2. The Concept of Convergence: Ultrafilters and Nets

NOTE: AS OF 2008, SOME OF THIS STUFF IS A BIT OUTDATED AND HAS A FEW TYPOS. I WILL REVISE THIS MATERIAL SOMETIME.

In this lecture we discuss two points of view on the notion of convergence. The first one employs a set theoretical concept, which turns out to be technically useful.

DEFINITION. Suppose X is a fixed (non-empty) set. A *filter in* X is a (non-empty) family $\mathcal{F}$ of non-empty subsets of X which has the property[2]:

(F) *Whenever F and G belong to $\mathcal{F}$, it follows that there exists $H \in \mathcal{F}$ with $F \cap G \supset H$.*

What is important here is that all the sets in the filter are assumed to be *non-empty.* The set of all filters in X can be ordered by inclusion. A simple application of Zorn's Lemma yields:

- *For each filter $\mathcal{F}$ there exists at least one* ***maximal*** *filter $\mathcal{U}$ with $\mathcal{U} \supset \mathcal{F}$.*

Maximal filters will be called *ultrafilters.*

An interesting feature of ultrafilters is given by the following:

LEMMA 2.1. *Let X be a non-empty set, and let $\mathcal{U}$ be a filter on X. The following are equivalent:*

(i) *$\mathcal{U}$ is an ultrafilter.*
(ii) *For any subsets $A \subset X$, it follows that either A or $X \smallsetminus A$ belongs to $\mathcal{U}$, but not both!*

PROOF. $(i) \Rightarrow (ii)$. Assume $\mathcal{U}$ is an ultrafilter. First remark that X always belongs to $\mathcal{U}$. (Otherwise, if X does not belong to $\mathcal{U}$, the family $\mathcal{U} \cup \{X\}$ will be obviously a new filter which will contradict the maximality of $\mathcal{U}$).

Let us assume that A is non-empty and it does not belong to $\mathcal{U}$. This means that the family

$$\mathcal{M} = \mathcal{U} \cup \{A \cap U \,|\, U \in \mathcal{U}\}$$

is no longer a filter (otherwise, the maximality of $\mathcal{U}$ will be contradicted). Note that if F and G belong to $\mathcal{M}$, then automatically $F \cap G$ belongs to $\mathcal{M}$. This means that the only thing that can prevent $\mathcal{M}$ from being a filter, must be the fact that one of the sets in $\mathcal{M}$ is empty. That is, there is some set $V \in \mathcal{U}$ such that $A \cap V = \varnothing$. In other words, $V \subset X \smallsetminus A$. But then, it follows that for any $U \in \mathcal{U}$ we have $U \cap (X \smallsetminus A) \supset U \cap V \neq \varnothing$ and then the set

$$\mathcal{N} = \mathcal{U} \cup \{U \cap (X \smallsetminus A) \,|\, U \in \mathcal{U}\}$$

will be a filter. By maximality, it follows that $\mathcal{N} = \mathcal{U}$, in particular, $X \smallsetminus A$ belongs to $\mathcal{U}$. It is obvious that A and $X \smallsetminus A$ cannot simultaneously belong to $\mathcal{U}$, because this will force $\varnothing = A \cap (X \smallsetminus A)$ to belong to $\mathcal{U}$.

$(ii) \Rightarrow (i)$. Assume property (ii) holds, but $\mathcal{U}$ is not maximal, which means that there exists some ultrafilter $\mathcal{V}$ with $\mathcal{V} \supsetneq \mathcal{U}$. Pick then some set $A \in \mathcal{V} \smallsetminus \mathcal{U}$.

[2] Some textbooks may use a slightly different definition.

Since $A \notin \mathcal{U}$, by (ii) we must have $X \smallsetminus A \in \mathcal{U}$. This would force both A and $X \smallsetminus A$ to belong to $\mathcal{V}$, which is impossible. □

Exercise 1. Let $\mathcal{U}$ be an ultrafilter on X, and let $A \in \mathcal{U}$. Prove that the collection

$$\mathcal{U}\big|_A = \{U \cap A : U \in \mathcal{U}\}$$

is an ultrafilter on A.

Remark 2.1. If $\mathcal{U}$ is an ultrafilter on X, and $A \in \mathcal{U}$, then $\mathcal{U}$ contains all sets B with $A \subset B \subset X$. Indeed, if we start with such a B, then by the above result, either $B \in \mathcal{U}$ or $X \smallsetminus B \in \mathcal{U}$. Notice however that in the case $X \smallsetminus B \in \mathcal{U}$ we would get

$$\mathcal{U} \ni (X \smallsetminus B) \cap A = \varnothing,$$

which is impossible. Therefore B must belong to $\mathcal{U}$.

Remark 2.2. Maps between sets can be put to act on ultrafilters. More explicitly one has the following construction. Suppose $f : X \to Y$ is a map and $\mathcal{U}$ is a ultrafilter in X. Consider the collection

$$f_*(\mathcal{U}) = \{V \subset Y \mid f^{-1}(V) \in \mathcal{U}\}.$$

Then $f_*(\mathcal{U})$ is a ultrafilter on Y. Indeed, it is easy to show that $f_*(\mathcal{U})$ is a filter. To prove that it is in fact an ultrafilter, we use Lemma 2.1. Start with some arbitrary set $A \subset Y$, and let us show that exactly one of the sets A or $Y \smallsetminus A$ belongs to $f_*(\mathcal{U})$. This is pretty obvious however from the equality $f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)$, combined with Lemma 2.1 applied to $\mathcal{U}$, which gives the fact that exactly one of the sets $f^{-1}(A)$ or $f^{-1}(Y \smallsetminus A) = X \smallsetminus f^{-1}(A)$ belongs to $\mathcal{U}$.

Remark 2.3. With the above notations, one has

$$f(U) \in f_*(\mathcal{U}), \ \ \forall\, U \in \mathcal{U}.$$

One can prove this property by contradiction. Assume $f(U)$ does not belong to $f_*(\mathcal{U})$, for some $U \in \mathcal{U}$. Then $Y \smallsetminus f(U)$ belongs to $f_*(\mathcal{U})$, which means that the set

$$M = f^{-1}\big(Y \smallsetminus f(U)\big) = X \smallsetminus f^{-1}\big(f(U)\big)$$

belongs to $\mathcal{U}$. But using the obvious inclusion $U \subset f^{-1}\big(f(U)\big)$, this gives $M \cap U = \varnothing$, which is impossible.

