CHERN–SIMONS THEORY AND THE ASYMPTOTIC EXPANSION OF WITTEN–RESHETIKHIN–TURAEV'S INVARIANT OF 3–MANIFOLDS

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Abstract. In this report we make a thorough study of the Chern–Simons field theory for compact oriented 3–manifolds associated to a compact simply connected simple Lie group G mainly following [F]. The action of the classical Wess–Zumino–Witten theory in $1+1$ dimensions appears in Chern–Simons theory in the definition of the Hermitian line corresponding to the boundary of a 3–manifold. A part of the report is concerned with the study of the moduli space of flat G–connections on a compact oriented 3–manifold, the flat connections being the so-called classical solutions in the Chern–Simons field theory. We also give an outline of how to construct (in a rigorous way) the 2–dimensional part (the modular functor) of a $2+1$–dimensional TQFT using geometric quantization of the moduli space of flat connections on Riemann surfaces. In the final part of the report, we begin a rigorous calculation of the large quantum level asymptotics of the SU(2) Reshetikhin–Turaev invariants of 3–fibered Seifert manifolds with base S^2 following [Ro1]. The report supplements the calculations of Rozansky by obtaining analytic estimates needed to justify the method of Rozansky.

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(3 figures are missing in the last section of this text.)

1. Introduction

This report reflects the work done in a project begun in the autumn of 1996 under supervision of Jørgen Ellegaard Andersen. Everywhere a 3–manifold will be a compact oriented 3–manifold, except otherwise stated. In particular, a Seifert manifold is an oriented Seifert manifold.

In 1988 E. Witten introduced new invariants of 3–manifolds and of links in these 3–manifolds, by quantizing the Chern–Simons field theory by means of the Feynman path integral, cf. [W]. If X is a 3–manifold the invariant of X is given by integrating the Chern–Simons action over the space of gauge equivalence classes of connections on G bundles over X, G being some nice Lie group, e.g. equal to SU(n). There is however from a mathematical point of view a major problem with this approach, namely that the path integral is not well-defined in this situation. In 1991, N. Reshetikhin and V. G. Turaev [RT] published a paper where they in a mathematical rigorous way defined invariants of 3–manifolds and links by combinatorial means, which they conjectured to be identical with Witten's invariants. However this conjecture can of course not be proved (or rejected) before the path integral has been given a mathematical rigorous definition.

Actually both in Witten's and in Reshetikhin and Turaev's theory one has an invariant for every positive integer k, the so-called quantum level. Let $\mathrm{W}_k(X)$ and $\mathrm{RT}_k(X)$ be respectively the Witten invariant and the Reshetikhin–Turaev invariant of X at level k, and assume that X is closed, i.e. without boundary. By using stationary phase approximation techniques on the Chern–Simons path integral $\mathrm{W}_k(X)$, one finds an asymptotic expansion for $\mathrm{W}_k(X)$ as a power series in $1/k$ (in [W] this asymptotic formula is only calculated in the case where the space of stationary points, the so-called moduli space of flat G–connections, is discrete). We emphasize here that these calculations inherently are nonrigorous, since they build on certain calculation rules the physicists use to handle the path integral. There is no such natural asymptotic expansion of $\mathrm{RT}_k(X)$ in the large k limit. However, if we believe in that the invariants of Witten and of Reshetikhin and Turaev are equal, there should be such an expansion also for $\mathrm{RT}_k(X)$. It is however not an easy task to find this asymptotic formula for a given closed oriented 3–manifold. The problem is that first one has to calculate $\mathrm{RT}_k(X)$ for every k and thereafter one has to rewrite this function of k in a way so that one finally can determine the large k asymptotics of that function. Until now this program has only been carried through for certain classes of 3–manifolds.

Chern–Simons gauge theory plays a major role in Witten's path integral approach. Not only does the Chern–Simons action play the role as the *Lagrangian* in the field theory behind the invariant, but the Chern–Simons action also plays a prominent role in the asymptotic formula. In Sect. 2 we therefore take up a rather detailed study of the Chern–Simons theory for 3–manifolds ending up with the construction of the hermitian Chern–Simons line bundle and the determination of the laws satisfied by the so-called Chern–Simons Lagrangian field theory (Theorem 2.15). These laws have formally great similarities to the axioms for an axiomatic topological quantum field theory (TQFT), and can be taken as the axioms for topological classical field theories. In Sect. 3 we continue the study of the Chern–Simons gauge theory. In this section we concentrate on the moduli space of flat G–connections, these flat connections being the classical solutions in the Chern–Simons field theory associated to G, that is the stationary points of the Chern–Simons action. These two sections are based on Freed's recent paper [F].

Sect. 4 is a motivating section with nonrigorous arguments. We briefly give the axioms for a TQFT and then proceed by saing a few words about the physicists way of obtaining a (topological) quantum field theory. We give some vague arguments for that Witten's approach leads to a TQFT. In this section we also conjecture an expression for the large k asymptotic expansion of Witten–Reshetikhin–Turaev's invariants of an arbitrary closed oriented 3–manifold.

Inspired by Jeffrey's work [J], Rozansky studies in [Ro] the case of 3–fibered and more generally n–fibered Seifert manifolds with base S^2. However, to justify Rozansky's method certain analytic estimates need to be done. In Sect. 5 we begin a thorough examination of Rozansky's paper. We make some minor corrections and prove that (some of) the methods he uses give exact results. We are not yet done with these calculations, and at the end of the section we make some comments about what still has to be done here.

The long term aim of this PhD project is to get as close as possible to an understanding of the relation between the path integral invariants of Witten and the combinatorial invariants of Reshetikhin and Turaev. One of the goals is to find an asymptotic formula working for all closed oriented 3–manifolds, which (maybe) coincide with the asymptotic formula predicted by the path-integral. It is then the hope that one can show that this asymptotic formula in itself is a 3–manifold invariant with deep topological information hitten in its coefficients. Finally it is the hope that this can give us a deeper understanding of the geometry and topology of 3–manifolds. The first main goal in this project will be to obtain a full asymptotic expansion of the SU(2) Reshetikhin–Turaev invariants of all Seifert manifolds.

I would here like to thank my supervisor Jørgen Ellegaard Andersen for suggesting this project and for many inspiring and instructive conversations.

2. Chern–Simons gauge theory

In this section we introduce the so-called classical Chern–Simons gauge theory. We will primarily be interested in the theory for 3–manifolds. We will follow the paper [F] closely and mainly give proofs for aspects which are not proved explicitly in that paper.

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. We let $\langle \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be an Ad–invariant symmetric bilinear form on $\mathfrak{g}$ and let $[\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be the Lie bracket. For a smooth manifold M and a vector bundle ξ we let $\Omega^0(\xi)$ be the smooth sections of ξ and $\Omega^k(M,\xi) = \Omega^0(\Lambda^k(T^*M) \otimes \xi)$ be the space of smooth k-forms on M with values in ξ. For the trivial bundle $\xi = M \times V$, $\Omega^k(M,\xi)$ is identified with the space $\Omega^k(M,V)$ of smooth k-forms with values in V. We write $\Omega^k(M)$ for $\Omega^k(M,\mathbb{R})$.

Now let $P \xrightarrow{\pi} X$ be a smooth principal G bundle over a smooth manifold X, in the following often just called a G bundle, and fix a connection Θ on P. For a vector space V and a fixed linear representation $\rho : G \to GL(V)$, $\omega \in \Omega^k(P,V)$ is called equivariant (with respect to ρ) if $R_g^*(\omega) = \rho(g^{-1})\omega$, where $R_g : P \to P$ is the right action of $g \in G$. If ρ is the trivial representation an equivariant form is called an invariant form. We say that ω is basic if ω is horizontal and equivariant with respect to the fixed representation. The basic k–forms on P with values in V are denoted $\Omega^k_{\mathrm{bas}}(P,V)$. If $V = \mathbb{R}$ we just write $\Omega^k_{bas}(P)$. The invariant horizontal forms on P with values in $\mathbb{R}$ are exactly the forms in the image of $\pi^* : \Omega^*(X) \to \Omega^*(P)$. Let Ω be the curvature of Θ and let $\langle \Omega \wedge \Omega \rangle \in \Omega^4(P)$ be the Chern-Weil 4–form associated with the bilinear form $\langle \cdot \rangle$. Since Ω is horizontal and $\langle \cdot \rangle$ is Ad–invariant, $\langle \Omega \wedge \Omega \rangle$ is an invariant horizontal form with values in $\mathbb{R}$, hence the lift of a 4–form on X, which we also denote $\langle \Omega \wedge \Omega \rangle$. The Chern–Simons form is an antiderivative of $\langle \Omega \wedge \Omega \rangle$ on P given by

$$\alpha(\Theta) = \langle \Theta \wedge \Omega \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta \wedge \Theta] \rangle \in \Omega^3(P). \tag{1}$$

A bundle map between G bundles $P \xrightarrow{\pi} X$ and $P' \xrightarrow{\pi'} X'$ is a smooth map $\varphi : P' \to P$ which commutes with the G action. Hence it induces a smooth map $\bar{\varphi} : X' \to X$ such that the diagram

$$\begin{array}{ccc} P' & \xrightarrow{\varphi} & P \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[\bar{\varphi}]{} & X \end{array}$$

commutes. If $X = X'$ and $\bar{\varphi}$ is the identity, φ is invertible and in that case we term φ a *morphism* of principal G bundles over X. If, in addition $P = P'$ then φ is termed an automorphism of P, or a *gauge transformation*. The group of gauge transformations on P is denoted $\mathcal{G}_P$. If $\varphi \in \mathcal{G}_P$ there is an associated smooth map $g_\varphi : P \to G$ defined by the equation

$$\varphi(p) = p \cdot g_\varphi(p). \tag{2}$$

The map g_φ satisfies $g_\varphi(p \cdot a) = a^{-1} g_\varphi(p) a$ for every $a \in G$ and $p \in P$. On the other hand every smooth map $g_\varphi : P \to G$, which satisfies this, defines an element $\varphi \in \mathcal{G}_P$ by (2). If $\varphi : P' \to P$ is a bundle map and Θ is a connection on P with curvature Ω, then $\varphi^*(\Theta)$ is a connection on P' with curvature $\varphi^*(\Omega)$. For a smooth manifold M and a smooth map $g : M \to G$ we let $\phi_g = g^*(\theta)$ where θ is the left invariant Maurer-Cartan 1-form on G. If $\varphi \in \mathcal{G}_P$ then

$$\varphi^*(\Theta) = Ad_{g_\varphi^{-1}} \Theta + \phi_{g_\varphi} \quad , \quad \varphi^*(\Omega) = Ad_{g_\varphi^{-1}} \Omega. \tag{3}$$

It is clear that for a bundle map $\varphi : P' \to P$ and a connection Θ on P then $\alpha(\varphi^*(\Theta)) = \varphi^*(\alpha(\Theta))$. If $\varphi \in \mathcal{G}_P$ with associated map $g_\varphi : P \to G$ and $\phi = \phi_{g_\varphi}$, then

$$\varphi^*(\alpha(\Theta)) = \alpha(\Theta) + d\langle Ad_{g_\varphi^{-1}}\Theta \wedge \phi\rangle - \frac{1}{6}\langle \phi \wedge [\phi \wedge \phi]\rangle. \tag{4}$$

To prove this identity one uses (3) and the cyclic permutation rule

$$\langle [\omega \wedge \tau] \wedge \delta\rangle = (-1)^{(k+l)q}\langle [\delta \wedge \omega] \wedge \tau\rangle, \tag{5}$$

which holds for $\omega \in \Omega^k(P,\mathfrak{g})$, $\tau \in \Omega^l(P,\mathfrak{g})$ and $\delta \in \Omega^q(P,\mathfrak{g})$.

