has the inequality*

$$|\omega(\phi)| \le \varepsilon^{-1} \cdot \max\big\{|\phi(x_1)|, \dots, |\phi(x_n)|\big\}, \ \forall \phi \in \mathcal{X}^*.$$

Fix an arbitrary $\phi \in \mathcal{X}^*$, and put $M = \max\big\{|\phi(x_1)|, \dots, |\phi(x_n)|\big\}$. For every integer $k \ge 1$, define

$$\phi_k = \varepsilon\big(M + \tfrac{1}{k}\big)^{-1}\phi,$$

so that

$$|\phi_k(x_j)| = \varepsilon\big(M + \tfrac{1}{k}\big)^{-1}|\phi(x_j)| \le \varepsilon M\big(M + \tfrac{1}{k}\big)^{-1} < \varepsilon, \ \forall k \ge 1, \ j \in \{1, \dots, n\}.$$

This proves that $\phi_k \in W(0; \varepsilon, x_j)$, for all $k \ge 1$, and all $j \in \{1, \dots, n\}$. By (1) this will give

$$|\omega(\phi_k)| < 1, \ \forall k \ge 1,$$

which reads

$$\varepsilon\big(M + \tfrac{1}{k}\big)^{-1}|\omega(\phi)| < 1, \ \forall k \ge 1.$$

This gives

$$|\omega(\phi)| \le \varepsilon^{-1}\big(M + \tfrac{1}{k}\big), \ \forall k \ge 1,$$

and it will obviously force

$$|\omega(\phi)| \le \varepsilon^{-1}M.$$

Having proven the Claim, we now define the linear map $T : \mathcal{X}^* \to \mathbb{K}^n$, by

$$T\phi = \big(\phi(x_1), \dots, \phi(x_n)\big), \ \forall \phi \in \mathcal{X}^*.$$

Claim 2: *There exists a linear map $\sigma : \mathbb{K}^n \to \mathbb{K}$, such that $\omega = \sigma \circ T$.*

First we show that we have the inclusion

$$\operatorname{Ker}\omega \supset \operatorname{Ker} T.$$

If we start with $\phi \in \operatorname{Ker} T$, then $\phi(x_1) = \dots = \phi(x_n) = 0$, and then by Claim 1 we immediately get $\omega(\phi) = 0$, so ϕ indeed belongs to $\operatorname{Ker}\omega$. We us now a bit of linear algebra. On the one hand, since $\omega\big|_{\operatorname{Ker} T} = 0$, there exists a linear map $\hat{\omega} : \mathcal{X}/\operatorname{Ker} T \to \mathbb{K}$, such that $\omega = \hat{\omega} \circ \pi$, where $\pi : \mathcal{X} \to \mathcal{X}/\operatorname{Ker} T$ denotes the quoatient map. On the other hand, by the Isomorphism Theorem for linear maps, there exists a linear isomorphism $\hat{T} : \mathcal{X}/\mathrm{Ker}\, T \xrightarrow{\sim} \mathrm{Ran}\, T$, such that $\hat{T} \circ \pi = T$. We then define

$$\sigma_0 = \hat{\omega} \circ \hat{T}^{-1} : \mathrm{Ran}\, T \to \mathbb{K},$$

and we will have

$$\sigma_0 \circ T = (\hat{\omega} \circ \hat{T}^{-1}) \circ (\hat{T} \circ \pi) = \hat{\omega} \circ \pi = \omega.$$

We finally extend[1] $\sigma_0 : \mathrm{Ran}\, T \to \mathbb{K}$ to a linear map $\sigma : \mathbb{K}^n \to \mathbb{K}$.

Having proven Claim 2, we choose scalars $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, such that

$$\sigma(\lambda_1, \dots, \lambda_n) = \alpha_1\lambda_1 + \dots + \alpha_n\lambda_n, \ \ \forall\, (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n.$$