We are in position now to define the notion of convergence for ultrafilters, by means of the following.

Proposition 2.1. *Let $(X, \mathcal{T})$ be a topological space, let $\mathcal{U}$ be an ultrafilter in X, and let x be a point in X. The following are equivalent:*

- (i) *Every neighborhood of x belongs to $\mathcal{U}$.*
- (ii) *There exists $\mathcal{N}$ a basic system of neighborhoods of x, with $\mathcal{N} \subset \mathcal{U}$.*
- (iii) *There exists $\mathcal{V}$ a fundamental system of neighborhoods of x, with $\mathcal{V} \subset \mathcal{U}$.*

If the ultrafilter $\mathcal{U}$ satisfies one of the equivalent conditions above, we say that $\mathcal{U}$ is *convergent to x*, and we write $\mathcal{U} \to x$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

$(iii) \Rightarrow (i)$. Let $\mathcal{V}$ be a fundamental system of neighborhoods of x, with $\mathcal{V} \subset \mathcal{U}$. Start with an arbitrary neighborhood M of x. By the properties of $\mathcal{V}$, there exists a finite sequence $V_1, \dots, V_n \in \mathcal{V}$, with

$$x \in V_1 \cap \dots \cap V_n \subset M.$$

Since $\mathcal{V} \subset \mathcal{U}$, and $\mathcal{U}$ is a filter, it follows that the intersection $W = V_1 \cap \dots \cap V_n$ belongs to $\mathcal{U}$. By Remark 2.1 it follows that M itself belong to $\mathcal{U}$. Since M was arbitrary, it follows that $\mathcal{U}$ indeed satisfies condition (i). □

Examples 2.1. A. Let x be a point in X. We can consider the collection $\mathcal{U}_x = \{U \subset X \mid U \ni x\}$. Clearly $\mathcal{U}_x$ is an ultrafilter in X. This is called a *constant* ultrafilter at x. If $(X, \mathcal{T})$ is a topological space, then it is obvious that $\mathcal{U}_x$ is convergent to x.

B. (Example of a convergent non-constant ultrafilter.) Suppose $(X, \mathcal{T})$ is a topological space and x is a point in X such that for any neighborhood N of x, we have $N \smallsetminus \{x\} \neq \varnothing$. Consider the collection

$$\mathcal{F} = \{N \smallsetminus \{x\} \mid N \text{ neighborhood of } x\}.$$

Then $\mathcal{F}$ is a filter. If we take $\mathcal{U}$ any ultrafilter which contains $\mathcal{F}$, we get a non-constant (sometimes called *free*) ultrafilter. It is again clear that $\mathcal{U}$ is again convergent to x.

C. (Example of a non-convergent ultrafilter.) Let $\mathbb{N}$ be the set of non-negative integers. Equip $\mathbb{N}$ with the *discrete topology* (in which *every* subset is open). Consider the collection $\mathcal{F}$ consisting of all subsets $F \subset \mathbb{N}$ which have *finite* complement $\mathbb{N} \smallsetminus F$. It is easy to check that $\mathcal{F}$ is a filter. Pick then $\mathcal{U}$ to be any ultrafilter with $\mathcal{U} \supset \mathcal{F}$. Since on $\mathbb{N}$ we use the discrete topology, it follows that the only convergent ultrafilters are the constant ones. Note however, that if $n \in \mathbb{N}$, then the set $\mathbb{N} \smallsetminus \{n\}$ belongs to $\mathcal{F}$, hence to $\mathcal{U}$. This means that the singleton set $\{n\}$ cannot belong to $\mathcal{U}$. Therefore $\mathcal{U}$ cannot be constant.

The ultrafilter point of view on convergence is extremely useful for streamlining some proofs. As we shall see later in this section, ultrafilters can also be used for characterizing several features such as closure, the Hausdorff property, and continuity. The drawback is of course the fact that, apart from constant ultrafilters, we "cannot put our hand" on an ultrafilter. One can somehow circumvent this complication by the introduction of a new concept: *nets*.

Definitions. A. An *directed set* is an ordered set $(\Lambda, \succ)$ with the property:

- *for any $\lambda, \mu \in \Lambda$, there exists some $\nu \in \Lambda$ with $\nu \succ \lambda$ and $\nu \succ \mu$.*

B. Given two directed sets $(\Lambda, \succ)$ and $(\Delta, \succ)$, a map $\phi : \Lambda \to \Delta$ is said to be a *directed map*, if it has the property:

- *for every $\delta \in \Lambda$, there exists some $\lambda(\delta) \in \Lambda$, such that*

$$\phi(\lambda) \succ \delta, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_\delta.$$

C. Given a set X, we define a *net in* X to be any map $\Lambda \to X$, where Λ is some directed set. Such an object is typically denoted like $\boldsymbol{x} = (x_\lambda)_{\lambda \in \Lambda}$.

D. Given two nets $\boldsymbol{x} = (x_\lambda)_{\lambda \in \Lambda}$ and $\boldsymbol{y} = (y_\delta)_{\delta \in \Delta}$ in X, we say that $\boldsymbol{x}$ is a *sub-net of* $\boldsymbol{y}$, if there exists some directed map $\phi : \Lambda \to \Delta$, such that

$$x_\lambda = y_{\phi(\lambda)}, \ \ \forall \lambda \in \Lambda.$$

In this case we shall use the notation $(x_\lambda)_{\lambda\in\Lambda} \underset{\phi}{\subset} (y_\delta)_{\delta\in\Delta}$.

REMARK 2.4. Given directed sets Λ, Δ, and Σ, and directed maps $\Lambda \xrightarrow{\phi} \Delta \xrightarrow{\psi} \Sigma$, the composition $\psi \circ \phi : \Lambda \to \Sigma$ is again a directed map. As a consequence of this fact, we see that if $(x_\lambda)_{\lambda\in\Lambda}$ is a sub-net of $(y_\delta)_{\delta\in\Delta}$, and if $(y_\delta)_{\delta\in\Delta}$ is a sub-net of $(z_\sigma)_{\sigma\in\Sigma}$, then $(x_\lambda)_{\lambda\in\Lambda}$ is a sub-net of $(z_\sigma)_{\sigma\in\Sigma}$.