From now on we assume that G is connected, simply connected and compact. We then have

Lemma 2.1. *Any principal G bundle over a manifold of dimension ≤ 3 is trivializable.*

Proof. By asumption $\pi_0 G = \pi_1 G = \pi_2 G = 0$. Let BG be a classifying space for G. By the long exact homotopy sequence induced by the fibration $G \to EG \to BG$ we get

$$\pi_i BG \cong \pi_{i-1} G, \quad i \geq 1,$$

since EG is contractible, which follows by the Milnor construction of EG, see [Hu]. Now if X is a manifold of dimension≤ 3, X is a CW-complex with all cells of dimension≤ 3. We therefore get that $[X, BG]$ is a space with only one point, where $[X, BG]$ is the space of homotopy classes of maps from X to BG. Since there is a one to one correspondance between the isomorphism classes of principal G bundles over X and $[X, BG]$ the result follows. □

Note that a principal G bundle $P \to X$ is trivializable if and only if it admits a global section $p : X \to P$: p defines a global trivialization $\varphi : X \times G \to P$, $\varphi(x,g) = p(x)\cdot g$, and a global trivialization $\varphi : X \times G \to P$ induces a global section $p : X \to P$, $p(x) = \varphi(x,1)$, where 1 is the unit in G. In the following we will omit the word global.

Suppose Θ is a connection on a principal G bundle $P \to X$, where X is a closed oriented 3–manifold. Let $p : X \to P$ be a section. Define

$$S_X(p,\Theta) := \int_X p^*\alpha(\Theta). \tag{6}$$

If $\varphi \in \mathcal{G}_P$ with associated map $g_\varphi : P \to G$, then $\varphi p : X \to P$ is also a section and

$$S_X(\varphi p,\Theta) = \int_X p^*\varphi^*(\alpha(\Theta)) = \int_X p^*(\alpha(\varphi^*(\Theta))) = S_X(p,\varphi^*(\Theta)).$$

From (4) and Stokes' theorem we get

$$S_X(\varphi p,\Theta) = \int_X p^*\varphi^*(\alpha(\Theta)) = S_X(p,\Theta) - \frac{1}{6}\int_X \langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle, \tag{7}$$

where $g = g_\varphi p : X \to G$. We now make the following integrality hypothesis on the bilinear symmetric form $\langle \cdot \rangle : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{R}$:

Hypothesis 2.2. Let θ be the Maurer-Cartan form on G. Then $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle$ represents an integral class in $H^3(G;\mathbb{R})$.

Remark 2.3. 1) The differential form $-\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle \in \Omega^3(G)$ is closed since

$$\begin{aligned} d\langle \theta \wedge [\theta \wedge \theta]\rangle &= \langle d\theta \wedge [\theta \wedge \theta]\rangle - \langle \theta \wedge d[\theta \wedge \theta]\rangle = \langle d\theta \wedge [\theta \wedge \theta]\rangle + 2\langle \theta \wedge [\theta \wedge d\theta]\rangle \\ &= 3\langle d\theta \wedge [\theta \wedge \theta]\rangle = -\frac{3}{2}\langle [\theta \wedge \theta] \wedge [\theta \wedge \theta]\rangle = 0. \end{aligned}$$

2) The hypothesis above has the consequence, that the last integral in (7) is an integer. To see this let M be an arbitrary compact oriented smooth n-manifold and let Λ be the integers or the real numbers. Let $H^k(M;\Lambda)$ and $H_k(M;\Lambda)$ be the k-th singular cohomology group and homology group respectively with coefficients in Λ. Let $H^k_{dR}(M;\mathbb{R})$ be the k-th de Rham cohomology group. We let $[M]_\Lambda \in H_n(M;\Lambda)$ be the fundamental class over Λ induced by the orientation on M. The inclusion map $j : \mathbb{Z} \to \mathbb{R}$ induces a chain map (between chain complexes over $\mathbb{Z}$)

$$1 \otimes j : S_*(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \to S_*(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} =: S_*(M;\mathbb{R}),$$

where $S_*(M;\Lambda)$ is the singular chain complex with coefficients in Λ. Note that $S_k(M;\Lambda)$ is the free module over Λ with basis the continuous maps from the standard k-simplex into M. Actually we get the same homology and cohomology groups by considering only the smooth maps from the k-simplex into M. The restriction to the smooth chains is necessary when we use de Rham's isomorphism below. We let $\beta_M = (1\otimes j)_* : H_k(M;\mathbb{Z}) \to H_k(M;\mathbb{R})$ be the induced homomorphism. Now let $S^*(M;\Lambda) = \mathrm{Hom}_\Lambda(S_*(M;\Lambda),\Lambda)$ be the singular cochain complex with coeffitients in Λ, and let

$$F_M : S^*(M;\mathbb{Z}) \to S^*(M;\mathbb{R})$$

be the natural cochain map (between cochain complexes over $\mathbb{Z}$) induced by the inclusion $j : \mathbb{Z} \to \mathbb{R}$. Explicitly $F_M(f)(\sigma \otimes x) = f(\sigma)x$. We let $\beta^1_M = F^*_M : H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{R})$ be the induced homomorphism. The evaluation paring $\langle\,\cdot\,\rangle_\Lambda : S^k(M;\Lambda) \otimes S_k(M;\Lambda) \to \Lambda$ induces the Kronecker pairing $\langle\,\cdot\,\rangle_\Lambda : H^k(M;\Lambda) \otimes H_k(M;\Lambda) \to \Lambda$, $\langle [f] \otimes [\sigma]\rangle_\Lambda = \langle f \otimes \sigma\rangle_\Lambda = f(\sigma)$ and we have a commutative diagram

$$\begin{array}{ccc} H^k(M;\mathbb{Z}) \otimes H_k(M;\mathbb{Z}) & \xrightarrow{\langle\,\cdot\,\rangle_{\mathbb{Z}}} & \mathbb{Z} \\ {\scriptstyle \beta^1_M \otimes \beta_M}\downarrow & & \downarrow{\scriptstyle j} \\ H^k(M;\mathbb{R}) \otimes H_k(M;\mathbb{R}) & \xrightarrow[\langle\,\cdot\,\rangle_{\mathbb{R}}]{} & \mathbb{R}. \end{array}$$

Note that $\beta_M([M]_{\mathbb{Z}}) = [M]_{\mathbb{R}}$. By de Rham's theorem we have an isomorphism of graded algebras $I : H^*_{dR}(M;\mathbb{R}) \to H^*(M;\mathbb{R})$ induced by the natural chain map $I : \Omega^*(M) \to S^*(M;\mathbb{R})$, $I(\omega)(\sigma) = \int_\sigma \omega$, $\omega \in \Omega^k(M)$, $\sigma \in S^k_\infty(M;\mathbb{R})$. The map I satisfies

$$\int_M \omega = \langle I([\omega]), [M]_{\mathbb{R}}\rangle_{\mathbb{R}}$$

for all closed $\omega \in \Omega^n(M)$. Now let $\omega = -\frac{1}{6}\langle \theta \wedge [\theta \wedge \theta]\rangle \in \Omega^3(G)$. By the above hypothesis there exists a $[\Omega] \in H^3(G;\mathbb{Z})$ such that $\beta^1_G([\Omega]) = I([\omega])$. Now $g^*(\omega) = -\frac{1}{6}\langle \phi_g \wedge [\phi_g \wedge \phi_g]\rangle$ and we get $\int_X g^*(\omega) = \langle I([g^*(\omega)]), [X]_{\mathbb{R}}\rangle_{\mathbb{R}}$. By naturality of I and β^1 we have $I([g^*(\omega)]) = g^*(I([\omega])) = g^* \circ \beta^1_G([\Omega]) = \beta^1_X \circ g^*([\Omega])$ so finally we get

$$\int_X g^*(\omega) = \langle \beta^1_X \circ g^*([\Omega]), \beta_X([X]_{\mathbb{Z}})\rangle_{\mathbb{R}} = j(\langle g^*([\Omega]), [X]_{\mathbb{Z}}\rangle_{\mathbb{Z}}) \in \mathbb{Z}.$$

3) In our applications G will be simple. In that case the Lie algebra $\mathfrak{g}$ is simple and an Ad–invariant symmetric bilinear form on $\mathfrak{g}$ is a multiple of the killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$. Here $\kappa(x \otimes y) = \mathrm{tr}(ad(x)ad(y))$, and in the case of simple Lie groups it is costum to write $\mathrm{tr}(\cdot)$ or $\lambda\mathrm{tr}(\cdot)$ for a $\lambda \in \mathbb{R}$ instead of our $\langle\,\cdot\,\rangle$.

Since G is connected and simply connected with $\pi_2(G) = 0$ it follows by Hurewicz that $\pi_3(G,g) \cong H_3(G,g;\mathbb{Z})$ for an arbitrary $g \in G$ and that $H_q(G,g;\mathbb{Z}) = 0$ for $q = 1,2$. By the long exact homology sequence for the pair (G,g) we have $H_q(G,g;\mathbb{Z}) \cong H_q(G;\mathbb{Z})$ for $q \geq 2$. Now assume that $H_3(G;\mathbb{Z}) \cong \mathbb{Z}$. Then $H_q(G;\mathbb{R}) \cong H_q(G;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ by the universal coefficient theorem, since

$\mathbb{R}$ is a torsion free $\mathbb{Z}$-module and the singular chain complex is free. But then $H_3(G;\mathbb{R}) \cong \mathbb{R}$, and $H_2(G;\mathbb{R}) = 0$, which implies that $H^3(G;\mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(H_3(G;\mathbb{R}),\mathbb{R}) \cong \mathbb{R}$. If $x = -\frac{1}{6}\langle\theta \wedge [\theta \wedge \theta]\rangle \in \mathbb{R} \setminus \mathbb{Z}$ under this isomorphism we let $\langle\ \cdot\ \rangle_1 = \frac{1}{x}\langle\ \cdot\ \rangle$. Then $-\frac{1}{6}\langle\theta \wedge [\theta \wedge \theta]\rangle_1$ represents an integral class in $H^3(G;\mathbb{R})$. It is known that $\pi_3(G) \cong \mathbb{Z}$ if G is a connected, simply connected compact simple Lie group, so in that case we can always choose our form $\langle\ \cdot\ \rangle$ so that Hypothesis 2.2 is satisfied.

We will now make some comments on the space of connections. For a G bundle $P \xrightarrow{\pi} X$ we let $\mathcal{A}_P$ be the space of connection 1-forms on P. $\mathcal{A}_P$ is an affine subspace of $\Omega^1(P,\mathfrak{g})$. If Θ_0 is a fixed connection then $\mathcal{A}_P = \Theta_0 + \Omega^1_{bas}(P,\mathfrak{g})$. Here $\Omega^1_{bas}(P,\mathfrak{g})$ is isomorphic with $\Omega^1(X,\mathfrak{g}_P)$ as vector spaces, where $\mathfrak{g}_P = P \times_G \mathfrak{g}$ is the adjoint bundle. We can therefore make $\mathcal{A}_P$ a smooth manifold modelled on the (pre-) Hilbert space $\Omega^0(T^*M \otimes \mathfrak{g}_P) \cong \Omega^1(X,\mathfrak{g}_P)$ equipped with an appropriate Sobolev norm. We will not bother with technicalities regarding the smooth structure of infinite dimensional manifolds here. Rather, we will consider smooth families of connections parametrized by a smooth manifold, which we will take to be finite dimensional. Let U be a smooth finite dimensional manifold. A family $\{\Theta_u\}_{u\in U} \in \mathcal{A}_P$ induces a 1-form Θ on the G bundle $U \times P \xrightarrow{id_U \times \pi} U \times X$ by

$$\Theta_{(u,p)}(v,w) = (\Theta_u)_p(w).$$

It is easy to see that Θ is equivariant and that $\Theta(A^*) = A$ for all $A \in \mathfrak{g}$, where A^* is the fundametal vector field on $U \times P$ associated to A. We will say that $\{\Theta_u\}_{u\in U}$ is a smooth family if Θ is smooth, hence a connection 1-form on $U \times P$. In that case we will call Θ the associated connection to the smooth family. Note that $i_u^*(\Theta) = \Theta_u$ for all $u \in U$, where $i_u : P \to U \times P$, $i_u(p) = (u,p)$. We shall say that a function $f : \mathcal{A}_P \to M$ into a finite dimensional smooth manifold M is smooth (with respect to smooth families of connections) if $f_U : U \to M$ given by $f_U(u) = f(\Theta_u)$ is smooth for any smooth family $\{\Theta_u\}_{u\in U} \in \mathcal{A}_P$.

