A FUNCTIONAL ANALYSIS POINT OF VIEW ON ARZELA-ASCOLI THEOREM

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ABSTRACT. We discuss the Arzela-Ascoli pre-compactness Theorem from the point of view of Functional Analysis, using compactness in ℓ^∞ and its dual.

The Arzela-Ascoli Theorem is a very important technical result, used in many branches of mathematics. Aside from its numerous applications to Partial Differential Equations, the Arzela-Ascoli Theorem is also used as a tool in obtaining Functional Analysis results, such as the compactness for duals of compact operators, as presented for example [1]. The purpose of this note is to offer a new perspective on the Arzela-Ascoli Theorem based on a functional analytic proof.

The theorem of Arzela and Ascoli deals with (relative) compactness in the Banach space $C(K)$ of complex valued continuous functions on a compact Hausdorff space K . One helpful characterization of pre-compactness for sets in a complete metric space is the following well-known criterion, which we state without proof.

Proposition 1. *Let $(\mathcal{Y}, d)$ be a complete metric space. For a subset $\mathcal{M} \subset \mathcal{Y}$, the following are equivalent:*

- (i) $\mathcal{M}$ is relatively compact in $\mathcal{Y}$, i.e. its closure $\overline{\mathcal{M}}$ in $\mathcal{Y}$ is compact;
- (ii) $\mathcal{M}$ contains no infinite subsets $\mathcal{T}$, satisfying

$$\inf \{d(x, y) : x, y \in \mathcal{T}, x \neq y\} > 0.$$

Proposition 2. *Let $\mathcal{X}$ be a normed vector space, let $\mathcal{S} \subset \mathcal{X}$ be a compact subset, and let $\mathcal{B}$ be the unit ball in $\mathcal{X}^*$ – the (topological) dual of $\mathcal{X}$ – equipped with the w^* -topology. If we consider the Banach algebra $\mathcal{A} = C(\mathcal{S})$, equipped with the uniform topology, then the restriction map $\Theta : (\mathcal{B}, w^*) \ni \phi \mapsto \phi|_{\mathcal{S}} \in (\mathcal{A}, \|\cdot\|)$ is continuous. In particular, the set $\Theta(\mathcal{B})$ is compact in $\mathcal{A}$.*

Proof. To prove continuity, we start with a net $(\phi_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{B}$, that converges to some $\phi \in \mathcal{B}$ in the w^* -topology, and let us show that the net $(\phi_\lambda|_{\mathcal{S}})_{\lambda \in \Lambda}$ converges to $\phi|_{\mathcal{S}}$ uniformly on $\mathcal{S}$. Fix some $\varepsilon > 0$, and (use compactness of $\mathcal{S}$) choose points $s_1, \dots, s_n \in \mathcal{S}$, such that

$$(*) \text{ for every } s \in \mathcal{S}, \text{ there exists } k \in \{1, \dots, n\}, \text{ with } \|s - s_k\| < \varepsilon/3.$$

Using the condition $\phi_\lambda \xrightarrow{w^*} \phi$, there exists $\lambda_\varepsilon \in \Lambda$, such that

$$(1) \quad |\phi_\lambda(s_k) - \phi(s_k)| < \varepsilon/3, \quad \forall \lambda \geq \lambda_\varepsilon, k \in \{1, \dots, n\}.$$

Now we are done, since if we start with some arbitrary $s \in \mathcal{S}$, and we choose $k \in \{1, \dots, n\}$, such that $\|s - s_k\| < \varepsilon/3$, then using (1) we get

$$\begin{aligned} |\phi_\lambda(s) - \phi(s)| &\leq |\phi_\lambda(s) - \phi_\lambda(s_k)| + |\phi_\lambda(s_k) - \phi(s_k)| + |\phi(s_k) - \phi(s)| \leq \\ &\leq \|\phi_\lambda\| \cdot \|s - s_k\| + |\phi_\lambda(s_k) - \phi(s_k)| + \|\phi\| \cdot \|s - s_k\| \leq \\ &\leq 2\|s - s_k\| + |\phi_\lambda(s_k) - \phi(s_k)| \leq \varepsilon, \quad \forall \lambda \geq \lambda_\varepsilon. \end{aligned}$$