EXAMPLES 2.2. A. A *sequence* $(x_n)_{n=1}^\infty \subset X$ can be regarded as a net, where the indexing set is $\mathbb{N}$, equipped with the usual order. This is the reason why nets are sometimes termed "generalized sequences." It should be noted here that a subsequence of a sequence is a special kind of a sub-net, where the directed map is strictly increasing.

B. Any filter $\mathcal{F}$ on X is a directed set, with the order being the reverse inclusion: $F \succ G \Leftrightarrow F \subset G$. (With this notation, the filter axiom yields $F \cap G \succ F, G$.)

DEFINITION. Given a directed set Λ, and a filter $\mathcal{F}$, we say that a net $(x_\lambda)_{\lambda\in\Lambda}$ is a *section for* $\mathcal{F}$, if it satisfies the condition

- *for every $F \in \mathcal{F}$, there exists $\lambda_F \in \Lambda$, such that*

$$x_\lambda \in F, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_F.$$

EXAMPLES 2.3. A. Given a filter $\mathcal{F}$, if we think it as a directed set as above, then any element $(x_F)_{F\in\mathcal{F}} \in \prod_{F\in\mathcal{F}} F$ is a section for $\mathcal{F}$. Such a net is called a *selection for* $\mathcal{F}$

B. Conversely, given a net $(x_\lambda)_{\lambda\in\Lambda}$ in X, if we define

$$F_\lambda = \{x_\mu : \mu \in \Lambda,\ \mu \succ \lambda\},\ \ \forall\, \lambda \in \Lambda,$$

then the collection $\mathcal{F} = \big\{F_\lambda\big\}_{\lambda\in\Lambda}$ defines a filter in X, for which the net $(x_\lambda)_{\lambda\in\Lambda}$ constitutes a selection.

COMMENT. As we shall see shortly, nets can be used for defining a notion of convergence very similar to the usual one for ordinary sequences. The only drawback of this approach is the fact that *all nets in a set X do not constitute a set*. (This is due to the fact that there is no such thing as the "set of all sets," and to the fact that any set can be equipped with a direct ordering, for example a total ordering.) Instead, one should deal with the *class* of all nets.

Apart from this minor problem, the notion of convergence for nets is modeled after the corresponding one for ultrafilters, having in mind the examples 2.2.B-D above.

DEFINITION. Let $(X, \mathcal{T})$ be a topological space, and let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. Given a point $x \in X$, we say that the net $(x_\lambda)_{\lambda\in\Lambda}$ is *convergent to* x, if it is a section for the filter of all neighborhoods of x. This means that

- *for every neighborhood N of x, there exists some $\lambda_N \in \Lambda$, such that*

$$x_\lambda \in N, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_N.$$

In this case we use the notation $\lim_{\lambda\in\Lambda} x_\lambda = x$.

The following result is a direct analogue of Proposition 2.1.

PROPOSITION 2.2. *Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in the topological space $(X, \mathcal{T})$. For a point $x \in X$, the following are equivalent.*

(i) $\lim_{\lambda\in\Lambda} x_\lambda = x$;

(ii) *there exists a basic system of neighborhoods* $\mathcal{N}$ *for* x*, with the property that for every* $N \in \mathcal{N}$*, there exists some* $\lambda_N \in \Lambda$*,*

$$x_\lambda \in N, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_N;$$

(iii) *there exists a fundamental system of neighborhoods* $\mathcal{V}$ *for* x*, with the property that for every* $V \in \mathcal{V}$*, there exists some* $\lambda_V \in \Lambda$*,*

$$x_\lambda \in V, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_V.$$

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$. Let $\mathcal{V}$ be a fundamental system of neighborhoods of x, as in (iii). Start with an arbitrary neighborhood N of x. By the properties of $\mathcal{V}$, there exists a finite sequence $V_1, \dots, V_n \in \mathcal{V}$, with

$$x \in V_1 \cap \dots \cap V_n \subset N.$$

Using condition (iii), for each $k \in \{1, \dots, n\}$ there exists some $\lambda_k \in \Lambda$, such that

$$x_\lambda \in V_k, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_k. \tag{1}$$

Using the fact that Λ is directed, we can find some $\lambda' \in \Lambda$, such that $\lambda \succ \lambda_k$, $\forall\, k = 1, \dots, n$. Using (1) it follows that

$$x_\lambda \in V_1 \cap \dots \cap V_n \subset N, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda',$$

so by the definition of convergence (put $\lambda_N = \lambda'$) it follows that $(x_\lambda)_{\lambda \in \Lambda}$ indeed converges to x. □

Example 2.4. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, equipped with the standard topology. For a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{K}$ and a point $x \in \mathbb{K}$, the condition that $(x_\lambda)_{\lambda \in \Lambda}$ is convergent to x is equivalent to

- *for every* $\varepsilon > 0$*, there exists* $\lambda_\varepsilon \in \Lambda$*, such that* $|x_\lambda - x| < \varepsilon,\ \forall\, \lambda \succ \lambda_\varepsilon$*.*

This is immediate, for if one defines the sets $V_\varepsilon = \{y \in \mathbb{K} : |y - x| < \varepsilon\}$, $\varepsilon > 0$, then the collection $\mathcal{V} = \{V_\varepsilon\}_{\varepsilon > 0}$ is a basic system of neighborhoods of x.

The following result establishes the link between the two notions of convergence.

Lemma 2.2. *Let* $(X, \mathcal{T})$ *be a topological space, let* $\mathcal{U}$ *be an ultrafilter on* X*, and let* $(x_\lambda)_{\lambda \in \Lambda}$ *be a net which is a section for* $\mathcal{U}$*. For a point* $x \in X$*, the following are equivalent.*

(i) *the net* $(x_\lambda)_{\lambda \in \Lambda}$ *is convergent to* x*;*
(ii) *the ultrafilter* $\mathcal{U}$ *is convergent to* x*.*

Proof. $(i) \Rightarrow (ii)$. $(i) \Rightarrow (ii)$. We argue by contradiction. Assume $\lim_{\lambda \in \Lambda} x_\lambda = x$, but $\mathcal{U}$ is not convergent to x, i.e. there exists some neighborhood N of x, with $N \notin \mathcal{U}$. By Lemma 2.1 this forces $X \smallsetminus N \in \mathcal{U}$. Since $(x_\lambda)_{\lambda \in \Lambda}$ is a section for $\mathcal{U}$, there exists some $\mu \in \Lambda$, such that $x_\lambda \in X \smallsetminus N$, $\forall\, \lambda \succ \mu$, which clearly contradicts condition (i).