We now have

$$\omega(\phi) = \sigma(T\phi) = \sigma\big(\phi(x_1), \dots, \phi(x_n)\big) = \alpha_1\phi(x_1) + \dots + \alpha_n\phi(x_n), \ \ \forall\, \phi \in \mathcal{X}^*,$$

so if we define $x = \alpha_1 x_1 + \dots + \alpha_n x_n$, we claerly have

$$\omega(\phi) = \phi(x), \ \ \forall\, \phi \in \mathcal{X}^*.$$

$(ii) \Rightarrow (i)$. This implication is trivial. □

Corollary 4.2. *Let $\mathcal{X}$ be a normed vector space, let $\mathcal{C} \subset \mathcal{X}^*$ be a convex set, and let $\phi \in \mathcal{X}^* \smallsetminus \overline{\mathcal{C}}^{w^*}$. (Here $\overline{\mathcal{C}}^{w^*}$ denotes the w^*-closure of $\mathcal{C}$.) Then there exists an element $x \in \mathcal{X}$, and a real number α, such that*

$$\mathrm{Re}\, \phi(x) < \alpha \leq \mathrm{Re}\, \psi(x), \ \ \forall\, \psi \in \mathcal{C}.$$

Proof. Since the w^* topology on $\mathcal{X}^*$ is locally convex, there exists a convex w^*-open set $\mathcal{A} \subset \mathcal{X}^*$, such that $\phi \in \mathcal{A} \subset \mathcal{X}^* \smallsetminus \overline{\mathcal{C}}^{w^*}$. In particular, we have $\mathcal{A} \cap \mathcal{C} = \varnothing$. Apply the Hahn-Banach separation theorem to find a linear map $\omega : \mathcal{X}^* \to \mathbb{K}$, which is w^*-continuous, and a real number α, such that

$$\mathrm{Re}\, \omega(\rho) < \alpha \leq \mathrm{Re}\, \omega(\psi), \ \ \forall\, \rho \in \mathcal{A},\, \psi \in \mathcal{C}.$$

We then apply the above Proposition. □

Comment. The definition of the w^* topology can be used in a more general setting, when $\mathcal{X}$ is just a topological vector space. The above results are still vaild in this general setting.

The definition of the weak dual topology can be naturally realized using product topologies. To clarify this statement, it will be helpful to introduce the following.

Notations. Suppose $\mathcal{Z}$ is a normed $\mathbb{K}$-vector space, where $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$. We denote by $(\mathcal{Z})_1$ the closed unit ball in $\mathcal{Z}$, that is

$$(\mathcal{Z})_1 = \big\{ z \in \mathcal{Z} \, : \, \|z\| \leq 1 \big\}.$$

We also define the product space

$$\mathbf{P}_{\mathcal{Z}} = \prod_{z \in (\mathcal{Z})_1} \mathbb{K}.$$

With these notations, we have the following.

[1] One can invoke the Hahn-Banach Theorem here. In fact this is not necessary, since $\mathrm{Ran}\, T \subset \mathbb{K}^n$ are finite dimensional vector spaces.

Proposition 4.4. *Let $\mathcal{X}$ be a normed $\mathbb{K}$ vector space. Consider the map $\Theta : \mathcal{X}^* \to \mathbf{P}_\mathcal{X}$, defined by*

$$\Theta(\phi) = \big(\phi(x)\big)_{x\in(\mathcal{X})_1},\ \ \phi \in \mathcal{X}^*.$$

Equip the space $\mathbf{P}_\mathcal{X}$ with the product topology. When we equip the range $\Theta(\mathcal{X}^*) \subset \mathbf{P}_\mathcal{X}$ with the induced topology, the map $\Theta : \mathcal{X}^* \to \Theta(\mathcal{X}^*)$ is a homeomorphism.

Proof. What we need to prove here is the fact that, for a net $(\phi_\lambda)_{\lambda\in\Lambda} \subset \mathcal{X}^*$, and an element $\phi \in \mathcal{X}^*$, the conditions

(*a*) $\phi = w^*\text{-}\lim_{\lambda\in\Lambda} \phi_\lambda$,
(*b*) $\Theta(\phi) = \lim_{\lambda\in\Lambda} \Theta(\phi_\lambda)$

are equivalent. (In condition (*b*) the convegence is meant in the product topology.) By definition, condition (*a*) is equivalent to

$$\lim_{\lambda\in\Lambda} \phi_\lambda(x) = \phi(x),\ \ \forall\, x \in \mathcal{X}, \tag{2}$$

while condition (*b*) is equivalent to

$$\lim_{\lambda\in\Lambda} \phi_\lambda(x) = \phi(x),\ \ \forall\, x \in (\mathcal{X})_1. \tag{3}$$