If X is manifold of dimension≤ 3 we have seen that all G bundles over X are trivializable, and at a first glance it should be enough to consider $\mathcal{A}_P$ for a fixed G bundle $P \to X$. The gauge group $\mathcal{G}_P$ then plays the role as the *symmetries* of the Chern–Simons theory. As we shall see, the Chern–Simons action defined below is invariant under these symmetries. However we need a more detailed way of considering symmetries in the Chern–Simons theory, since there is no canonical identification of two isomorphic G bundles with connections. The need for extra symmetries is particular important when we glue together bundles and connections as the following result from [F] shows.

Proposition 2.4. *Suppose $P \to X$ is a G bundle over an oriented manifold X and let Y be an oriented codimension one submanifold of X. Let X^{cut} be the manifold obtained by cutting X along Y. There is a gluing map $\bar{g} : X^{cut} \to X$ which is a diffeomorphism off of Y and maps distinct submanifolds Y_1,Y_2 of ∂X^{cut} diffeomorphically onto Y. Let $P^{cut} = \bar{g}^*(P)$ be the cut bundle and $g : P^{cut} \to P$ the induced gluing map. Now suppose Θ^{cut} is a connection on P^{cut} such that there exists a connection η on $P|_Y$ with $g^*(\eta) = \Theta^{cut}|_{P_1^{cut} \sqcup P_2^{cut}}$, where P_i^{cut} is the restriction of P^{cut} to Y_i. Then η extends to a connection Θ on P such that $g^*(\Theta)$ is gauge equivalent to Θ^{cut}. Θ is smooth on $P|_{X\setminus Y}$ and continuous on all of P. If Θ' is another connection on P which extends η and such that $g^*(\Theta')$ is gauge equivalent to Θ^{cut} then Θ and Θ' are gauge equivalent.* □

Note that the gauge equivalence class of the glued connection depends on an identification of the boundary connections. As there is no canonical identification, we need to keep track of all possible identifications. This is done in the following way. Let $\mathcal{C}_X^G$ be the category which objects are connections on principal G bundles over X. A morphism $\Theta' \xrightarrow{\varphi} \Theta$ is a bundle map $\varphi : P' \to P$ covering id_X (i.e. a bundle morphism) such that $\Theta' = \varphi^*(\Theta)$. The symmetries in the Chern–Simons theory are thus described by the morphisms of the category $\mathcal{C}_X$.

We will say that two connections Θ and Θ' are equivalent if and only if there exists a morphism $\Theta' \xrightarrow{\varphi} \Theta$. We denote the set of equivalence classes by $\overline{\mathcal{C}_X^G}$. Note that this is actually a set. In the following, we will suppress G from the notation and write $\mathcal{C}_X$ and $\overline{\mathcal{C}_X}$ when no confusion can arise. We will only be interested in the smooth category, that is smooth connections on smooth bundles and smooth bundle morphisms. The above description is only used to keep track of the additional symmetries in certain situations such as when we glue together bundles and connections. However the objects in $\mathcal{C}_X$ does not form a set and when we consider smoothness of maps defined on connections, we will use the more commen set up by considering $\mathcal{A}_P$ and the orbit space $\mathcal{A}_P/\mathcal{G}_P$ after fixing a G bundle $P \to X$. ($\mathcal{G}_P$ acts on $\mathcal{A}_P$ by pull back, see (2).) Note in this connection that if $\{P_i\}$ is a set of representatives of isomorphism classes of smooth G bundles over X, then there is a bijection

$$\overline{\mathcal{C}_X} \cong \bigsqcup_{P_i} \mathcal{A}_{P_i}/\mathcal{G}_{P_i}.$$

By our choise of G, we have in particular $\overline{\mathcal{C}_X} \cong \mathcal{A}_P/\mathcal{G}_P$ for an arbitrary G bundle P over a manifold X of dimension$\leq$3.
Under the Hypothesis 2.2 we can make the following

Definition 2.5. Let X be a closed oriented 3–manifold. Then there is a map

$$S_X : \mathcal{C}_X \to \mathbb{R}/\mathbb{Z}$$

given by $S_X(\Theta) = S_X(p, \Theta) \pmod 1$, where Θ is a connection on a G bundle $P \to X$, and $p : X \to P$ is an arbitrary section. This map is the Chern–Simons action on X.

We then have the following easy result from [F].

Proposition 2.6. *Let X be a closed oriented 3–manifold. The Chern–Simons action $S_X : \mathcal{C}_X \to \mathbb{R}/\mathbb{Z}$ satisfies*

i) *If $P \to X$ is a smooth G bundle then $S_X : \mathcal{A}_P \to \mathbb{R}/\mathbb{Z}$ is smooth.*

ii) *(Functoriality) If $\varphi : P' \to P$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\varphi} : X' \to X$, and Θ is a connection on P then*
$$S_{X'}(\varphi^*\Theta) = S_X(\Theta).$$

iii) *(Orientation) Let $-X$ denote X with the opposite orientation. Then*
$$S_{-X}(\Theta) = -S_X(\Theta).$$

iv) *(Additivity) If $X = X_1 \sqcup X_2$ is a disjoint union, and Θ_i are connections over X_i, then*
$$S_X(\Theta_1 \sqcup \Theta_2) = S_{X_1}(\Theta_1) + S_{X_2}(\Theta_2).$$

□

From ii) we get an induced action $S_X : \overline{\mathcal{C}_X} \to \mathbb{R}/\mathbb{Z}$ on the fields modulo symmetries.

If our 3–manifold is the boundary of an oriented compact 4–manifold the Chern–Simons action can be computed by integrating the Chern-Weil 4–form. Precisely we have

Proposition 2.7. *Assume that W is a 4–manifold with nonempty boundary ∂W and $\tilde{\Theta}$ is a connection on a trivializable G bundle $\tilde{P} \xrightarrow{\tilde{\pi}} W$. Let $\tilde{\Omega}$ be the curvature of $\tilde{\Theta}$. Then*

$$S_{\partial W}(\partial\tilde{\Theta}) = \int_W \langle \tilde{\Omega} \wedge \tilde{\Omega} \rangle \qquad \pmod 1,$$

where $\partial\tilde{\Theta}$ is the restriction of $\tilde{\Theta}$ to $\partial\tilde{P} = \tilde{P}|_{\partial W}$.

This immediately follows from Stokes' theorem and the fact that $d\alpha(\tilde{\Theta}) = \langle \tilde{\Omega} \wedge \tilde{\Omega} \rangle$.

Next we generalies to compact oriented 3–manifolds X with boundary. Suppose Θ is a connection on the principal G bundle $P \to X$. We define $S_X(p, \Theta)$ as before for a section $p : X \to P$, see (6). Exactly as we proved (7) we get

Proposition 2.8. *Let $\varphi \in \mathcal{G}_P$ be a gauge transformation with associated map $g_\varphi : P \to G$ and let $g = g_\varphi p : X \to G$. Then*

$$\begin{aligned} S_X(\varphi p, \Theta) &= S_X(p, \varphi^*(\Theta)) \\ &= S_X(p, \Theta) + \int_{\partial X} j^* \langle Ad_{g^{-1}} p^* \Theta \wedge \phi_g \rangle + \int_X -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle \end{aligned}$$

where $j : \partial X \to X$ is the inclusion. □

If $\partial X \neq \emptyset$ it is no longer true that the last two terms in the above expression vanish modulo integers. To handle the situation we need the so-called Wess–Zumino–Witten functional associated to closed oriented 2–manifolds. For manifolds M and N we let $\mathrm{Map}_\infty(M, N)$ be the set of smooth maps from M into N.

Definition 2.9. Let Y be a closed oriented 2–manifold. Then the Wess–Zumino–Witten functional $W_Y : Map_\infty(Y, G) \to \mathbb{R}/\mathbb{Z}$ is defined by

$$W_Y(g) = \int_X -\frac{1}{6} \langle \phi_{\tilde{g}} \wedge [\phi_{\tilde{g}} \wedge \phi_{\tilde{g}}] \rangle \qquad (\mathrm{mod}\ 1),$$

where X is an arbitrary compact oriented smooth 3–manifold with boundary Y and $\tilde{g} : X \to G$ is an arbitrary smooth extension of $g : Y \to G$ to X. For a smooth 3–manifold X, $\omega : Map_\infty(X, G) \to \Omega^3(X)$ defined by $\omega(g) = -\frac{1}{6} \langle \phi_g \wedge [\phi_g \wedge \phi_g] \rangle$ is the Wess–Zumino–Witten lagrangian.

Remark. If X is a compact oriented manifold with boundary $\partial X = Y$, it is always possible to extend $g : Y \to G$ to X for our choises of G. Since the closed oriented 2–manifolds up to diffeomorphism are the 2–sphere or a connected sum of tori, every closed oriented 2–manifold is the boundary of a compact oriented 3–manifold.

To secure that the Wess–Zumino–Witten functional is well defined we prove

Lemma 2.10. *The Wess–Zumino–Witten functional $W_Y(g)$ is independent of the choise of X with $\partial X = Y$ and the extension $\tilde{g} : X \to G$ of $g : Y \to G$.*

Proof. Let X and X' be two compact oriented 3–manifolds and let $g : X \to G$ and $g' : X' \to G$ be smooth maps and assume that we have an orientation reversing diffeomorphism $\psi : \partial X \to \partial X'$ such that $g'|_{\partial X'} \circ \psi = g|_{\partial X}$. Let $\kappa : [0, 1) \times \partial X \to X$ and $\kappa' : [0, 1) \times \partial X' \to X'$ be collars. Then $N = \mathrm{Im}(\kappa)$ and $N' = \mathrm{Im}(\kappa')$ are open neighbourhoods of ∂X in X and of $\partial X'$ in X' respectively. Let $\epsilon \in (0, 1)$ and let $\delta_\epsilon : [0, 1) \to [0, 1)$ be a monotone increasing smooth function with $\delta_\epsilon(t) = 0$, $t \in [0, \epsilon/2]$ and $\delta_\epsilon(t) = t$, $t \in [\epsilon, 1)$. Define smooth functions $g_\epsilon : X \to G$ and $g'_\epsilon : X' \to G$ by

$$\begin{aligned} g_\epsilon(\kappa(t, x)) &= g(\kappa(\delta_\epsilon(t), x)) \quad , \ (t, x) \in [0, 1) \times \partial X, \\ g'_\epsilon(\kappa(t, x')) &= g'(\kappa'(\delta_\epsilon(t), x')) \quad , \ (t, x') \in [0, 1) \times \partial X', \end{aligned}$$

and by letting $g_\epsilon = g$ on $X \backslash N$ and $g'_\epsilon = g'$ on $X' \backslash N'$. Let $-X'$ be X' with the opposite orientation, and let $Z = X \sqcup -X'$ be the disjoint union. Let $\beta : [0, 1) \times \partial Z \to Z$ be a collar for Z and let $\bar{Z} = Z/_{x \sim \psi(x)}$ be the quotient space. We define a map $\tau : \partial Z \to \partial Z$ by $\tau|_{\partial X} = \psi$ and $\tau|_{\partial X'} = \psi^{-1}$.