$(ii) \Rightarrow (i)$. This is pretty obvious, since every neighborhood N of x belongs to $\mathcal{U}$, and the condition that $(x_\lambda)_{\lambda \in \Lambda}$ is a section for $\mathcal{U}$ immediately gives the existence of some λ_N such that $x_\lambda \in N$, $\forall\, \lambda \succ \lambda_N$. □

We conclude this section with a sequence of results which characterize various features in terms of convergence.

PROPOSITION 2.3 (Characterization of closure). *Let* $(X, \mathcal{T})$ *be a topological space, let* $A \subset X$ *be some non-empty set. For a point* $x \in X$ *the following are equivalent*

(i) *x belongs to $\overline{A}$, the closure of A;*
(ii) *there exists an ultrafilter $\mathcal{U}$ convergent to x, such that $A \in \mathcal{U}$;*
(iii) *there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in A, such that $\lim_{\lambda \in \Lambda} x_\lambda = x$.*

PROOF. $(i) \Rightarrow (ii)$. Assume $x \in \overline{A}$. Consider the filter $\mathcal{N}_x$ of all neighborhoods of x, and the collection

$$\mathcal{F} = \{A \cap N : N \in \mathcal{N}_x\} \cup \mathcal{N}_x,$$

which is obviously a filter, containing A. Take $\mathcal{U}$ to be an ultrafilter containing $\mathcal{F}$. Then $\mathcal{U}$ contains both A and all neighborhoods of x, so $\mathcal{U}$ is indeed convergent to x.

$(ii) \Rightarrow (iii)$. Let $\mathcal{U}$ be an ultrafilter which is convergent to x, and which contains A. In particular the collection

$$\mathcal{F} = \{U \cap A : U \in \mathcal{U}\}$$

is a filter. Let $(x_F)_{F \in \mathcal{F}}$ be a selection for $\mathcal{F}$. By construction, this net is contained in A. Moreover, this net is a section for $\mathcal{U}$. Indeed, if we start with some arbitrary $U \in \mathcal{U}$ and we take $F_U = U \cap A$, then

$$x_F \in F \subset F_U \subset U, \text{ for all } F \in \mathcal{F} \text{ with } F \succ F_U \text{ (i.e. } F \subset F_U).$$

By Lemma 2.2 it follows that $\lim_{F \in \mathcal{F}} x_F = x$.

$(iii) \Rightarrow (i)$. Assume there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in A, which is convergent to x, and let us show that x belongs to the closure of A, which amount to showing that $N \cap A \neq \varnothing$, for every neighborhood N of x. This is however obvious, since for each N there exists some $\lambda_N \in \Lambda$ such that $x_{\lambda_N} \in N$. □

The Hausdorff property has the following characterization in terms of convergence:

PROPOSITION 2.4. *For a topological space $(X, \mathcal{T})$, the following are equivalent:*

(i) *The topology $\mathcal{T}$ is Hausdorff.*
(ii) *Every convergent ultrafilter in X has a unique limit.*
(iii) *Every convergent net has a unique limit.*

PROOF. $(i) \Rightarrow (iii)$. Assume the topology is Hausdorff. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X which is convergent to both x and y. If $x \neq y$, then by the Hausdorff property, there exist two open sets $U, V \subset X$, with $x \in U$, $y \in V$, and $U \cap V = \varnothing$. Since U is a neighborhood of x, there exists some $\lambda_U \in \Lambda$, such that

$$x_\lambda \in U, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_U. \tag{2}$$

Likewise, there exists some $\lambda_V \in \Lambda$, such that

$$x_\lambda \in V, \text{ for all } \lambda \in \Lambda \text{ with } \lambda \succ \lambda_V. \tag{3}$$

Using the fact that Λ is directed, there exists some $\mu \in \Lambda$ with $\mu \succ \lambda_U$ and $\mu \succ \lambda_V$. By (2) and (3) however, this will force $x_\mu \in U \cap V$, thus contradicting the fact that $U \cap V = \varnothing$.

$(iii) \Rightarrow (ii)$. Let $\mathcal{U}$ be an ultrafilter in X which is convergent to both x and y. Take then a selection net $(x_U)_{U \in \mathcal{U}}$ for $\mathcal{U}$. By Lemma 2.2 it follows that the net $(x_U)_{U \in \mathcal{U}}$ is convergent to both x and y, so condition (iii) forces $x = y$.

(ii) $\Rightarrow$ *(i)*. Assume X satisfies condition (ii), but the topology is not Hausdorff. This means that there exist two points $x, y \in X$, with $x \neq y$, such that

$(*)$ *for any open sets* $U, V \subset X$, *with* $U \ni x$ *and* $V \ni y$, *we have* $U \cap V \neq \varnothing$.

Let $\mathcal{N}_x$ denote the collection of all neighborhoods of x, and $\mathcal{N}_y$ denote the collection of all neighborhoods of y. By condition $(*)$ we have

$$M \cap N \neq \varnothing, \ \forall M \in \mathcal{N}_x, N \in \mathcal{N}_y.$$

This proves that the collection

$$\mathcal{F} = \{M \cap N : M \in \mathcal{N}_x, N \in \mathcal{N}_y\}$$

is a filter in X. Notice that, since X is a neighborhood for both x and y, we have the inclusion $\mathcal{F} \supset \mathcal{N}_x \cup \mathcal{N}_y$. So if we take $\mathcal{U}$ to be an ultrafilter, with $\mathcal{U} \supset \mathcal{F}$, it follows that $\mathcal{U} \supset \mathcal{N}_x$, hence $\mathcal{U}$ converges to x, but also $\mathcal{U} \supset \mathcal{N}_y$, hence $\mathcal{U}$ is also convergent to y. By condition (ii) this is impossible. □

Continuity can also be nicely characterized using convergence:

Proposition 2.5. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{S})$ *be topological spaces, and let* x *be element in* X. *For a function* $f : X \to Y$, *the following are equivalent:*

(i) f *is continuous at* x.
(ii) *Whenever* $\mathcal{U}$ *is an ultrafilter on* X *convergent to* x, *it follows that the ultrafilter* $f_*(\mathcal{U})$ *in* Y, *convergent to* $f(x)$.
(iii) *Whenever* $(x_\lambda)_{\lambda\in\Lambda}$ *is a net in* X, *which is convergent to* x, *it follows that the net* $(f(x_\lambda))_{\lambda\in\Lambda}$ *is convergent to* $f(x)$.

