Convexity "Warm-up" II: The Minkowski Functional

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

Convention. Throughout this note $\mathbb{K}$ will be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and all vector spaces are over $\mathbb{K}$. In this section, however, when $\mathbb{K} = \mathbb{C}$, it will be only the *real* linear structure that will play a significant role.

Proposition-Definition 1. *Let $\mathcal{X}$ be a vector space, and let $\mathcal{A} \subset \mathcal{X}$ be a convex absorbing set. Define, for every element $x \in \mathcal{X}$ the quantity*[1]

$$q_{\mathcal{A}}(x) = \inf\{\tau > 0 \,:\, x \in \tau\mathcal{A}\}. \tag{1}$$

(i) *The map $q_{\mathcal{A}} : \mathcal{X} \to \mathbb{R}$ is a quasi-seminorm, i.e.*

$$q_{\mathcal{A}}(x+y) \leq q_{\mathcal{A}}(x) + q_{\mathcal{A}}(y), \;\; \forall\, x, y \in \mathcal{X}; \tag{2}$$

$$q_{\mathcal{A}}(tx) = tq_{\mathcal{A}}(x), \;\; \forall\, x \in \mathcal{X},\, t \geq 0. \tag{3}$$

(ii) *One has the inclusions:*

$$\{x \in \mathcal{X} \,:\, q_{\mathcal{A}}(x) < 1\} \subset \mathcal{A} \subset \{x \in \mathcal{X} \,:\, q_{\mathcal{A}}(x) \leq 1\}. \tag{4}$$

(iii) *If, in addition to the above hypothesis, $\mathcal{A}$ is balanced, then $q_{\mathcal{A}}$ is a seminorm, i.e. besides* (2) *and* (3) *it also satisfies the condition:*

$$q_{\mathcal{A}}(\alpha x) = |\alpha| \cdot q_{\mathcal{A}}(x), \;\; \forall\, x \in \mathcal{X},\, \alpha \in \mathbb{K}. \tag{5}$$

The map $q_{\mathcal{A}} : \mathcal{X} \to \mathbb{R}$ is called the *Minkowski functional associated to $\mathcal{A}$*.

Proof. (i). To prove (2) start with two elements $x, y \in \mathcal{X}$ and some $\varepsilon > 0$. By the definition (1) there exist positive real numbers $\alpha < q_{\mathcal{A}}(x) + \varepsilon$ and $\beta < q_{\mathcal{A}}(y) + \varepsilon$, such that $x \in \alpha\mathcal{A}$ and $y \in \beta\mathcal{A}$. By Exercise ?? from LCTVS I, it follows that

$$x + y \in \alpha\mathcal{A} + \beta\mathcal{A} = (\alpha + \beta)\mathcal{A},$$

so by the definition (1) we get

$$q_{\mathcal{A}}(x+y) \leq \alpha + \beta < q_{\mathcal{A}}(x) + q_{\mathcal{A}}(x) + 2\varepsilon.$$

Since the inequality $q_{\mathcal{A}}(x+y) < q_{\mathcal{A}}(x) + q_{\mathcal{A}}(x) + 2\varepsilon$ holds for all $\varepsilon > 0$, the desired inequality (2) follows.

[1] Since $\mathcal{A}$ is absorbing, the set on the right-hand side of (1) is non-empty.

To prove (3), let us first observe that, since $\mathcal{A}$ is absorbing, it contains the zero vector, so in fact $0 \in \tau\mathcal{A}$, $\forall \tau > 0$, which means that $q_{\mathcal{A}}(0) = 0$. This shows that in order to prove (3) we may restrict to the case when $t > 0$. In that case we clearly have the equivalence:

$$tx \in \tau\mathcal{A} \Leftrightarrow x \in (\tau/t)\mathcal{A},$$

which shows that the map

$$\Theta : \{\tau > 0 \,:\, tx \in \tau\mathcal{A}\} \ni \tau \longmapsto \tau/t \in \{\gamma > 0 \,:\, x \in \gamma\mathcal{A}\}$$

is a bijection. Taking the infimum then yields the equality

$$q_{\mathcal{A}}(tx)/t = q_{\mathcal{A}}(x),$$

which is exactly (3).