Then there exists exactly one differentiabel structure $\mathcal{F}_\beta$ on $\bar{Z}$ such that the inclusion $Z \setminus \partial Z \to \bar{Z}$ and $h_\beta : (-1,1) \times \partial Z/_{(t,p)\sim(-t,\tau(p))} \to \bar{Z}$ given by

$$h_\beta([t,p]) = \begin{cases} [\beta(t,p)] & ,t \geq 0 \\ [\beta(-t,\tau(p))] & ,t \leq 0 \end{cases}$$

are embeddings. Here the projection $\pi : (-1,1) \times \partial Z \to (-1,1) \times \partial Z/_{(t,p)\sim(-t,\tau(p))}$ is topologically a two leaved covering and $(-1,1) \times \partial Z/_{(t,p)\sim(-t,\tau(p))}$ is given the unique differentiabel structure such that π is a local diffeomorphism. For any two collars β_1, β_2 for Z the manifolds $(\bar{Z}, \mathcal{F}_{\beta_1})$ and $(\bar{Z}, \mathcal{F}_{\beta_2})$ are diffeomorphic. We define $L = X \cup_\psi -X'$ to be $\bar{Z}$ with this differentiabel structure. Now define $\tilde{g}_\epsilon : L \to G$ by

$$\tilde{g}_\epsilon([p]) = \begin{cases} g_\epsilon(p) & ,p \in X \\ g'_\epsilon(p) & ,p \in X'. \end{cases}$$

Since $\kappa(0,x) = x$, $x \in \partial X$, and $\kappa'(0,x') = x'$, $x' \in X'$ we get $g'_\epsilon(\psi(x)) = g_\epsilon(x)$ for $x \in \partial X$ so $\tilde{g}_\epsilon$ is well defined and continuous on all of L. Now let $\beta : [0,1) \times \partial Z \to Z$ be the collar defined by κ and κ'. By using that the inclusion $Z \setminus \partial Z \to L$ and h_β are embeddings with open images in L we immediately get that $\tilde{g}_\epsilon : L \to G$ is smooth. Here we use that for $x \in \partial X$

$$\tilde{g}_\epsilon \circ h_\beta([t,x]) = \begin{cases} g(\kappa(\delta_\epsilon(t),x))) & ,t \geq 0 \\ g'(\kappa'(\delta_\epsilon(-t),\psi(x))) & ,t \leq 0, \end{cases}$$

so in particular

$$\tilde{g}_\epsilon \circ h_\beta([t,x]) = \begin{cases} g(x) & ,0 \leq t \leq \epsilon/2 \\ g'(\psi(x)) = g(x) & ,-\epsilon/2 \leq t \leq 0. \end{cases}$$

Since L is an oriented closed 3–manifold we have $\int_L (\tilde{g}_\epsilon)^*(\omega) \in \mathbb{Z}$ for all $\epsilon \in (0,1)$ by Remark 2.3, so $\lim_{\epsilon\to 0} \int_L (\tilde{g}_\epsilon)^*(\omega) \in \mathbb{Z}$ (note also that since L is closed $\int_L (\tilde{g}_\epsilon)^*(\omega)$ only depends on the cohomology class of $(\tilde{g}_\epsilon)^*(\omega)$ which is independent of ϵ since all the maps $\tilde{g}_\epsilon$ are homotopic). Now we have

$$\int_X (g_\epsilon)^*(\omega) - \int_{X'} (g'_\epsilon)^*(\omega) = \int_X (g_\epsilon)^*(\omega) + \int_{-X'} (g'_\epsilon)^*(\omega) = \int_L (\tilde{g}_\epsilon)^*(\omega)$$

since the inclusion $Z \setminus \partial Z \to L$ is an embedding and since $\tilde{g}_\epsilon$ coincide with g_ϵ on $X \setminus \partial X$ and with g'_ϵ on $-X' \setminus \partial(-X')$, and since the rest of L has Lebesgue measure zero. By Lebesgue's dominated convergence theorem it follows that

$$\begin{aligned} & \int_X g^*(\omega) - \int_{X'} (g')^*(\omega) = \int_X g^*(\omega) + \int_{-X'} (g')^*(\omega) \\ = & \lim_{\epsilon\to 0} \left(\int_X (g_\epsilon)^*(\omega) + \int_{-X'} (g'_\epsilon)^*(\omega) \right) = \lim_{\epsilon\to 0} \int_L (\tilde{g}_\epsilon)^*(\omega) \in \mathbb{Z} \end{aligned}$$

since $g = g_\epsilon$ on $X \setminus (\kappa([0,\epsilon] \times \partial X)$ and $g' = g'_\epsilon$ on $X' \setminus (\kappa'([0,\epsilon] \times \partial X')$. □

To establish the Chern–Simons line bundle we will need the following lemma giving some properties of the Wess–Zumino–Witten lagrangian and functional.

Lemma 2.11.

i) *Let M be a smooth manifold. For $g, g_1, g_2 \in \mathrm{Map}_\infty(M,G)$ we have*

(8) $$\phi_{g_1 g_2} = Ad_{g_2^{-1}} \phi_{g_1} + \phi_{g_2} \ , \ \ \phi_{g^{-1}} = -Ad_g \phi_g.$$

ii) *Let X be a smooth (3–)manifold. The Wess–Zumino–Witten lagrangian satisfies*

(9) $$\omega(g_1 g_2) = \omega(g_1) + \omega(g_2) + d\sigma(g_1,g_2)$$

for $g_1, g_2 \in \mathrm{Map}_\infty(X,G)$, *where* $\sigma(g_1,g_2) = \langle Ad_{g_2^{-1}}\phi_{g_1} \wedge \phi_{g_2}\rangle$.

iii) *For a closed oriented 2–manifold* Y *the Wess–Zumino–Witten functional satisfies*

$$W_Y(g_1g_2) = W_Y(g_1) + W_Y(g_2) + (\int_Y \nu(\phi_{g_1}, g_2) \pmod 1) \tag{10}$$

for $g_1, g_2 \in \mathrm{Map}_\infty(Y,G)$, *where* $\nu(\tau, g) = \langle Ad_{g^{-1}}\tau \wedge \phi_g\rangle$ *for* $g \in \mathrm{Map}_\infty(Y,G)$ *and* $\tau \in \Omega^1(Y,\mathfrak{g})$. *For* $g \in \mathrm{Map}_\infty(Y,G)$ *we have*

$$W_Y(g^{-1}) = -W_Y(g). \tag{11}$$

Here $g_1g_2(x) = g_1(x)g_2(x)$.

Proof. i) Let $g = g_1g_2$. Then

$$(dg)_x(v) = (dR_{g_2(x)})_{g_1(x)}((dg_1)_x(v)) + (dL_{g_1(x)})_{g_2(x)}((dg_2)_x(v))$$

for $x \in X$ and $v \in T_xX$, and therefore

$$g^*(\theta)_x(v) = (dL_{g(x)^{-1}})_{g(x)}((dg)_x(v)) = (Ad_{g_2^{-1}}g_1^*(\theta) + g_2^*(\theta))_x(v).$$

This proves the first identity in (8). Now use this and $gg^{-1} \equiv 1$ to get $0 = \phi_{gg^{-1}} = Ad_g\phi_g + \phi_{g^{-1}}$.
ii) Let $g = g_1g_2$. By the Ad–invariance of $\langle\,\cdot\,\rangle$ and the first identity in (8) we get

$$\omega(g) = -\frac{1}{6}\langle\phi_g \wedge [\phi_g \wedge \phi_g]\rangle = \omega(g_1) + \omega(g_2) + \tau(g_1,g_2),$$

where

$$\tau(g_1,g_2) = -\frac{1}{2}\langle\phi_{g_2} \wedge [Ad_{g_2^{-1}}\phi_{g_1} \wedge Ad_{g_2^{-1}}\phi_{g_1}]\rangle - \frac{1}{2}\langle Ad_{g_2^{-1}}\phi_{g_1} \wedge [\phi_{g_2} \wedge \phi_{g_2}\rangle.$$

Here we have used (5). Now since $Ad_{g_2^{-1}}\phi_{g_1} = \phi_g - \phi_{g_2}$ and $d\phi_g = -\frac{1}{2}[\phi_g, \phi_g]$ and equivalently for ϕ_{g_2} we get that

$$\begin{aligned} d\langle Ad_{g_2^{-1}}\phi_{g_1} \wedge \phi_{g_2}\rangle &= \langle d(\phi_g - \phi_{g_2}) \wedge \phi_{g_2} - Ad_{g_2^{-1}}\phi_{g_1} \wedge d\phi_{g_2}\rangle \\ &= -\frac{1}{2}\langle[\phi_g \wedge \phi_g] \wedge \phi_{g_2}\rangle + \frac{1}{2}\langle[\phi_{g_2} \wedge \phi_{g_2}] \wedge \phi_{g_2}\rangle \\ &\quad + \frac{1}{2}\langle Ad_{g_2^{-1}}\phi_{g_1} \wedge [\phi_{g_2} \wedge \phi_{g_2}]\rangle, \end{aligned}$$

and finally by using the first identity in (8) once more we get

$$d\langle Ad_{g_2^{-1}}\phi_{g_1} \wedge \phi_{g_2}\rangle = \tau(g_1,g_2).$$

iii) Let X be an oriented compact 3–manifold with boundary Y and let $\tilde{g_1}, \tilde{g_2} : X \to G$ be smooth extensions of g_1 and g_2. By ii) and Stokes' theorem we get

$$\begin{aligned} W_Y(g_1g_2) &= W_Y(g_1) + W_Y(g_2) + (\int_Y j^*\langle Ad_{\tilde{g}_2^{-1}}\phi_{\tilde{g_1}} \wedge \phi_{\tilde{g_2}}\rangle \pmod 1) \\ &= W_Y(g_1) + W_Y(g_2) + (\int_Y \langle Ad_{g_2^{-1}}\phi_{g_1} \wedge \phi_{g_2}\rangle \pmod 1), \end{aligned}$$

where $j : Y \to X$ is the inclusion. Now let $g \in \mathrm{Map}_\infty(Y,G)$ and let $\tilde{g} : X \to G$ be a smooth extension to X. Since $\tilde{g}\tilde{g}^{-1} \equiv 1$ we have $\omega(\tilde{g}\tilde{g}^{-1}) = 0$ so $W_Y(gg^{-1}) = 0$. By the above result we then have

$$W_Y(g) + W_Y(g^{-1}) + (\int_Y \langle Ad_g\phi_g \wedge \phi_{g^{-1}}\rangle \pmod 1) = 0.$$

But $\langle Ad_g\phi_g \wedge \phi_{g^{-1}}\rangle = -\langle Ad_g\phi_g \wedge Ad_g\phi_g\rangle = 0.$ $\square$

The Chern–Simons action on X can be interpreted as a section of a hermitian line bundle. We make this precise in the following. We will follow [F]. Let $\mathcal{L}$ be the category whose objects are hermitian lines and whose morphisms are unitary isomorphisms, and let $\mathcal{G}$ be a groupoid, i.e. a category for which every morphism is invertible. Suppose we have a functor $\mathcal{F} : \mathcal{G} \to \mathcal{L}$. We now associate to $\mathcal{F}$ a certain set $V_{\mathcal{F}}$ termed the space of invariant sections of $\mathcal{F}$. An element $v \in V_{\mathcal{F}}$ is a collection $\{v(C) \in \mathcal{F}(C)\}_{C\in \mathrm{Obj}(\mathcal{G})}$ such that if $C_1 \xrightarrow{\psi} C_2$ is a morphism in $\mathcal{G}$ then $\mathcal{F}(\psi)v(C_1) = v(C_2)$. We say the category $\mathcal{G}$ is connected if $\mathrm{Mor}(C_1, C_2) \neq \emptyset$ for all $C_1, C_2 \in \mathrm{Obj}(\mathcal{G})$. If $\mathcal{F}(\psi) = id$ for every automorphism $C \xrightarrow{\psi} C$ we say that the functor $\mathcal{F}$ has no holonomy. For an explanation of this terminology, see [F]. $V_{\mathcal{F}}$ is a vector space over $\mathbb{C}$ in an obvious way. If $\mathcal{G}$ is connected we have

i) $\dim_{\mathbb{C}} V_{\mathcal{F}} \in \{0, 1\}$,
ii) $\dim_{\mathbb{C}} V_{\mathcal{F}} = 1$ if and only if the functor $\mathcal{F}$ has no holonomy.