Proof. *(i)* $\Rightarrow$ *(iii)*. Assume that f is continuous at x. Start with some net $(x_\lambda)_{\lambda\in\Lambda}$, which is convergent to x, and let us show that the net $(f(x_\lambda))_{\lambda\in\Lambda}$ is convergent to $f(x)$. Start with some neighborhood N of $f(x)$. The continuity of f at x gives the fact that $f^{-1}(N)$ is a neighborhood of x. The fact that $\lim_{\lambda\in\Lambda} x_\lambda = x$ then gives the existence of some $\lambda_N \in \Lambda$, such that $x_\lambda \in f^{-1}(N)$, $\forall \lambda \succ \lambda_N$, meaning of course that $f(x_\lambda) \in N$, $\forall \lambda \succ \lambda_N$.

(iii) $\Rightarrow$ *(ii)*. Assume condition (iii) is true, and let $\mathcal{U}$ be an ultrafilter on X which converges to x, and let us show that the ultrafilter $f_*(\mathcal{U})$ is convergent to $f(x)$. Fix a selection net $(x_U)_{U\in\mathcal{U}}$ for $\mathcal{U}$, which by Lemma 2.2. converges to x. By condition (iii) it follows that the net $(f(x_U))_{U\in\mathcal{U}}$ converges to $f(x)$, so by Lemma 2.2, in order to prove that $f_*(\mathcal{U})$ converges to $f(x)$, it suffices to show that the set $(f(x_U))_{U\in\mathcal{U}}$ is a section for $f_*(\mathcal{U})$. Start with some $V \in f_*(\mathcal{U})$, and let $U_V = f^{-1}(V) \in \mathcal{U}$. If $U \in \mathcal{U}$ is such that $U \succ U_V$, i.e. $U \subset U_V$, then we get $x_U \in U \subset U_V = f^{-1}(V)$. Therefore we have

$$f(x_U) \in V, \text{ for all } U \in \mathcal{U} \text{ with } U \succ U_V,$$

and we are done.

(ii) $\Rightarrow$ *(i)*. Assume f satisfies condition (ii), but f is not continuous at x. This means that there exists some neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of x. Consider the collection

$$\mathcal{F} = \{N \smallsetminus f^{-1}(V) : N \text{ neighborhood of } x\}.$$

Our assumption on V shows that all the sets in $\mathcal{F}$ are non-empty. (Otherwise $f^{-1}(V)$ would contain some neighborhood of x, which would force $f^{-1}(V)$ itself to be a neighborhood of x.) It is also clear that $\mathcal{F}$ is a filter. Let $\mathcal{U}$ be an ultrafilter with $\mathcal{U} \supset \mathcal{F}$.

Claim: *The ultrafilter* $\mathcal{U}$ *is convergent to* x.

To prove this, start with some arbitrary neighborhood N of x. If N does not belong to $\mathcal{U}$, then $X \smallsetminus N$ belongs to $\mathcal{U}$. But then $(X \smallsetminus N) \cap (N \smallsetminus f^{-1}(V)) = \varnothing$ belongs to $\mathcal{U}$, which is impossible. So $\mathcal{U}$ contains all neighborhoods of x, which means that indeed $\mathcal{U}$ is convergent to x.

Using our assumption on V, plus condition (ii), it follows that $V \in f_*(\mathcal{U})$, which means that $f^{-1}(V) \in \mathcal{U}$. But this leads to a contradiction, since $X \smallsetminus f^{-1}(V)$ clearly belongs to $\mathcal{F} \subset \mathcal{U}$. □

The following exercises[3] below explain how one can construct the topology back from its "convergent objects."

Exercise 2$^\heartsuit$. The following construction will be needed in the formulation of the next exercise. Suppose $\Delta = (\Delta_\omega)_{\omega\in\Omega}$ is a system of directed sets, indexed by a directed set Ω. One defines the set Consider the set

$$\text{Diag}(\Delta) = \big\{(\omega, \boldsymbol{\theta}) \,:\, \omega \in \Omega,\ \boldsymbol{\theta} = (\theta_\sigma)_{\sigma\succ\omega} \in \prod_{\sigma\succ\omega} \Delta_\sigma\big\},$$

equipped with the relation

$$(\omega, \boldsymbol{\theta}) \succ (\omega', \boldsymbol{\theta}') \iff \begin{cases} \omega \succ \omega' \\ \theta_\sigma \succ \theta'_\sigma,\ \forall\, \sigma \succ \omega \end{cases}$$

Prove that $(\text{Diag}(\Delta), \succ)$ is a directed set. This directed set is called the *diagonal of* Δ.

Exercise 3$^\heartsuit$. Let $(X, \mathcal{T})$ be a topological space. Prove the following.

A. If $(x_\lambda)_{\lambda\in\Lambda}$ is a constant net, i.e. $x_\lambda = x$, $\forall\, \lambda \in \Lambda$, then $(x_\lambda)_{\lambda\in\Lambda}$ is convergent to x.

B. (Sub-net convergence) If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X, which is convergent to some point $x \in X$, then every subnet of $(x_\lambda)_{\lambda\in\Lambda}$ is again convergent to x.

C. (Sub-sub-net convergence criterion) For a net $(x_\lambda)_{\lambda\in\Lambda}$ in X, and a point $x \in X$, the following are equivalent:
   (i) $(x_\lambda)_{\lambda\in\Lambda}$ is convergent to x;
   (ii) every sub-net of $(x_\lambda)_{\lambda\in\Lambda}$ has a sub-net which is convergent to x.

D. (Convergence along diagonal) Let $\Delta = (\Delta_\omega)_{\omega\in\Omega}$ be a system of directed sets, indexed by a directed set Ω, and assume that for each $\omega \in \Omega$, a net $(y_{\omega,\delta})_{\delta\in\Delta_\omega}$ in X is given (indexed by Δ_ω), such that
   (*a*) for each $\omega \in \Omega$, the net $(y_{\omega,\delta})_{\delta\in\Delta_\omega}$ is convergent to some point y_ω;
   (*b*) the net $(y_\omega)_{\omega\in\Omega}$ is convergent to some point $x \in X$.
   Let $\Lambda = \text{Diag}(\Delta)$ be the diagonal of Δ, and define, for each $\lambda = (\omega, \boldsymbol{\theta}) \in \Lambda$, the point $x_\lambda = y_{\omega,\theta_\omega}$. Then the net $(x_\lambda)_{\lambda\in\Lambda}$ is convergent to x.