(ii). To prove the first inclusion in (4), suppose $x \in \mathcal{X}$ satisfies the inequality $q_{\mathcal{A}}(x) < 1$, and let us show that $x \in \mathcal{A}$. By the definition (1), there exists some $\tau \in (0, 1)$, such that $x \in \tau\mathcal{A}$. In particular, the vector $a = \frac{1}{\tau}x$ belongs to $\mathcal{A}$, so the vector $\tau a + (1 - \tau)0 = x$ also belongs to $\mathcal{A}$.

The second inclusion in (4) is quite obvious, since for any $a \in \mathcal{A}$, the set $\{\tau > 0 \,:\, a \in \tau\mathcal{A}\}$ clearly contains 1.

(iii). Assume $\mathcal{A}$ is balanced. In particular, if we consider the *multiplicative group*

$$G = \{\gamma \in \mathbb{K} \,:\, |\alpha| = 1\},$$

we know that

$$\gamma\mathcal{A} = \mathcal{A}, \ \ \forall \gamma \in G.$$

Therefore, for any $x \in \mathcal{X}$, $\tau > 0$, and $\gamma \in G$, one has the equivalences

$$x \in \tau\mathcal{A} \Leftrightarrow \gamma x \in \tau\gamma\mathcal{A} \Leftrightarrow \gamma x \in \tau\mathcal{A}.$$

By the definition (1) this yields

$$q_{\mathcal{A}}(\gamma x) = q_{\mathcal{A}}(x), \ \ \forall x \in \mathcal{X}, \ \gamma \in G. \tag{6}$$

To prove (5) we notice that for any $\alpha \in \mathbb{K}$ we can find $\gamma \in G$ and $t \geq 0$, with

$$\alpha = \gamma t, \tag{7}$$

so using (6) and (3) we get

$$q_{\mathcal{A}}(\alpha x) = q_{\mathcal{A}}(\gamma t x) = q_{\mathcal{A}}(tx) = t q_{\mathcal{A}}(x),$$

and the desired equality follows from (7) which forces $t = |\alpha|$. □

Remark. Call a subset $\mathcal{A} \subset \mathcal{X}$ *openly absorbing*, if for every $x \in \mathcal{X}$, the set

$$T(x) = \{t > 0 \,:\, tx \in \mathcal{A}\}$$

is non-empty[2] and *open* in $(0, \infty)$.

With this terminology, statement (ii) from Proposition-Definition 1 can be slightly improved, in the following sense.

Proposition 2. *For a convex absorbing subset $\mathcal{A} \subset \mathcal{X}$, the following are equivalent:*

[2] Of course, the condition $T(x) \neq \varnothing$, $\forall x \in \mathcal{X}$, means that $\mathcal{A}$ is absorbing.

(i) $\mathcal{A}$ *is openly absorbing;*

(ii) $\mathcal{A} = \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}$.

Proof. $(i) \Rightarrow (ii)$. Assume $\mathcal{A}$ is openly absorbing. By (4) we only need to prove the inclusion

$$\mathcal{A} \subset \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}.$$

Start with some $x \in \mathcal{A}$, so that, using the notaion from the above Definition, the set $T(x)$ contains 1. Since $T(x)$ is open, it will contain a small open interval around 1. In particular, $T(x)$ contains some $t > 1$. For such a t, it follows that $x \in t^{-1}\mathcal{A}$, so $q_{\mathcal{A}}(x) \leq t^{-1} < 1$.