Now let Y be a closed oriented 2–manifold and let $Q \to Y$ be a G bundle. Note that $Q \to Y$ is trivializable by Lemma 2.1. Let $\mathcal{C}_Q$ be the category whose objects are sections $q : Y \to Q$. For any two sections q, q' there is a unique gauge transformation $\psi : Q \to Q$ such that $q' = \psi q$, namely $\psi = \delta_{q'} \circ \delta_q^{-1}$ where $\delta_s : Y \times G \to Q$, $\delta_s(y, g) = s(y)g$ is the trivialization associated to the section $s : Y \to Q$. The set of morphisms between q and q' in $\mathcal{C}_Q$ is $\{\psi\}$. We see immediately that $\mathcal{C}_Q$ is a connected groupoid.

Fix a connection η on Q. Define the functor $\mathcal{F}_\eta : \mathcal{C}_Q \to \mathcal{L}$ in the following way: for all sections $q : Y \to Q$ we let $\mathcal{F}_\eta(q) = \mathbb{C}$, where $\mathbb{C}$ has its standard hermitian metric. $\mathcal{F}_\eta(q \xrightarrow{\psi} q') : \mathbb{C} \to \mathbb{C}$ is multiplication by $c_Y(q^*\eta, g_\psi q)$ where $g_\psi : Q \to G$ is the associated map to ψ and c_Y is the cocycle defined by

$$c_Y(\tau, g) = \exp(2\pi i[\int_Y \nu(\tau, g) + W_Y(g)]) \qquad , \tau \in \Omega^1(Y, \mathfrak{g}),\ g \in \mathrm{Map}_\infty(Y, G).$$

Here $W_Y : \mathrm{Map}_\infty(Y, G) \to \mathbb{R}/\mathbb{Z}$ is the Wess–Zumino–Witten functional and $\nu(\tau, g) = \langle Ad_{g^{-1}}\tau \wedge \phi_g\rangle$. We have to show that $\mathcal{F}_\eta$ is actually a functor. To this end we need the following

Lemma 2.12. *The cocycle c_Y satisfies the following cocycle identity*

$$c_Y(\tau, g_1g_2) = c_Y(\tau, g_1)c_Y(\tau^{g_1}, g_2) \tag{12}$$

for $g_1, g_2 \in \mathrm{Map}_\infty(Y, G)$ and $\tau \in \Omega^1(Y, \mathfrak{g})$. Here $\tau^g = Ad_{g^{-1}}\tau + \phi_g$ for $g \in \mathrm{Map}_\infty(Y, G)$.

Proof. We have

$$\begin{aligned} \nu(\tau, g_1g_2) &= \langle Ad_{(g_1g_2)^{-1}}\tau \wedge \phi_{g_1g_2}\rangle = \langle Ad_{g_2^{-1}} \circ Ad_{g_1^{-1}}\tau \wedge (Ad_{g_2^{-1}}\phi_{g_1} + \phi_{g_2})\rangle \\ &= \langle Ad_{g_1^{-1}}\tau \wedge \phi_{g_1}\rangle + \langle Ad_{g_2^{-1}} \circ Ad_{g_1^{-1}}\tau \wedge \phi_{g_2}\rangle \\ &= \nu(\tau, g_1) + \nu(Ad_{g_1^{-1}}\tau, g_2), \end{aligned}$$

so by using Lemma 2.11 we get

$$\begin{aligned} c_Y(\tau, g_1g_2) &= \exp(2\pi i[\int_Y \nu(\tau, g_1g_2) + W_Y(g_1g_2)]) \\ &= c_Y(\tau, g_1)\exp(2\pi i[\int_Y \nu(Ad_{g_1^{-1}}\tau, g_2) + \int_Y \nu(\phi_{g_1}, g_2) + W_Y(g_2)]) \\ &= c_Y(\tau, g_1)\exp(2\pi i[\int_Y \nu(\tau^{g_1}, g_2) + W_Y(g_2)]) = c_Y(\tau, g_1)c_Y(\tau^{g_1}, g_2) \end{aligned}$$

since $\nu(\tau_1, g) + \nu(\tau_2, g) = \nu(\tau_1 + \tau_2, g)$. □

Corollary 2.13. *The category map $\mathcal{F}_\eta : \mathcal{C}_Q \to \mathcal{L}$ is a covariant functor.*

Proof. Let $q \stackrel{\psi_1}{\to} q_1$ and $q_1 \stackrel{\psi_2}{\to} q_2$ be two morphisms, i.e. $\psi_1, \psi_2 : Q \to Q$ are gauge transformations with $q_1 = \psi_1 q$ and $q_2 = \psi_2 q_1$. Let $g_\nu : Q \to G$ be the associated map to ψ_ν and let $g : Q \to G$ be the associated map to $\psi = \psi_2 \circ \psi_1$. Note that ψ is our morphism from q to q_2 and that $g = g_2 g_1$. We have to show that

$$c_Y(q^*\eta, gq) = c_Y(q_1^*\eta, g_2 q_1) c_Y(q^*\eta, g_1 q).$$

Since $gq = (g_2 q)(g_1 q)$ we have by the cocycle identity (12), that

$$\begin{aligned} c_Y(q^*\eta, gq) &= c_Y(q^*\eta, g_2 q) c_Y((q^*\eta)^{(g_2 q)}, g_1 q) \\ &= \exp(2\pi i(W_Y(g_2 q) + W_Y(g_1 q))) \exp(2\pi i \int_Y M), \end{aligned}$$

where

$$M = \langle Ad_{(g_2 q)^{-1}} q^*\eta \wedge \phi_{g_2 q}\rangle + \langle Ad_{(g_1 q)^{-1}} Ad_{(g_2 q)^{-1}} q^*\eta \wedge \phi_{g_1 q}\rangle + \langle Ad_{(g_1 q)^{-1}} \phi_{g_2 q} \wedge \phi_{g_1 q}\rangle.$$

Now note that

$$\begin{aligned} q_1^*\eta &= (\psi_1 q)^*\eta = q^*(\psi_1^*\eta) = q^*(Ad_{g_1^{-1}}\eta + \phi_{g_1}) = Ad_{(g_1 q)^{-1}} q^*\eta + \phi_{g_1 q}, \\ g_2 q_1 &= (g_1 q)^{-1}(g_2 q)(g_1 q). \end{aligned}$$

By the cocycle identity (12) we then get

$$\begin{aligned} c_Y(q_1^*\eta, g_2 q_1) &= c_Y(Ad_{(g_1 q)^{-1}} q^*\eta + \phi_{(g_1 q)^{-1}}, (g_1 q)^{-1}) \\ &\quad \cdot c_Y(q^*\eta + Ad_{(g_1 q)}\phi_{(g_1 q)} + \phi_{(g_1 q)^{-1}}, g_2 q) \\ &\quad \cdot c_Y(Ad_{(g_1 q)^{-1}} q^*\eta + Ad_{(g_2 q)^{-1}} Ad_{(g_1 q)}\phi_{(g_1 q)} + \phi_{(g_1 q)^{-1}(g_2 q)}, g_1 q). \end{aligned}$$

By using (11) in Lemma 2.11 we therefore find that

$$c_Y(q_1^*\eta, g_2 q_1) c_Y(q^*\eta, g_1 q) = \exp(2\pi i(W_Y(g_1 q) + W_Y(g_2 q))) \exp(2\pi i \int_Y L),$$

where a straight forward calculation shows that $L = M$, using

$$\phi_{(g_1 q)^{-1}(g_2 q)} = Ad_{(g_2 q)^{-1}} \phi_{(g_1 q)^{-1}} + \phi_{g_2 q} = -Ad_{(g_2 q)^{-1}} Ad_{g_1 q} \phi_{g_1 q} + \phi_{g_2 q}.$$

This last identity follows from (8) in Lemma 2.11.

Finally we have to show that $c_Y(q^*\eta, gq) = 1$ for any section $q : Y \to Q$ where $g : Q \to G$ is the associated map to $id : Q \to Q$. But this follows simply by the fact that $g \equiv 1$ so $W_Y(g) = 0$ and $\nu(q^*\eta, gq) = 0$. □

Since $\mathcal{C}_Q$ is a connected groupoid, and $\mathcal{F}_\eta : \mathcal{C}_Q \to \mathcal{L}$ is a functor with no holonomy (the only automorphism from q to q is $id : Q \to Q$) we obtain a complex line $L_\eta = L_{Y,\eta} := V_{\mathcal{F}_\eta}$ of invariant sections, which we will term the Chern–Simons line over η. If $q : Y \to Q$ is a section we get an isomorphism $f_\eta^q : L_\eta \to \mathbb{C}$, $f_\eta^q(v) = v(q)$. Now we define a hermitian line bundle $\pi : L_Q \to \mathcal{A}_Q$ by $L_Q = \bigsqcup_{\eta \in \mathcal{A}_Q} L_\eta$ and $\pi(L_\eta) = \eta$. The hermitian metric on L_η is defined by $\langle v, w\rangle_\eta = \langle v(q), w(q)\rangle_{\mathbb{C}} = v(q)\overline{w(q)}$. It is important to observe that this metric is independent of the section q, which follows by the fact that $\mathcal{F}_\eta(\psi) \in U(1)$ for any morphism $q \stackrel{\psi}{\to} q'$. Note that the section q induces a trivialisation $h_q : L_Q \to \mathcal{A}_Q \times \mathbb{C}$ given by $h_q(v) = (\eta, v(q))$, $v \in L_\eta$. $pr_2 h_q|_{L_\eta}$ is just the linear isometry f^q_η. If $q' : Y \to Q$ is another section we write

$$h_{q'} \circ h_q^{-1}(\eta, z) = (\eta, g(\eta)z),$$

where $g = g_{q',q} : \mathcal{A}_Q \to GL(1, \mathbb{C}) \cong \mathbb{C}$ is the transition function between the trivialisations h_q and $h_{q'}$. If $z \in \mathbb{C}$ and $\eta \in \mathcal{A}_Q$ choose the unique $v \in L_\eta$ such that $v(q) = z$. Then $h_{q'} \circ h_q^{-1}(\eta, z) = h_{q'}(v) = (\eta, v(q'))$. Here $v(q') = \mathcal{F}_\eta(q \xrightarrow{\psi} q')v(q) = c_Y(q^*\eta, g_\psi q)z$, so

$$g_{q',q}(\eta) = c_Y(q^*\eta, g_\psi q), \tag{13}$$

where $\psi : Q \to Q$ is the unique gauge transformation with $q' = \psi q$. These transition functions satisfies the cocycle identity

$$g_{q_i,q_k} = g_{q_i,q_j} g_{q_j,q_k}. \tag{14}$$

We could of course have started by defining funktions $g_{q',q}$ for $q, q' \in \Gamma(Q)$ by (13) and then have shown that the cocycle identity (14) is satisfied. Here $\Gamma(Q)$ are the smooth sections of Q. Then we obtain the line bundle $L_Q \to \mathcal{A}_Q$ by defining $L_Q = \mathcal{A}_Q \times \mathbb{C} \times \Gamma(Q)/_\sim$, where $\sim$ is the equivalence relation $(\eta, z, q) \sim (\eta, g_{q',q}(\eta)z, q')$. Note that we actually have $g_{q',q} : \mathcal{A}_Q \to U(1)$.