Exercise 4$^\heartsuit$. Suppose X is some non-empty set. Let $\mathbf{C}$ be a class[4] consisting of pairs of the form $(\boldsymbol{x}, p)$, with $\boldsymbol{x}$ net in X and p point in X. Assume $\mathbf{C}$ has properties A-D from the preceding exercise (with "$\boldsymbol{x}$ is convergent to p" replaced with "$(\boldsymbol{x}, p) \in \mathbf{C}$"). Prove that there exists a unique topology on X, with respect to which the condition "$\boldsymbol{x}$ is convergent to p" is *equivalent* to "$(\boldsymbol{x}, p) \in \mathbf{C}$."

HINT: Define, for each non-empty set $A \subset X$ the set

$$\text{cl}(A) = \{p \in X \,:\, \text{ there exists a net } \boldsymbol{x} \subset A \text{ with } (\boldsymbol{x}, p) \in \mathbf{C}\},$$

[3] The exercises marked $^\heartsuit$ are optional.

[4] See the comment following Examples 2.3 on why the term "class" is used here.

and show that this defines a closure operator.

Exercise 5$^\heartsuit$. Suppose X is some non-empty set. Let $\mathbf{C}$ be a set consisting of pairs of the form $(\mathcal{U}, p)$, with $\mathcal{U}$ ultrafilter in X and p point in X. Assume $\mathbf{C}$ has properties:

(i) For every $p \in X$, if one considers the constant ultrafilter $\mathcal{U}(p)$, then $(\mathcal{U}(p), p) \in \mathbf{C}$.

(ii) Suppose $(\mathcal{V}_i, q_i)$, $i \in I$, are elements in $\mathbf{C}$, and $(\mathcal{U}, p) \in \mathbf{C}$ is such that $\{q_i : i \in I\} \in \mathcal{U}$. Then

$$\mathcal{U} \supset \bigcap_{i \in I} \mathcal{V}_i.$$

Prove that there exists a unique topology on X, with respect to which the condition "$\mathcal{U} \to p$" is *equivalent* to "$(\mathcal{U}, p) \in \mathbf{C}$."

HINT: Define, for each non-empty set $A \subset X$ the set

$$\text{cl}(A) = \{p \in X : \text{ there exists an ultrafilter } \mathcal{U} \text{ with } \mathcal{U} \ni A \text{ and } (\mathcal{U}, p) \in \mathbf{C}\},$$

and show that this defines a closure operator.

In the remainder of this section we discuss some special type of nets of numbers, with emphasis on the notion of summability. Many key results are left as exercise.

We start off by adopting the following conventions (analogous to the usual one for sequences of real numbers).

CONVENTIONS. A. We say that a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ *has limit* ∞ if

- *for every $M \in \mathbb{R}$ there exists some $\lambda_M \in \Lambda$, such that $x_\lambda > M$, $\forall \lambda \succ \lambda_M$.*

In this case we write $\lim_{\lambda \in \Lambda} x_\lambda = \infty$.

B. We say that a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ *has limit* $-\infty$, if

- *for every $M \in \mathbb{R}$ there exists some $\lambda_M \in \Lambda$, such that $x_\lambda < M$, $\forall \lambda \succ \lambda_M$.*

In this case we write $\lim_{\lambda \in \Lambda} x_\lambda = -\infty$.

C. We say that a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ *has limit*, if either $(x_\lambda)_{\lambda \in \Lambda}$ is convergent, or it has limit $\pm\infty$.

The following exercise is a generalization of a well known result for ordinary sequences.

Exercise 6$^\diamondsuit$. A. Let us call a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ *monotone increasing*, if it satisfies the condition

($\uparrow$) $\lambda \succ \mu \Rightarrow x_\lambda \geq x_\mu$.

Prove that, a monotone increasing net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ always has limit, and one has the equality $\lim_{\lambda \in \Lambda} x_\lambda = \sup \{x_\lambda : \lambda \in \Lambda\}$.

B. Let us call a net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ *monotone decreasing*, if it satisfies the condition

($\downarrow$) $\lambda \succ \mu \Rightarrow x_\lambda \leq x_\mu$.

Prove that, a monotone decreasing net $(x_\lambda)_{\lambda \in \Lambda} \subset \mathbb{R}$ always has limit, and one has the equality $\lim_{\lambda \in \Lambda} x_\lambda = \inf \{x_\lambda : \lambda \in \Lambda\}$.

NOTATIONS. Let J be some non-empty set. We define

$$\mathcal{P}_{\text{fin}}(J) = \big\{F \subset J : F \text{ finite}\big\}.$$

We equip $\mathcal{P}_{\text{fin}}(I)$ with the order given by inclusion ($F \succ G \Leftrightarrow F \supset G$), and it is clear that $\mathcal{P}_{\text{fin}}(J)$ becomes a directed set.

If $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$, and α is a map : $J \to \mathbb{K}$, then we define its *support* to be the set

$$[[\alpha]] = \{j \in J : \alpha(j) \neq 0\}.$$

Definition. Let J be some non-empty set, and let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. A map $\alpha : J \to \mathbb{K}$ is said to be *summable*, if the net of partial sums $(s_F)_{F\in\mathcal{P}_{\text{fin}}(J)}$, defined by

$$s_F = \sum_{j\in F} \alpha(j), \quad F \in \mathcal{P}_{\text{fin}}(J),$$

is convergent to some number $s \in \mathbb{K}$. This means that

- *for every $\varepsilon > 0$, there exists some $F_\varepsilon \in \mathcal{P}_{\text{fin}}(J)$, such that*

$$\Big| s - \sum_{j\in F} \alpha(j) \Big| < \varepsilon, \text{ for all } F \in \mathcal{P}_{\text{fin}}(J), \text{ with } F \supset F_\varepsilon.$$

If this is the case, the number s will be denoted by $\sum_{j\in J} \alpha(j)$. In the case when J is *finite*, every map $\alpha : J \to \mathbb{K}$ is summable, and this notation agrees with the usual notation for the sum. Remark also that if the support $[[\alpha]]$ is finite, then α is automatically summable, and one has $\sum_{j\in J} \alpha(j) = \sum_{j\in[[\alpha]]} \alpha(j)$.