$(ii) \Rightarrow (i)$. Assume now condition (ii), and let us prove that $\mathcal{A}$ is openly absorbing. Fix some $x \in \mathcal{X}$, and some $t \in T(x)$. We wish to produce an open interval J, such that $t \in J \subset T(x)$. On the one hand, since $tx \in \mathcal{A}$, by (ii) it follows that $q_{\mathcal{A}}(tx) < 1$, or equivalently $t < q_{\mathcal{A}}(x)^{-1}$ (with the convention that $q_{\mathcal{A}}(x)^{-1} = \infty$, if $q_{\mathcal{A}}(x) = 0$). This suggests that we could define J to be the interval $(0, q_{\mathcal{A}}(x)^{-1})$. To check that this works, start with some $s \in J$, i.e. $0 < s < q_{\mathcal{A}}(x)^{-1}$. This forces $q_{\mathcal{A}}(sx) = sq_{\mathcal{A}}(x) < 1$, so by (ii) $sx \in \mathcal{A}$, which means that s indeed belongs to $T(x)$. $\square$

Exercises 1-9. Let $\mathcal{X}$ be a vector space.

1. Suppose $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ are convex absorbing sets, such that $\mathcal{A} \subset \mathcal{B}$. Show that

$$q_{\mathcal{A}}(x) \geq q_{\mathcal{B}}(x), \ \forall x \in \mathcal{X}.$$

2. Prove that if $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{X}$ are convex and absorbing, then $\bigcap_{i=1}^n \mathcal{A}_i$ is also absorbing. Give an example of two (non-convex) absorbing sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$, such that $\mathcal{A}_1 \cap \mathcal{A}_2$ is not absorbing.

3. Prove that, if $\mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathcal{X}$ are openly absorbing, then so is $\bigcap_{i=1}^n \mathcal{A}_i$.

4. Prove that given an arbitrary collection $(\mathcal{A}_i)_{i \in I}$ of openly absorbing subsets, the union $\bigcup_{i \in I} \mathcal{A}_i$ is again openly absorbing.

5. Show that, if $\mathcal{A}$ is openly absorbing, then so is its convex hull $\text{conv}(\mathcal{A})$.

6. Suppose $\mathcal{X}$ is equipped with a linear topology, and $\mathcal{A} \subset \mathcal{X}$ is an open set containing 0. Prove that $\mathcal{A}$ is openly absorbing.

7. Suppose $\mathcal{A} \subset \mathcal{X}$ is convex and absorbing. Define the *pseudo-interior of* $\mathcal{A}$ to be the set $\mathcal{A}^{\diamondsuit} = \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}$. Prove the following statements.

(i) $\mathcal{A}^{\diamondsuit}$ is the largest openly absorbing set contained in $\mathcal{A}$.

(ii) $\mathcal{A}^{\diamondsuit}$ is convex.

(iii) $q_{\mathcal{A}^{\diamondsuit}} = q_{\mathcal{A}}$.

8. Let $p : \mathcal{X} \to \mathbb{R}$ be a quasi-seminorm on a real vector space $\mathcal{X}$. Consider the set $\mathcal{A} = \{x \in \mathcal{X} : p(x) < 1\}$. Show that:

(i) $\mathcal{A}$ is convex and openly absorbing.

(ii) If p is a seminorm, then $\mathcal{A}$ is also balanced.

(ii) The Minkowski functional $q_{\mathcal{A}}$ is given by

$$q_{\mathcal{A}}(x) = \max\{p(x), 0\}, \ \ \forall\, x \in \mathcal{X}.$$

9$^{\heartsuit}$. Assume $\mathcal{A} \subset \mathcal{X}$ is non-empty, convex and balanced. Prove that the following conditions are equivalent:

(i) the Minkowski functional $q_{\mathcal{A}}$ is a *norm*, i.e. $q_{\mathcal{A}}(x) = 0 \Rightarrow x = 0$;

(ii) $\bigcap_{t>0} t\mathcal{A} = \{0\}$.

Exercise 10. Let $n \geq 1$ be an integer. Consider the following sets in $\mathbb{K}^n$:

$$\mathcal{A} = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \, : \, x_1^2 + \cdots + x_n^2 < 1\big\};$$
$$\mathcal{B} = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \, : \, x_1^2 + \cdots + x_n^2 \leq 1\big\}.$$

Show that for any $x = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, one has the equalities

$$q_{\mathcal{A}}(x) = q_{\mathcal{B}}(x) = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}.$$