The Chern–Simons lines vary smoothly in smooth families of connections. To see this let $\{\eta_u\}_{u \in U}$ be a smooth family in $\mathcal{A}_Q$ and let $\pi : L_U \to U$ be the hermitian line bundle defined by $L_U = \bigsqcup_{u \in U} L_{\eta_u}$, $\pi(L_{\eta_u}) = u$. Then L_U is a trivializable smooth hermitian line bundle. A section $q : Y \to Q$ defines a trivialization $k_q : L_U \to U \times \mathbb{C}$, $k_q(v) = (u, v(q))$, $v \in L_{\eta_u}$, and the transition function between the trivialisations k_q and $k_{q'}$ is the smooth function $g_{q',q} : U \to U(1)$ given by

$$g_{q',q}(u) = c_Y(q^*\eta_u, g_\psi q) = \exp(2\pi i[\int_Y \langle Ad_{g^{-1}}(i_u \circ q)^*(\omega) \wedge \phi_g\rangle + W_Y(g)]),$$

where $g = g_\psi q : Y \to G$ and ω is the associated connection on $U \times Q$ to the smooth family.

Let X be an oriented smooth compact 3–manifold with nonempty boundary ∂X and let Θ be a connection on a G bundle $P \to X$. We let $\partial\Theta$ be the restriction of Θ to $\partial P = P|_{\partial X}$. If $q : \partial X \to \partial P$ is a section, we can extend it to a section $p : X \to P$ by our choises of G, and we let $v(q) = \exp(2\pi i S_X(p, \Theta))$.

Lemma 2.14. *We have that v is a well-defined element in $L_{\partial\Theta}$. We will denote it by $e^{2\pi i S_X(\Theta)}$.*

Proof. We have to show that $v(q)$ is independent of the choise of extension p. Let $\varphi : P \to P$ be a gauge transformation, and let $p : X \to P$ be a section. Then by Proposition 2.8 and Definition 2.9

$$S_X(\varphi p, \Theta) \pmod 1 = (S_X(p, \Theta) + \int_{\partial X} \nu((\partial p)^*(\partial\Theta), \partial g) \pmod 1) + W_{\partial X}(\partial g),$$

where $g = g_\varphi p : X \to G$ and ∂g, ∂p and $\partial\Theta$ are the restrictions to the boundary. But then

$$\exp(2\pi i S_X(\varphi p, \Theta)) = c_{\partial X}((\partial p)^*(\partial\Theta), \partial g) \exp(2\pi i S_X(p, \Theta)).$$

Now for any two extensions $p_1, p_2 : X \to P$ of $q : \partial X \to \partial P$ there is a unique gauge transformation $\varphi : P \to P$ such that $p_2 = \varphi p_1$. Since $p_1|_{\partial X} = p_2|_{\partial X} = q$ we have that $\partial g \equiv 1$, where $g = g_\varphi p_1$. But then $c_{\partial X}((\partial p_1)^*(\partial\Theta), \partial g) = 1$ so v is well defined. It also follows from these calculations that $v \in L_{\partial\Theta}$. □

Chern–Simons theory is a so-called local Lagrangian field theory. A Lagrangian field theory is defined by a functional of the fields, called the *lagrangian*, and its integral over "spacetime" called the *action*. The fields can be smooth functions on a manifold, sections on a bundle etc. Here in our setting the fields are connections on smooth principal bundles over compact oriented 3–manifolds, and the lagrangian is $\alpha : \mathcal{C}_X \to \Omega^3(X)$ where $\alpha(\Theta)$ is the Chern–Simons 3–form defined in (1). The term "spacetime" is a misnormer here since we integrate over general compact oriented 3–manifolds. There is no preferred time dimension. If X has no boundary the action is simply $S_X : \mathcal{C}_X \to \mathbb{R}/\mathbb{Z}$ defined in Definition 2.5. In this case we also write the action as $e^{2\pi i S_X(-)} : \mathcal{C}_X \to \mathbb{C}$ in which case iv) in Proposition 2.6 appears as a multiplicative property. Note that since S_X is real the Chern–Simons action has unit norm in this form. In the case where the boundary of X is nonempty we have seen that S_X is no longer well defined. However, as it follows by the above result, we can interprete the Chern–Simons action as an invariant section of unit norm $\Theta \mapsto e^{2\pi i S_X(\Theta)} \in L_{\partial\Theta}$. If we let $L_\emptyset = \mathbb{C}$ with the standard hermitian metric, the case $\partial X \neq \emptyset$ can be thougt of as a generalization of the case $\partial X = \emptyset$. Finally the fields, here connections, can be cut and pasted, see Proposition 2.4, and the action behaves properly under gluing fields together, see iv) in the theorem below. This is the *locality* of the theory. The following theorem generalize Proposition 2.6 and collect the properties of the classical local Chern–Simons Lagrangian field theory.

Theorem 2.15. *[Freed] Let G be a connected, simply connected compact Lie group and $\langle \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ an Ad–invariant symmetric bilinear form on its Lie algebra $\mathfrak{g}$ which satisfies the Hypothesis 2.2. Then the assignments*

$$\eta \mapsto L_\eta , \quad \eta \in \mathcal{C}_Y,$$
$$\Theta \mapsto e^{2\pi i S_X(\Theta)} , \quad \Theta \in \mathcal{C}_X$$

defined above for closed oriented 2–manifolds Y and compact oriented 3–manifolds X are smooth with respect to smooth families of connections and satisfy

i) *(Functoriality) If $\psi : Q' \to Q$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\psi} : Y' \to Y$, and η is a connection on Q, then there is an induced isometry*

$$\psi^* : L_\eta \to L_{\psi^*\eta},$$

and these compose properly. If $\varphi : P' \to P$ is a bundle map covering an orientation preserving diffeomorphism $\bar{\varphi} : X' \to X$, and Θ is a connection on P, then

$$(\partial\varphi)^*(e^{2\pi i S_X(\Theta)}) = e^{2\pi i S_{X'}(\varphi^*(\Theta))}, \tag{15}$$

where $\partial\varphi : \partial P' \to \partial P$ is the induced bundle map to the boundary.

ii) *(Orientation) There is a natural isometry*

$$L_{-Y,\eta} \cong \overline{L_{Y,\eta}},$$

where $\overline{L_{Y,\eta}}$ is the conjugate space, and

$$e^{2\pi i S_{-X}(\Theta)} = \overline{e^{2\pi i S_X(\Theta)}}$$

iii) *(Multiplicativity) If $Y = Y_1 \sqcup Y_2$ is a disjoint union, and η_i are connections on G bundles over Y_i, then there is a natural isometry*

$$L_{\eta_1 \sqcup \eta_2} \cong L_{\eta_1} \otimes L_{\eta_2}.$$

If $X = X_1 \sqcup X_2$ is a disjoint union, and Θ_i are connections on G bundles over X_i, then

$$e^{2\pi i S_{X_1 \sqcup X_2}(\Theta_1 \sqcup \Theta_2)} = e^{2\pi i S_{X_1}(\Theta_1)} \otimes e^{2\pi i S_{X_2}(\Theta_2)}.$$

iv) *(Gluing) Suppose $Y \hookrightarrow X$ is a closed, oriented submanifold of dimension 2 and X^{cut} is the manifold obtained by cutting X along Y. Then $\partial X^{cut} = \partial X \sqcup Y \sqcup -Y$. Let $\bar{g} : X^{cut} \to X$ be the gluing map, let $P \to X$ be a G bundle and let $P^{cut} = \bar{g}^*P$ be the cut bundle and*

$g : P^{cut} \to P$ the induced gluing bundle map. Suppose Θ is a connection on P, with $\Theta^{cut} = g^*(\Theta)$ the induced connection on P^{cut}, and η the restriction of Θ to Y. Then

$$e^{2\pi i S_X(\Theta)} = Tr_\eta(e^{2\pi i S_{X^{cut}}(\Theta^{cut})}),$$

where Tr_η is the contraction $Tr_\eta : L_{\partial\Theta^{cut}} \cong L_{\partial\Theta} \otimes L_\eta \otimes \overline{L_\eta} \to L_{\partial\Theta}$ using the hermitian metric on L_η.

Proof. We have already seen that the Chern–Simons lines vary smoothly in smooth families of connections, and the rest of the proof also follows from [F]. However two points need to be deepened, namely that the isomorphism in i) constructed in [F] is well defined, and the gluing result iv). We will give a full proof of these points here. For the readers convenience we also give the constructions of the isometries in ii) and iii).

i) Let $q' : Y' \to Q'$ be a fixed section and let $q = \psi q' \bar{\psi}^{-1} : Y \to Q$ be the induced section on Q. $\psi^*(\eta)$ is a connection on Q' and we have isometries $f^q_\eta : L_\eta \to \mathbb{C}$ and $f^{q'}_{\psi^*(\eta)} : L_{\psi^*(\eta)} \to \mathbb{C}$ which induce the isometry

$$\psi^* = \psi^*_{q'} := (f^{q'}_{\psi^*(\eta)})^{-1} \circ f^q_\eta : L_\eta \to L_{\psi^*(\eta)}.$$

This construction is independent of the choise of section q'. To see this let $\varphi' : Q' \to Q'$ be a gauge transformation and let $p' = \varphi' q'$ and $p = \psi p' \bar{\psi}^{-1} = \varphi q$ where $\varphi = \psi \varphi' \psi^{-1}$ is a gauge transformation on Q. Now let $w \in L_\eta$. Then $\psi^*_{q'}(w) = v_1$ and $\psi^*_{p'}(w) = v_2$ where v_1, v_2 are the unique elements in $L_{\psi^*(\eta)}$ such that $v_1(q') = w(q)$ and $v_2(p') = w(p)$. To show that $v_1 = v_2$ it is enough to show that they agree on a single point. Now $w(p) = w(\varphi q) = c_Y(q^*\eta, g^*_\varphi q) w(q)$ and $v_1(p') = v_1(\varphi' q') = c_{Y'}((q')^*\psi^*\eta, g_{\varphi'} q') v_1(q')$. Since $v_1(q') = w(q)$ we see that it is enough to show that $c_{Y'}((q')^*\psi^*\eta, g_{\varphi'} q') = c_Y(q^*\eta, g^*_\varphi q)$. Now use that $(q')^*\psi^*\eta = \bar{\psi}^*(q^*\eta)$ and $g_{\varphi'} q' = (g_\varphi q)\bar{\psi}$. Since $\bar{\psi}$ is orientation preserving one gets

$$W_{Y'}(g_\varphi q \bar{\psi}) = W_Y(g_\varphi q),$$
$$\int_{Y'} \nu(\bar{\psi}^*(q^*\eta), g_\varphi q \bar{\psi}) = \int_Y \nu(q^*\eta, g_\varphi q).$$

ii) There is an induced hermitian metric on the conjugate space $\overline{L_{Y,\eta}}$ given by $\langle v, w\rangle^-_\eta := \overline{\langle v, w\rangle_\eta}$. Define $h : \overline{L_{Y,\eta}} \to L_{-Y,\eta}$ by $h(v) = \bar{v}$ where $\bar{v}(q) = \overline{v(q)}$ for any section $q : Y \to Q$. Then h is our isometry.