Having proven the continuity of Θ , the second assertion follows from Alaoglu's Theorem (which states that $\mathcal{B}$ is compact in the w^* -topology). $\square$

Theorem (Arzela-Ascoli). *Let K be a compact Hausdorff space, and let $\mathcal{M} \subset C(K)$ be a set which is*

- **pointwise bounded:** $\sup \{|f(p)| : f \in \mathcal{M}\} < \infty, \forall p \in K$;
- **equicontinuous:** *for every $p \in K$ and $\varepsilon > 0$, there exists a neighborhood $N_{p,\varepsilon}$ of p in K , such that: $\sup_{f \in \mathcal{M}} |f(q) - f(p)| \leq \varepsilon, \forall q \in N_{p,\varepsilon}$.*

Then $\mathcal{M}$ is relatively compact in $C(K)$ in the uniform topology.

Proof. The main observation is that, using pointwise boundedness, any set $\mathcal{T} \subset \mathcal{M}$ gives rise to a map

$$\Phi_{\mathcal{T}} : K \ni p \longmapsto [f(p)]_{f \in \mathcal{T}} \in \ell^{\infty}(\mathcal{T}).$$

(Here $\ell^{\infty}(\mathcal{T})$ denotes the Banach space of all bounded functions from $\mathcal{T}$ to $\mathbb{C}$.), Furthermore, by equicontinuity the map $\Phi_{\mathcal{T}}$ is in fact continuous, when $\ell^{\infty}(\mathcal{T})$ is equipped with the norm topology. Since K is compact, so is the set

$$\mathcal{S}_{\mathcal{T}} = \Phi_{\mathcal{T}}(K) \subset \ell^{\infty}(\mathcal{T}).$$

We are going to argue by contradiction (see Proposition 1), assuming the existence of some $\rho > 0$, and of an infinite set $\mathcal{T} \subset \mathcal{M}$, such that

$$(2) \quad \|f - g\| > \rho, \quad \forall f, g \in \mathcal{T}, f \neq g.$$

Consider now the unit ball $\mathcal{B}$ in the dual space $\ell^{\infty}(\mathcal{T})^*$, and use Proposition 2 to conclude that the set

$$\Theta_{\mathcal{T}} = \{\phi|_{\mathcal{S}_{\mathcal{T}}} : \phi \in \mathcal{B}\} \subset C(\mathcal{S}_{\mathcal{T}})$$

is compact in $C(\mathcal{S}_{\mathcal{T}})$ in the norm topology. Consider now the coordinate maps $e_f : \ell^{\infty}(\mathcal{T}) \rightarrow \mathbb{C}, f \in \mathcal{T}$, and their restrictions $\theta_f = e_f|_{\mathcal{S}_{\mathcal{T}}} \in C(\mathcal{S}_{\mathcal{T}})$, which satisfy

$$(3) \quad \theta_f(\Phi_{\mathcal{T}}(p)) = f(p), \quad \forall p \in K, f \in \mathcal{T}.$$

Now, if we start with $f, g \in \mathcal{T}, f \neq g$, then using (2) there exists $p \in K$, such that $|f(p) - g(p)| > \rho$, so by (3) we also get

$$|\theta_f(\Phi_{\mathcal{T}}(p)) - \theta_g(\Phi_{\mathcal{T}}(p))| > \rho.$$

This way we have shown that

$$\|\theta_f - \theta_g\|_{C(\mathcal{S}_{\mathcal{T}})} > \rho, \quad \forall f, g \in \mathcal{T}, f \neq g,$$

and therefore the set $\Theta_{\mathcal{T}}$ – which contains all the θ_f 's – cannot be compact in the norm topology. $\square$

REFERENCES

- [1] W. Rudin, *Functional Analysis*, McGraw-Hill, Springer-Verlag, New York, 1973

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