Convention. In the case $\mathbb{K} = \mathbb{R}$, we use the notation $\sum_{j\in J} \alpha(j) = \pm\infty$ to signify the case when the net of partial sums $(s_F)_{F\in\mathcal{P}_{\text{fin}}(J)}$ has limit $\pm\infty$.

Remark 2.5. If $\alpha(j) \geq 0$, $\forall j \in J$, then the net $(s_F)_{F\in\mathcal{P}_{\text{fin}}(J)}$ is monotone increasing, so the symbol $\sum_{j\in J} \alpha(j)$ always has a meaning. Its value is either ∞, or a non-negative number. In either case, by Exercise ??, one has the equality

$$\sum_{j\in J} \alpha(j) = \sup \Big\{ \sum_{j\in F} \alpha(j) : F \in \mathcal{P}_{\text{fin}}(J) \Big\}.$$

The map α is then summable, precisely when the right hand side is finite.

Exercise 7$^\diamondsuit$. Assume $\alpha : J \to \mathbb{K}$ is summable. Prove that, for every $\lambda \in \mathbb{K}$, the map $\lambda\alpha : J \to \mathbb{K}$ is summable, and

$$\sum_{j\in J} \lambda\alpha(j) = \lambda \sum_{j\in J} \alpha(j).$$

If $\beta : J \to \mathbb{K}$ is another summable map, prove that $\alpha + \beta : J \to \mathbb{K}$ is summable, and

$$\sum_{j\in J} [\alpha(j) + \beta(j)] = \big[\sum_{j\in J} \alpha(j)\big] + \big[\sum_{j\in J} \beta(j)\big].$$

Exercise 8$^\diamondsuit$. Let J be a non-empty set. For a function $\alpha : J \to \mathbb{C}$, the following are equivalent:

(i) α is summable;
(ii) both functions Re α, Im $\alpha : J \to \mathbb{R}$ are summable.

Moreover, in this case we have the equality

$$\sum_{j\in J} \alpha(j) = \sum_{j\in J} \text{Re } \alpha(j) + i \sum_{j\in J} \text{Im } \alpha(j).$$

Notation. By the above Exercise 7, the space

$$\ell^1_{\mathbb{K}}(J) = \big\{\alpha : J \to \mathbb{K} \,:\, \alpha \text{ summable}\,\big\}$$

is a vector space, when equipped with pointwise addition and scalar multiplication. We shall adopt the convention that, when $\mathbb{K} = \mathbb{C}$ the subscript will be omitted from the notation.

Exercise 9$^\diamondsuit$. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and let J be a non-empty set. Suppose one has two non-empty disjoint sets J_1, J_2 with $J = J_1 \cup J_2$. Suppose $\alpha : J \to \mathbb{K}$ has the property that both $\alpha\big|_{J_1} : J_1 \to \mathbb{K}$ and $\alpha\big|_{J_2} : J_2 \to \mathbb{K}$ are summable. Prove that α is summable, and

$$\sum_{j\in J}\alpha(j) = \sum_{j\in J_1}\alpha(j) + \sum_{j\in J_2}\alpha(j).$$

Exercise 10$^\diamondsuit$. Let $\alpha, \beta : J \to [0,\infty)$ be two maps, such that

$$\beta(j) \le \alpha(j), \ \ \forall\, j \in J.$$

Show that $\sum_{j\in J_1}\beta(j) \le \sum_{j\in J}\alpha(j)$. In particular, if α is summable, then so is β.

At this point it is instructive to analyze the case when $J = \mathbb{N}$, so we are talking about sequences in $\mathbb{K}$. One is tempted to believe that summability for a sequence $(\alpha_n)_{n=1}^\infty \subset \mathbb{K}$ is the same as the convergence of the series $\sum_{n=1}^\infty \alpha_n$. As we shall see later (see ??), this is not quite the case. The following two exercises deal with the two features that are true.

Exercise 11$^\diamondsuit$. Let $(\alpha_n)_{n=1}^\infty$ be a summable sequence in $\mathbb{K}$. Prove that the series $\sum_{n=1}^\infty \alpha_n$ is convergent, and one has the equality

$$\sum_{n=1}^\infty \alpha_n = \sum_{n\in\mathbb{N}}\alpha_n,$$

where the right hand side is defined as above.

Exercise 12$^\diamondsuit$. Let $(\alpha_n)_{n=1}^\infty$ be a sequence of *non-negative* numbers. Prove the equality

$$\sum_{n=1}^\infty \alpha_n = \sum_{n\in\mathbb{N}}\alpha_n,$$

where the right hand side is either ∞ or a non-negative number (according to the above conventions). In particular, the series $\sum_{n=1}^\infty \alpha_n$ is convergent, if and only if the sequence $(\alpha_n)_{n=1}^\infty$ is summable. (It should be noted here that the condition $\alpha_n \ge 0$, $\forall\, n \in \mathbb{N}$ is to be used in an essential way.)

Proposition 2.6. *Let J be a non-empty set, let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. For a map $\alpha : J \to \mathbb{K}$, the following are equivalent:*

(i) *α is summable;*
(ii) *$|\alpha|$ is summable.*

Moreover, in this case one has the inequality

$$\left|\sum_{j\in J}\alpha(j)\right| \le \sum_{j\in J}\big|\alpha(j)\big|. \tag{4}$$

Proof. $(i) \Rightarrow (ii)$. Assume α is summable. We divide the proof in two cases depending on $\mathbb{K}$.

Case $\mathbb{K} = \mathbb{R}$. Define the sets

$$J^+ = \{j \in J : \alpha(j) > 0\},$$
$$J^- = \{j \in J : \alpha(j) < 0\},$$
$$J^0 = \{j \in J : \alpha(j) = 0\}.$$

More generally, for any subset $F \subset J$ we define $F^\pm = F \cap J^\pm$ and $F^0 = F \cap J^0$.