iii) Let $v_i \in L_{Y_i,\eta_i}$ and let $v = v_1 v_2$ be defined by $v(q_1 \sqcup q_2) = v_1(q_1) v_2(q_2)$. Then

$$v(\psi q) = v_1(\psi_1 q_1) v_2(\psi_2 q_2) = \prod_{i=1}^2 c_{Y_i}(q_i^*\eta_i, g_{\psi_i} q_i) v_i(q_i) = c_Y(q^*\eta, g_\psi q) v(q)$$

so $v \in L_{Y,\eta}$. We therefore have a map $k : L_{\eta_1} \times L_{\eta_2} \to L_{\eta_1 \sqcup \eta_2}$, $k(v_1, v_2) = v_1 v_2$. This map is clearly $\mathbb{C}$-bilinear and induces a $\mathbb{C}$-linear map $k : L_{\eta_1} \otimes L_{\eta_2} \to L_{\eta_1 \sqcup \eta_2}$. Since $L_{\eta_1} \otimes L_{\eta_2}$ and $L_{\eta_1 \sqcup \eta_2}$ are one dimensional and k is not identical zero, k is an isomorphism. The metrics $\langle \cdot, \cdot \rangle_{\eta_i}$ on L_{η_i} induce a hermitian metric $\langle \cdot, \cdot \rangle_{\eta_1 \otimes \eta_2}$ on $L_{\eta_1} \otimes L_{\eta_2}$ given by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle_{\eta_1 \otimes \eta_2} := \langle v_1, w_1 \rangle_{\eta_1} \langle v_2, w_2 \rangle_{\eta_2}$, and it follows that k is an isometry.

iv) Use the following notation: $Y_3 = \partial X^{cut} \setminus Y_1 \sqcup Y_2$, $Z_1 = Y$, $Z_2 = -Y$, $Z_3 = \partial X$, $P^{cut}_i = P^{cut}|_{Y_i}$, $P_i = P|_{Z_i}$, $\bar{g}_i = \bar{g}|_{Y_i} : Y_i \to Z_i$ and $g_i = g|_{P^{cut}_i} : P^{cut}_i \to P_i$, $i = 1, 2, 3$. $g_i : P^{cut}_i \to P_i$ is a bundle map over the orientation preserving diffeomorphism $\bar{g}_i : Y_i \to Z_i$, so by i) we have isometries

$$g^*_i : L_{Z_i,\eta_i} \to L_{Y_i, g^*_i(\eta_i)}$$

$i = 1, 2, 3$, where $\eta_1 = \eta_2 = \eta$ and $\eta_3 = \partial\Theta$. Now $\partial\Theta^{cut} = g_1^*(\eta_1) \sqcup g_2^*(\eta_2) \sqcup g_3^*(\eta_3)$ and Tr_η is given by the composition

$$Tr_\eta : L_{\partial X^{cut},\partial\Theta^{cut}} \cong \otimes_{i=1}^3 L_{Y_i,g_i^*(\eta_i)} \xrightarrow{r} \otimes_{i=1}^3 L_{Z_i,\eta_i} \xrightarrow{s} L_{Y,\eta} \otimes \overline{L_{Y,\eta}} \otimes L_{\partial X,\partial\Theta} \xrightarrow{\overline{Tr_\eta}} L_{\partial X,\partial\Theta},$$

where $r = (g_1^{-1})^* \otimes (g_2^{-1})^* \otimes (g_3^{-1})^*$ and $s = id \otimes h \otimes id$ where $h(v) = \bar{v}$. Here $\overline{Tr_\eta}(v_1 \otimes v_2 \otimes w) = \langle v_1, v_2\rangle_{Y,\eta} w$. Now $e^{2\pi i S_X(\Theta)}$ is a basis for $L_{\partial X,\partial\Theta}$ so $w = g_3^*(e^{2\pi i S_X(\Theta)})$ is a basis for $L_{Y_3,g_3^*(\partial\Theta)}$. Choose $v_0 \in L_{Y,\eta}$ so that $\langle v_0, v_0\rangle_{Y,\eta} = 1$. Then $v_1 = g_1^*(v_0)$ and $v_2 = g_2^*(\bar{v}_0)$ are basis' for respectively $L_{Y_1,g_1^*(\eta)}$ and $L_{Y_2,g_2^*(\eta)}$. But then $v_1 \otimes v_2 \otimes w$ is a basis for $\otimes_{i=1}^3 L_{Y_i,g_i^*(\eta_i)}$ and hence $v_1 v_2 w$ is a basis for $L_{\partial X^{cut},\partial\Theta^{cut}}$. Choose $\lambda \in \mathbb{C}$ so that $e^{2\pi i S_{X^{cut}}(\Theta^{cut})} = \lambda v_1 v_2 w$. Then $Tr_\eta(e^{2\pi i S_{X^{cut}}(\Theta^{cut})}) = \lambda \overline{Tr_\eta}(v_0 \otimes v_0 \otimes e^{2\pi i S_X(\Theta)}) = \lambda e^{2\pi i S_X(\Theta)}$. Now let $q_1 \sqcup q_2 \sqcup q_3 : \partial X^{cut} \to \partial P^{cut}$ be a section, $q_i : Y_i \to P_i^{cut}$, and let $p^{cut} : X^{cut} \to P^{cut}$ be an extension. Let $\bar{\varphi} = \bar{g}|X^{cut} \setminus Y_2 \to X$ and $\varphi = g|P^{cut} \setminus P_2^{cut} \to P$. Then φ is a bundle map over the orientation preserving diffeomorphism $\bar{\varphi}$. Let $\tilde{p}^{cut} = p^{cut}|X^{cut} \setminus Y_2 \to P^{cut} \setminus P_2$ and let $p = \varphi\tilde{p}^{cut}\bar{\varphi}^{-1}$. Then p is a section on P and

$$\begin{aligned} S_X(p,\Theta) &= \int_X (\varphi^{-1})^*(\tilde{p}^{cut})^*\varphi^*(\alpha(\Theta)) = \int_{X^{cut}} (p^{cut})^*(g^*(\alpha(\Theta))) \\ &= \int_{X^{cut}} (p^{cut})^*(\alpha(\Theta^{cut})) = S_{X^{cut}}(p^{cut},\Theta^{cut}) \end{aligned}$$

since $g^*(\alpha(\Theta)) = \alpha(\Theta^{cut})$ and Y_2 has Lebesgue measure zero in X^{cut}. It follows that

$$e^{2\pi i S_{X^{cut}}(\Theta^{cut})}(q_1 \sqcup q_2 \sqcup q_3) = e^{2\pi i S_{X^{cut}}(p^{cut},\Theta^{cut})} = e^{2\pi i S_X(p,\Theta)} = e^{2\pi i S_X(\Theta)}(\partial p),$$

where $\partial p = g_3 q_3 \bar{g}_3^{-1} : \partial X \to P|_{\partial X}$. By definition of w and g_3^* we have that $w(q_3) = e^{2\pi i S_X(\Theta)}(\partial p)$, so

$$w(q_3) = e^{2\pi i S_{X^{cut}}(\Theta^{cut})}(q_1 \sqcup q_2 \sqcup q_3) = \lambda v_1(q_1)v_2(q_2)w(q_3)$$

which implies that $\lambda v_1(q_1)v_2(q_2) = 1$ for all sections $q_1 : Y_1 \to P_1^{cut}$ and $q_2 : Y_2 \to P_2^{cut}$. Now let $q : Y \to P|_Y$ be a section and let $q_i = g_i^{-1} q \bar{g}_i : Y_i \to P_i^{cut}$. Then

$$1 = \lambda v_1(q_1)v_2(q_2) = \lambda g_1^*(v_0)(q_1) g_2^*(\bar{v}_0)(q_2) = \lambda v_0(q)\bar{v}_0(q) = \lambda.$$

□

Remark 2.16. Let the situation be as in i) in the above theorem. Then we have a commutative diagram

$$\begin{array}{ccc} L_Q & \xrightarrow{\psi^*} & L_{Q'} \\ \pi\downarrow & & \downarrow\pi' \\ \mathcal{A}_Q & \xrightarrow[\psi^*]{} & \mathcal{A}_{Q'}, \end{array}$$

where π and π' are the bundle projections and the lower ψ^* is the pull back. Since $\psi^* : L_Q \to L_{Q'}$ is fiberwise an isometry it is an isomorphism of hermitian line bundles. If $\delta : Q'' \to Q'$ is another G bundle map over an orientation preserving diffeomorphism $\bar{\delta} : Y'' \to Y'$ we clearly have that $(\psi \circ \delta)^* = \delta^* \circ \psi^*$, and id_Q^* is the identity on L_Q.

We have a right action of the gauge group $\mathcal{G}_Q$ on $\mathcal{A}_Q$ by pull back. This action lifts to a right action of $\mathcal{G}_Q$ on L_Q such that the diagram

$$\begin{array}{ccc} L_Q \times \mathcal{G}_Q & \longrightarrow & L_Q \\ \pi \times id\downarrow & & \downarrow\pi \\ \mathcal{A}_Q \times \mathcal{G}_Q & \longrightarrow & \mathcal{A}_Q \end{array}$$

commutes. Explicitly $v \cdot \psi = \psi^*(v)$, $(v, \psi) \in L_Q \times \mathcal{G}_Q$. If $P \to X$ is a G bundle, we let $f : \mathcal{A}_P \to \mathcal{A}_{\partial P}$ be the restriction map and get an induced bundle $L_P = f^*(L_{\partial P}) \to \mathcal{A}_P$. The right action of $\mathcal{G}_P$ on $\mathcal{A}_P$ lifts to a right action on L_P such that the diagram

$$\begin{array}{ccc} L_P \times \mathcal{G}_P & \longrightarrow & L_P \\ \pi \times id \big\downarrow & & \big\downarrow \pi \\ \mathcal{A}_P \times \mathcal{G}_P & \longrightarrow & \mathcal{A}_P \end{array}$$

commutes. Explicitly $(\Theta, v) \cdot \varphi = (\varphi^*(\Theta), (\partial\varphi)^*(v))$ for $\varphi \in \mathcal{G}_P$ and $(\Theta, v) \in L_P = \{(\Theta, v) \in \mathcal{A}_P \times L_{\partial P} \mid v \in L_{\partial\Theta}\}$. Note that $\Theta \mapsto e^{2\pi i S_X(\Theta)}$ can be interpreted as a $\mathcal{G}_P$-equivariant section on $L_P \to \mathcal{A}_P$. Precisely we define $s : \mathcal{A}_P \to L_P$ by $s(\Theta) = (\Theta, e^{2\pi i S_X(\Theta)})$. Now by (15)

$$s(\Theta \cdot \varphi) = s(\varphi^*(\Theta)) = (\varphi^*(\Theta), (\partial\varphi)^*(e^{2\pi i S_X(\Theta)})) = s(\Theta) \cdot \varphi.$$

3. The moduli space

The (classical) solutions in a Lagrangian field theory are the critical points of the action. In the Chern–Simons theory these are the flat connections (up to equivalence). Before showing this, we give some comments on smooth curves in $\mathcal{A}_P$ where $P \to X$ is a smooth G bundle. By a smooth curve in $\mathcal{A}_P$ we shall mean a smooth family $\{\Theta_t\}_{t \in I}$ parametrised by an interval on the real axis. Let Θ be a fixed connection on P. Then $\mathcal{A}_P = \Theta + \Omega^1(X, \mathfrak{g}_P)$ and we have $\Theta_t = \Theta + \omega_t$, where $t \mapsto \omega_t$ is a curve in the vector space $\Omega^1(X, \mathfrak{g}_P)$ so $\dot{\Theta}_s = \frac{d}{dt}|_{t=s}\Theta_t = \frac{d}{dt}|_{t=s}\omega_t \in \Omega^1(X, \mathfrak{g}_P)$. If $\omega \in \Omega^1(X, \mathfrak{g}_P)$ we let $\Theta_t = \Theta + t\omega$. Then Θ_t is a smooth curve in $\mathcal{A}_P$, and $\dot{\Theta}_s = \omega$ for all $s \in I$ here. We can therefore think of $\Omega^1(X, \mathfrak{g}_P)$ as the tangent space to $\mathcal{A}_P$ in an arbitrary point:

$$T_\Theta \mathcal{A}_P \cong \Omega^1(X, \mathfrak{g}_P) \quad , \; \Theta \in \mathcal{A}_P.$$

Let Ω_t be the curvature of Θ_t. Then $\Omega_t \wedge \dot{\Theta}_t$ is a horizontal 3–form on P. Since $\langle \cdot \rangle$ is Ad–invariant, it follows that $\langle \Omega_t \wedge \dot{\Theta}_t \rangle$ is a horizontal invariant form on P with values in $\mathbb{R}$, hence a lift of a 3–form on the basis X which we also denote by $\langle \Omega_t \wedge \dot{\Theta}_t \rangle$.