It is now obvious that (*a*) ⇒ (*b*). Conversely, if (3) holds, then using linearity it follows that

$$\lim_{\lambda\in\Lambda} \phi_\lambda(\alpha y) = \lim_{\lambda\in\Lambda} \alpha\phi_\lambda(y) = \alpha\phi(y) = \phi(\alpha y),\ \ \forall\, y \in (\mathcal{X})_1,\ \alpha \in \mathbb{K}.$$

Since every $x \in \mathcal{X}$ can be written as $x = \alpha y$, with $\alpha \in \mathbb{K}$ and $y \in (\mathcal{X})_1$, condition (2) holds. □

Theorem 4.1 (Alaoglu). *If $\mathcal{X}$ is a normed vector space, then the unit ball $(\mathcal{X}^*)_1$, in the topological dual space, is compact in the w^* topology.*

Proof. The key ingredient in the proof will be the use of Proposition 4.4. Use the notations above. We know already that $\Theta : \mathcal{X}^* \to \Theta(\mathcal{X}^*)$ is a homeomorphism. Consider the set $K = \Theta\big((\mathcal{X}^*)_1\big)$, equipped with the induced topology from $\mathbf{P}_\mathcal{X}$, so that the map

$$\Theta : (\mathcal{X}^*)_1 \to K$$

is a homeomorphism. So in order to prove the theorem, it suffices to prove that K is compact. We prove this indirectly, first by constructing some compact subset $\mathbf{B}_\mathcal{X} \subset \mathbf{P}_\mathcal{X}$, with $K \subset \mathbf{B}_\mathcal{X}$, and then by showing that K is closed in $\mathbf{P}_\mathcal{X}$. To construct the set $\mathbf{B}_\mathcal{X}$, we consider the unit ball in $\mathbb{K}$:

$$\mathbb{B} = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\},$$

and we put

$$\mathbf{B}_\mathcal{X} = \prod_{x\in(\mathcal{X})_1} \mathbb{B}.$$

As a subset of $\mathbf{P}_\mathcal{X}$, the above set is

$$\mathbf{B}_\mathcal{X} = \big\{\boldsymbol{\alpha} = (\alpha_x)_{x\in(\mathcal{X})_1} \in \prod_{x\in(\mathcal{X})_1} \mathbb{K} : |\alpha_x| \leq 1,\ \ \forall\, x \in (\mathcal{X})_1\big\}.$$

Using *Tihonov's Theorem* (see I.4), it follows that $\mathbf{B}_\mathcal{X}$ is compact in $\mathbf{P}_\mathcal{X}$.

Let us observe now that we have the inclusion $K \subset \mathbf{B}_\mathcal{X}$. Indeed, if we start with some $\phi \in (\mathcal{X}^*)_1$, then we have

$$|\phi(x)| \leq \|\phi\| \cdot \|x\| \leq 1,\ \ \forall\, x \in (\mathcal{X})_1,$$

which clearly gives the fact that the system $\Theta(\phi) = \big(\phi(x)\big)_{x\in(\mathcal{X})_1}$ belongs to $\mathbf{B}_{\mathcal{X}}$.

Finally, let us prove that K is closed in $\mathbf{P}_{\mathcal{X}}$. This amounts to showing that, whenever one is given an element $\boldsymbol{\alpha} = (\alpha_x)_{x\in(\mathcal{X})_1} \in \mathbf{P}_{\mathcal{X}}$, and a net $(\phi_\lambda)_{\lambda\in\Lambda} \subset (\mathcal{X}^*)_1$, such that

$$\boldsymbol{\alpha} = \lim_{\lambda\in\Lambda} \Theta(\phi_\lambda) \text{ (in } \mathbf{P}_{\mathcal{X}}), \tag{4}$$

it follows that there exists some $\phi \in (\mathcal{X}^*)_1$, with $\boldsymbol{\alpha} = \Theta(\phi)$.