> *Claim*: *Both maps* $\alpha\big|_{J^+} : J^+ \to \mathbb{R}$ *and* $\alpha\big|_{J^-} : J^- \to \mathbb{R}$ *are summable. Moreover, one has the equality*

$$\sum_{j \in J} \alpha(j) = \sum_{j \in J^+} \alpha(j) + \sum_{j \in J^-} \alpha(j). \tag{5}$$

Denote the sum $\sum_{j \in J} \alpha(j)$ simply by s. Start by choosing some finite set $F \in \mathcal{P}_{\text{fin}}(J)$ such that

$$\Big| s - \sum_{j \in G} \alpha(j) \Big| < 1, \text{ for all } G \in \mathcal{P}_{\text{fin}}(J) \text{ with } G \supset F.$$

For each $E \in \mathcal{P}_{\text{fin}}(J^+)$, we define the set $\tilde{E} = E \cup F \in \mathcal{P}_{\text{fin}}(J)$. Since $\tilde{E} \supset F$, it is obvious that

$$\Big| s - \sum_{j \in \tilde{E}} \alpha(j) \Big| < 1, \ \ \forall E \in \mathcal{P}_{\text{fin}}(J^+)$$

so we get

$$\sum_{j \in E} \alpha(j) \leq \sum_{j \in E \cup F^+} \alpha(j) = \Big[\sum_{j \in E \cup F^+} \alpha(j) + \sum_{j \in F^0 \cup F^-} \alpha(j) \ \Big] - \Big[\sum_{j \in F^0 \cup F^-} \alpha(j) \ \Big] =$$
$$= \Big[\sum_{j \in \tilde{E}} \alpha(j) \Big] - \Big[\sum_{j \in F^-} \alpha(j) \Big] < s + 1 - \Big[\sum_{j \in F^-} \alpha(j) \Big].$$

In particular this gives

$$\sup \Big\{ \sum_{j \in E} \alpha(j) : E \in \mathcal{P}_{\text{fin}}(J^+) \Big\} \leq s + 1 - \Big[\sum_{j \in F^-} \alpha(j) \Big],$$

so by Remark 2.5, the map $\alpha\big|_{J^+} : I^+ \to [0, \infty)$ is indeed summable. The fact that the map $\alpha\big|_{J^-} : J^- \to (-\infty, 0]$ is summable is proven the exact same way. The equality (5) follows from Exercise 9

Having proven the Claim, we notice now that the map $-\alpha\big|_{J^-} : J^- \to [0, \infty)$ is also summable. Using Exercise 9, it is clear then that the map $|\alpha| : J \to [0, \infty)$ is summable, simply because all the three maps $(|\alpha|)\big|_{J^+} = \alpha\big|_{J^+}$, $(|\alpha|)\big|_{J^-} = -\alpha\big|_{J^-}$, and $(|\alpha|)\big|_{J^0} = 0$ are all summable.

Case $\mathbb{K} = \mathbb{C}$. By Exercise 8 we know that the maps $\operatorname{Re} \alpha$, $\operatorname{Im} \alpha : J \to \mathbb{R}$ are summable. In particular, using the real case, we get the fact that the maps $|\operatorname{Re} \alpha|$, $|\operatorname{Im} \alpha| : J \to [0, \infty)$ are summable. Using the obvious inequality

$$|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|, \ \ \forall z \in \mathbb{C},$$

we get

$$\sum_{j\in F}|\alpha(j)| \leq \sum_{j\in F}|\operatorname{Re}\alpha(j)| + \sum_{j\in F}|\operatorname{Im}\alpha(j)| \leq \sum_{j\in J}|\operatorname{Re}\alpha(j)| + \sum_{j\in J}|\operatorname{Im}\alpha(j)|,$$

for every $F \in \mathcal{P}_{\text{fin}}(J)$. Then we get

$$\sup\left\{\sum_{j\in F}|\alpha(j)| \,:\, F \in \mathcal{P}_{\text{fin}}(J)\right\} \leq \sum_{j\in J}|\operatorname{Re}\alpha(j)| + \sum_{j\in J}|\operatorname{Im}\alpha(j)| < \infty,$$

so $|\alpha| : J \to [0,\infty)$ is indeed summable.

Having proven the implication $(i) \Rightarrow (ii)$, let us prove the inequality (4). If s denotes the sum $\sum_{j\in J}\alpha(j)$, then for every $\varepsilon > 0$ there exists $F_\varepsilon \in \mathcal{P}_{\text{fin}}(J)$ such that

$$\left|s - \sum_{j\in F}\alpha(j)\right| < \varepsilon, \text{ for all } F \in \mathcal{P}_{\text{fin}}(J) \text{ with } F \supset F_\varepsilon.$$

In particular, we get

$$|s| \leq \varepsilon + \left|\sum_{j\in F_\varepsilon}\alpha(j)\right| \leq \varepsilon + \sum_{j\in F_\varepsilon}\big|\alpha(j)\big| \leq \varepsilon + \sum_{j\in J}\big|\alpha(j)\big|.$$

Since this inequality holds for all $\varepsilon > 0$, we then get

$$|s| \leq \sum_{j\in J}\big|\alpha(j)\big|.$$

$(ii) \Rightarrow (i)$. Assume now $|\alpha| : J \to [0,\infty)$ is summable.

Case $\mathbb{K} = \mathbb{R}$. It is obvious that $(|\alpha|)\big|_E : E \to [0,\infty)$ is summable, for any subset $E \subset J$. In particular, using the notations from the proof of $(i) \Rightarrow (ii)$, it follows that $\alpha\big|_{J^+} = (|\alpha|)\big|_{J^+}$, $\alpha\big|_{J^-} = (-|\alpha|)\big|_{J^-}$, and $\alpha\big|_{J^0} = 0$ are all summable. Then the summability of α follows from Exercise 9.

Case $\mathbb{K} = \mathbb{C}$. Using the inequality

$$\max\big\{|\operatorname{Re} z|,\ |\operatorname{Im} z|\big\} \leq |z|,\ \ \forall\, z \in \mathbb{C},$$

combined with Exercise 10, it follows that both maps $|\operatorname{Re}\alpha|,\ |\operatorname{Im}\alpha| : J \to [0,\infty)$ are summable. Using the real case it then follows that both maps $\operatorname{Re}\alpha,\ \operatorname{Im}\alpha : J \to \mathbb{R}$ are summable. Then the summability of α follows from Exercise 8. □

The following result shows that summability is essentially the same as the summability of sequences.

Proposition 2.7. *Suppose $\alpha : J \to \mathbb{K}$ is summable. Then the support set*

$$[[\alpha]] = \{j \in J \,:\, \alpha(j) \neq 0\}$$

is at most countable.

Proof. For every integer $n \geq 1$, we define the set $J_n = \{j \in J \,:\, |\alpha(j)| \geq \frac{1}{n}\}$. Since $|\alpha|$ is summable, the sets J_n, $n \geq 1$ are all finite. The desired result then follows from the obvious equality $[[\alpha]] = \bigcup_{n=1}^{\infty} J_n$. □