Fix $\boldsymbol{\alpha}$ and the net $(\phi_\lambda)_{\lambda\in\Lambda}$ as above. The condition (4) reads

$$\lim_{\lambda\in\Lambda} \phi_\lambda(x) = \alpha_x, \ \ \forall\, x \in (\mathcal{X})_1. \tag{5}$$

In order to define the map ϕ, we are going to use the following notation. For every $x \in \mathcal{X}$ we define

$$\check{x} = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{\|x\|}x & \text{if } x \neq 0 \end{cases}$$

It is clear that $\check{x} \in (\mathcal{X})_1$ and one also has the equality

$$x = \|x\|\check{x}, \ \ \forall\, x \in \mathcal{X}.$$

Let us define the map $\phi : \mathcal{X} \to \mathbb{K}$ by the formula

$$\phi(x) = \|x\| \cdot \alpha_{\check{x}}, \ \ \forall\, x \in \mathcal{X}.$$

The goal it to prove that ϕ is linear, continuous, with norm $\|\phi\| \leq 1$, and to show the equality $\boldsymbol{\alpha} = \Theta(\phi)$. To prove linearity, we start off by observing that, since $\check{0} = 0$, using (5), we immediately get

$$\phi(0) = \alpha_0 = \lim_{\lambda\in\Lambda} \phi_\lambda(0) = 0. \tag{6}$$

Let us prove the homogeneity of ϕ, i.e. the property

$$\phi(\zeta x) = \zeta\phi(x), \ \ \forall\, \zeta \in \mathbb{K},\, x \in \mathcal{X}. \tag{7}$$

The case when $\zeta = 0$, or $x = 0$ is already included in the equality (6), so we are going to assume that $\zeta \neq 0$ and $x \neq 0$. Consider the vector $y = \zeta x$. It is clear that $\check{y} = \omega\check{x}$, where $\omega = \zeta/|\zeta|$, so using (5) we have

$$\phi(\zeta x) = \|y\|\alpha_{\check{y}} = \|y\| \cdot \lim_{\lambda\in\Lambda} \phi(\check{y}) = \lim_{\lambda\in\Lambda} \big[\|y\| \cdot \phi_\lambda(\check{y})\big]. \tag{8}$$

Since each ϕ_λ, $\lambda \in \Lambda$ is linear, we have

$$\|y\| \cdot \phi_\lambda(\check{y}) = \|x\| \cdot |\zeta| \cdot \phi_\lambda(\omega\check{x}) = \|x\| \cdot |\zeta| \cdot \omega\phi_\lambda(\check{x}) = \zeta \cdot \|x\| \cdot \phi_\lambda(\check{x}), \ \ \forall\, \lambda \in \Lambda,$$

so, using again (5), the computation (8) continues as

$$\phi(\zeta x) = \lim_{\lambda\in\Lambda} \big[\zeta \cdot \|x\| \cdot \phi_\lambda(\check{x})\big] = \zeta \cdot \|x\| \cdot \lim_{\lambda\in\Lambda} \phi_\lambda(\check{x}) = \zeta \cdot \|x\| \cdot \alpha_{\check{x}} = \zeta\phi(x).$$

Having prove property (7), let us remark that among other things, one gets

$$\phi(x) = \alpha_x, \ \ \forall\, x \in (\mathcal{X})_1. \tag{9}$$

Indeed, if one starts with some arbitrary $x \in (\mathcal{X})_1$, then using (5), combined with the linearity of the ϕ_λ's, we get

$$\alpha_x = \lim_{\lambda\in\Lambda} \phi_\lambda(x) = \lim_{\lambda\in\Lambda} \phi_\lambda(\|x\|\check{x}) = \|x\| \cdot \lim_{\lambda\in\Lambda} \phi_\lambda(\check{x}) = \|x\| \cdot \alpha_{\check{x}} = \phi(x).$$

We now prove the fact that ϕ is additive, i.e. it has the property

$$\phi(x + y) = \phi(x) + \phi(y), \ \ \forall\, x, y \in \mathcal{X}. \tag{10}$$

In the case when $x = 0$ or $y = 0$, there is nothing to prove by (6), so we can assume $x, y \neq 0$. Put $t = \|x\| + \|y\|$ and remark that the vectors $u = t^{-1}x$, $v = t^{-1}y$, and $z = t^{-1}(x+y)$ all belong to $(\mathcal{X})_1$, so using the homogeneity property (7), combined with (9) and (5), we get

$$\phi(x+y) = \phi(tz) = t\phi(z) = t\alpha_z = t \cdot \lim_{\lambda \in \Lambda} \phi_\lambda(z).$$

Using the linearity of the ϕ_λ's, combined again with (7), (9) and (5), the above computation can be continued:

$$\begin{aligned}\phi(x+y) &= t \cdot \lim_{\lambda \in \Lambda} \phi_\lambda(z) = t \cdot \lim_{\lambda \in \Lambda} \phi_\lambda(u+v) = t \cdot \lim_{\lambda \in \Lambda} \big[\phi_\lambda(u) + \phi_\lambda(v)\big] = \\ &= t \cdot \big[\lim_{\lambda \in \Lambda} \phi_\lambda(u) + \lim_{\lambda \in \Lambda} \phi_\lambda(v)\big] = t \cdot [\alpha_u + \alpha_v] = \\ &= t \cdot [\phi(u) + \phi(v)] = \phi(tu) + \phi(tv) = \phi(x) + \phi(y).\end{aligned}$$

Having shown that ϕ is linear, let us prove now that it is continuous, and it belongs to $(\mathcal{X}^*)_1$. Using the fact that $\phi_\lambda \in (\mathcal{X}^*)_1$, $\forall\, \lambda \in \Lambda$, we have

$$|\phi_\lambda(x)| \leq 1, \ \ \forall\, x \in (\mathcal{X})_1,$$

so by (5) we have $|\alpha_x| \leq 1$, $\forall\, (\mathcal{X})_1$. Of course, the equalities (9) prove that

$$\sup\big\{|\phi(x)| \,:\, x \in (\mathcal{X})_1\big\} \leq 1,$$

and we are done.

Finally, the equality (9) gives $\alpha = \Theta(\phi)$. □

Proposition 4.5. *Suppose $\mathcal{X}$ is a normed vector space, which is separable in the norm topology. When equipped with the w^* topology, the compact space $(\mathcal{X}^*)_1$ is metrizable.*

Proof. Fix a countable dense subset $\mathcal{M} \subset \mathcal{X}$, and define $(\mathcal{M})_1 = (\mathcal{X})_1 \cap \mathcal{M}$. Notice that $(\mathcal{M})_1$ *is dense in* $(\mathcal{X})_1$. Indeed, if we start with some $x \in (\mathcal{X})_1$, and some $\varepsilon > 0$, then we set $x_\varepsilon = (1 - \frac{\varepsilon}{2})x$, and we choose $y \in \mathcal{M}$ such that $\|x_\varepsilon - y\| < \frac{\varepsilon}{2}$. On the one hand, we have

$$\|y\| \leq \|x_\varepsilon - y\| + \|x_\varepsilon\| < \frac{\varepsilon}{2} + \big(1 - \frac{\varepsilon}{2}\big) \cdot \|x\| \leq \frac{\varepsilon}{2} + 1 - \frac{\varepsilon}{2} = 1,$$

so $y \in (\mathcal{M})_1$. On the other hand, we have

$$\|y - x\| \leq \|y - x_\varepsilon\| + \|x - x_\varepsilon\| < \frac{\varepsilon}{2} + \big\|\frac{\varepsilon}{2}x\big\| \leq \frac{\varepsilon}{2} \cdot \big(1 + \|x\|\big) \leq \varepsilon.$$

Let us use the notations from the proof of Theorem 4.1. Let us then define the product space

$$\mathbf{P}_{\mathcal{M}} = \prod_{x \in (\mathcal{M})_1} \mathbb{K},$$

equipped with the product topology. Since $\mathcal{M}$, the product space $\mathbf{P}_{\mathcal{M}}$ is metrizable. Define also the map

$$\Upsilon : \mathbf{P}_{\mathcal{X}} \ni f \longmapsto f\big|_{(\mathcal{M})_1} \in \mathbf{P}_{\mathcal{M}}.$$

It is obvious that Υ is continuous, so the composition

$$\Upsilon \circ \Theta : (\mathcal{X}^*)_1 \to \mathbf{P}_{\mathcal{M}}$$

is continuous (of course, $(\mathcal{X}^*)_1$ is equipped with the w^*-topology). Since $(\mathcal{X}^*)_1$ is compact, the set $L = (\Upsilon \circ \Theta)\big((\mathcal{X}^*)_1\big)$ is compact in $\mathbf{P}_{\mathcal{M}}$, hence metrizable, when equipped with the induced topology. The desired property will then follow from the fact that $\Upsilon \circ \Theta : (\mathcal{X}^*)_1 \to L$ is a *homeomorphism*. Since we work with compact spaces, all we need to prove here is the fact that $\Upsilon \circ \Theta$ is *injective*. Indeed, if $\phi, \psi \in (\mathcal{X}^*)_1$ satisfy $(\Upsilon \circ \Theta)(\phi) = (\Upsilon \circ \Theta)(\psi)$, then we get $\phi\big|_{(\mathcal{M})_1} = \psi\big|_{(\mathcal{M})_1}$. Since $(\mathcal{M})_1$ is dense in $(\mathcal{X})_1$, this will force $\phi\big|_{(\mathcal{X})_1} = \psi\big|_{(\mathcal{X})_1}$, which finally forces $\phi = \psi$. $\square$

REMARK 4.2. Assuming $\mathcal{X}$ is separable, and $\mathcal{M} \subset \mathcal{X}$ is a countable dense subset. If we enumerate the countable set $(\mathcal{M})_1$ as

$$(\mathcal{M})_1 = \{y_n \,:\, n \geq 1\},$$

then a metric d that defines the w^* topology on $(\mathcal{X}^*)_1$ can be constructed as

$$d(\phi, \psi) = \sum_{n=1}^{\infty} \frac{|\phi(y_n) - \psi(y_n)|}{2^n}, \;\; \forall\, \phi, \psi \in (\mathcal{X}^*)_1.$$

COMMENTS. Using the notations above, the composition $\Upsilon \circ \Theta : \mathcal{X}^* \to \mathbf{P}_{\mathcal{M}}$ is continuous, injective (the proof is exactly the same as above), but in general the composition

$$\Upsilon \circ \Theta : \mathcal{X}^* \to (\Upsilon \circ \Theta)(\mathcal{X}^*)$$

may fail to be a homeomorphism. The exercise below explains exactly when this is the case.

*Exercise 1**. Let $\mathcal{X}$ be a normed vector space, which is of *uncountable* dimension (for example, a Banach space). Prove that the topological space $(\mathcal{X}^*, w^*)$ is not metrizable.

HINT: Assume $(\mathcal{X}^*, w^*)$ is metrizable. Let d be a metric which gives the w^*-topology. Then $0 \in \mathcal{X}^*$ will have a countable basic system of neighborhoods. In particular, there exist sequences $(x_n)_{n\geq 1} \subset \mathcal{X}$, and $(\varepsilon_n)_{n\geq 1} \in (0, \infty)$, such that the sets

$$B_n = \bigcap_{k=1}^{n} W(0; \varepsilon_n, x_k)$$

satisfy $B_n \subset \mathcal{B}_{1/n}(0)$, $\forall\, n \geq 1$, where $\mathcal{B}_{1/n}(0)$ denotes the d-open ball of center 0 and radius $1/n$. Consider the set $\mathcal{M} = \{x_n \,:\, n \in \mathbb{N}\}$. We know that $\operatorname{Span} \mathcal{M} \subsetneq \mathcal{X}$. Choose some vector $y \in \mathcal{X} \smallsetminus \operatorname{Span} \mathcal{M}$. For every $n \geq 1$, choose a linear map $\psi_n : \operatorname{Span}\{y, x_1, \dots, x_n\} \to \mathbb{K}$, such that $\psi_n(y) = 1$, and $\psi_n(x_k) = 0$, $\forall\, k \in \{1, \dots, n\}$. Extend (use Hahn-Banach) ψ_n to a linear continuous map $\phi_n : \mathcal{X} \to \mathbb{K}$. Notice now that $\phi_n \in B_n$, for all $n \geq 1$, which would then force $d\text{-}\lim_{n\to\infty} \phi_n = 0$. In particular, this would force $\lim_{n\to\infty} \phi_n(x) = 0$, $\forall\, x \in \mathcal{X}$. But this is impossible, since $\phi_n(y) = 1$, $\forall\, n \geq 1$.

COMMENT. If $\mathcal{X}$ is a normed vector space of *countable* dimension, then $(\mathcal{X}^*, w^*)$ is metrizable. Indeed, if we take a linear basis $\{b_n \,:\, n \in \mathbb{N}\}$ for $\mathcal{X}$, then the w^* topology on $\mathcal{X}^*$ is clearly defined by the metric

$$d(\phi, \psi) = \sum_{n=1}^{n} \frac{1}{2^n} \cdot \frac{|\phi(b_n) - \psi(b_n)|}{1 + |\phi(b_n) - \psi(b_n)|}, \;\; \phi, \psi \in \mathcal{X}^*.$$