Proposition 3.1. *Let the situation be as above and let $\Theta = \Theta_0$ and $\dot{\Theta} = \dot{\Theta}_0 = \frac{d}{dt}|_{t=0}\Theta_t$. Then*

$$\frac{d}{dt}|_{t=0} S_X(\Theta_t) = 2\int_X \langle \Omega \wedge \dot{\Theta} \rangle \qquad (\text{mod } 1),$$

where Ω is the curvature of Θ.

Proof. Let $\tilde{\Omega}$ be the curvature of $\tilde{\Theta}$, where $\tilde{\Theta}$ is the connection on $I \times P$ associated to the curve Θ_t. We then have

$$\tilde{\Omega} = \eta + ds \wedge \xi,$$

where $\eta \in \Omega^2(P, \mathfrak{g})$ and $\xi \in \Omega^1(P, \mathfrak{g})$ has no ds-component and satisfy $i_s^*(\eta) = \Omega_s$ and $i_s^*(\xi) = \dot{\Theta}_s$ for all $s \in I$. From this we get

$$\langle \tilde{\Omega} \wedge \tilde{\Omega} \rangle = \langle \eta \wedge \eta \rangle + 2ds \wedge \langle \eta \wedge \xi \rangle = 2ds \wedge \langle \eta \wedge \xi \rangle,$$

since $\langle \Omega_s \wedge \Omega_s \rangle$ is the pull back of a 4–form on X, so $\langle \Omega_s \wedge \Omega_s \rangle$ and therefore $\langle \eta \wedge \eta \rangle$ is zero, since $\dim X = 3$. By Proposition 2.7 we then get

$$S_{\partial(I \times X)}(\partial\tilde{\Theta}) = \int_{I \times X} 2ds \wedge \langle \eta \wedge \xi \rangle = 2\int_I ds \int_X \langle \Omega_s \wedge \dot{\Theta}_s \rangle.$$

Now let $I = [0,t]$. Then $\partial I \times X = \{0\} \times X \cup \{t\} \times X$. If $p : X \to P$ is a section $\tilde{p} = id \times p : I \times X \to I \times P$ is a section and if $p_0 = \tilde{p}|_{\{0\}\times X}$ and $p_t = \tilde{p}|_{\{t\}\times X}$ we get

$$\begin{aligned} S_{\partial(I\times X)}(\partial\tilde{\Theta}) &= \int_{\partial(I\times X)} (\partial\tilde{p})^*(\alpha(\partial\tilde{\Theta})) \\ &= \int_{\{0\}\times X} p_0^*(\alpha(\tilde{\Theta}|_{\{0\}\times X})) + \int_{\{t\}\times X} p_t^*(\alpha(\tilde{\Theta}|_{\{t\}\times X})) \\ &= \int_{-X} p^*(\alpha(\Theta)) + \int_X p^*(\alpha(\Theta_t)) = S_X(\Theta_t) - S_X(\Theta). \end{aligned}$$

That is, $S_X(\Theta_t) - S_X(\Theta) = 2\int_0^t ds \int_X \langle \Omega_s \wedge \dot{\Theta}_s \rangle$, where all calculations are modulo the integers. By differentiating at $t = 0$ the result follows. □

Remark. We call $\Theta \in \mathcal{A}_P$ a classical solution in the Chern–Simons theory if

$$\frac{d}{dt}|_{t=0} S_X(\Theta_t) = 0$$

for all smooth curves Θ_t in $\mathcal{A}_P$ with $\Theta_0 = \Theta$.

Corollary 3.2. *Suppose $\langle \cdot \rangle$ is nondegenerate. Then a connection Θ is a classical solution in the Chern–Simons theory if and only if it is flat, i.e. its curvature $\Omega = 0$.*

Proof. If $\Omega = 0$, Θ is clearly a classical solution. Now suppose that Θ is a solution. Then by the above proposition

$$\int_X \langle \Omega \wedge \dot{\Theta} \rangle = 0$$

for all $\dot{\Theta} \in \Omega^1_{bas}(P, \mathfrak{g})$. Now use that the bilinear form

$$\Omega^2(X, \mathfrak{g}_P) \times \Omega^1(X, \mathfrak{g}_P) \to \mathbb{C}$$
$$(\Omega, \tau) \mapsto \int_X \langle \Omega \wedge \tau \rangle$$

is a nonsingular paring (see the remark below). It follows that $\Omega = 0$. □

Remark 3.3. Let M be a smooth n-manifold with a fixed orientation and fixed Riemannian metric. Let $P \to M$ be a smooth G bundle and let $\langle \cdot \rangle$ be an *Ad*–invariant inner product on the Lie algebra $\mathfrak{g}$. We let $\langle \cdot \rangle_M$ be the metric on $\Lambda^*(T^*M)$ induced by the Riemannian metric on M. Let $\mathfrak{g}_P = P \times_G \mathfrak{g}$ be the adjoint bundle as usual and let $\langle \cdot \rangle_P$ be the metric induced on $\mathfrak{g}_P$ by the *Ad*–invariant inner product on $\mathfrak{g}$. (Simply define the inner product on each fiber such that the fibers in $\mathfrak{g}_P$ are isometric with $\mathfrak{g}$. This is possible exactly because our inner product is *Ad*–invariant.) We define a metric on $\Lambda^*(T^*M) \otimes \mathfrak{g}_P$ by $(x \otimes u, y \otimes v) = \langle x \otimes y \rangle_M \langle u \otimes v \rangle_P$. This induces an L^2–inner product structure on $\Omega^k(\mathfrak{g}_P) = \Omega^k(M, \mathfrak{g}_P)$ by

$$(\varphi, \psi) := \int_M (\varphi, \psi),$$

where $(\varphi, \psi)(x) = (\varphi(x), \psi(x))$ for $x \in M$. Now let $* : \Omega^*(M) \to \Omega^*(M)$ be the Hodge star operator associated to the orientation and metric on M. $*$ is characterized by

$$\tau \wedge *\omega = \langle \tau, \omega \rangle_M \mathrm{vol}(M) \qquad , \tau, \omega \in \Omega^k(M),$$

where vol(M) is the unique n-form of length 1 in the orientation of M. $*$ induces a $*$-operator on $\Omega^*(\mathfrak{g}_P) \cong \Omega^*(M) \otimes_{\Omega^0(M)} \Omega^0(\mathfrak{g}_P)$ by tensoring with the identity on $\Omega^0(\mathfrak{g}_P)$. We have a product

$$\wedge : \Omega^k(\mathfrak{g}_P) \times \Omega^l(\mathfrak{g}_P) \to \Omega^{k+l}(\mathfrak{g}_P \otimes \mathfrak{g}_P)$$

defined by $(\omega \otimes s) \wedge (\tau \otimes t) = (\omega \wedge \tau) \otimes (s \otimes t)$. Let $(\ \cdot\ )_P : \Omega^*(\mathfrak{g}_P \otimes \mathfrak{g}_P) \to \Omega^*(M)$ be the map defined by $(\omega \otimes (s \otimes t))_P = \omega \langle s \otimes t \rangle_P$. By these definitions we have

$$(\varphi, \psi) = \int_M (\varphi \wedge *\psi)_P.$$

In particular we have dual parings

$$\Omega^k(\mathfrak{g}_P) \otimes \Omega^{n-k}(\mathfrak{g}_P) \to \mathbb{R}$$

$$\varphi \otimes \psi \mapsto \int_M (\varphi \wedge \psi)_P \tag{16}$$

since $* \circ * = (-1)^{k(n-k)}$ on $\Omega^k(-)$. If ω and τ are equivariant forms on P with values in $\mathfrak{g}$, then $\langle \omega \wedge \tau \rangle$ is an invariant form on P with values in $\mathbb{R}$. If futhermore it is horizontal then it is the pullback of a form on the basis M. Under the isomorphism $\Omega^*_{bas}(P, \mathfrak{g}) \cong \Omega^*(\mathfrak{g}_P)$ this form coincide with $(\omega \wedge \tau)_P$. Therefore we also write the integral in (16) as $\int_M \langle \varphi \wedge \psi \rangle$ in this case.

If our Lie group G is simple, the Killing form of $\mathfrak{g}$ is nonsingular and all Ad–invariant symmetric bilinear forms on $\mathfrak{g}$ are proportional to the Killing form. It follows that if we use an arbitrary Ad–invariant symmetric bilinear form above, instead of our inner product $\langle \cdot \rangle$, the parings in (16) are still nonsingular.

For later use we also make some comments on the covariant derivative induced by a connection Θ on P. This is the $\mathbb{R}$-linear differential operator $d_\Theta : \Omega^0(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$ given by $d_\Theta = d + [\Theta \wedge -]$ under the isomorphism $\Omega^*(\mathfrak{g}_P) \cong \Omega^*_{bas}(P, \mathfrak{g})$. We can extend this operator to a $\mathbb{R}$-linear operator $d_\Theta : \Omega^*(\mathfrak{g}_P) \to \Omega^{*+1}(\mathfrak{g}_P)$ by simply letting $d_\Theta = d + [\Theta \wedge -]$ under the isomorphism $\Omega^*(\mathfrak{g}_P) \cong \Omega^*_{bas}(P, \mathfrak{g})$. Note that $d_\Theta(d_\Theta \beta) = [\Omega \wedge \beta]$ where Ω is the curvature of Θ, so when Θ is flat we actually have a complex

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_\Theta} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_\Theta} \dots \xrightarrow{d_\Theta} \Omega^n(\mathfrak{g}_P) \longrightarrow 0. \tag{17}$$

We denote the cohomology groups of this complex by $H^*(M, d_\Theta)$. Now let $\chi(\Theta)$ be the Euler characteristic of this complex. By index theory we know that $\chi(\Theta)$ is constant on a continuous family of connections, and since $\mathcal{A}_P$ is a path-connected space (it is an affine space), we just have to calculate $\chi(\Theta)$ for a single connection Θ. Now suppose $P \to M$ is trivializable and think of P as $M \times G$ and let $\Theta = pr_2^*(\theta)$, where θ is the Maurer-Cartan form on G and $pr_2 : M \times G \to G$ the projection. Then the complex (17) is isomorphic with

$$0 \longrightarrow \Omega^0(M) \otimes_{\mathbb{R}} \mathfrak{g} \xrightarrow{d \otimes id} \Omega^1(M) \otimes_{\mathbb{R}} \mathfrak{g} \xrightarrow{d \otimes id} \dots \xrightarrow{d \otimes id} \Omega^n(M) \otimes_{\mathbb{R}} \mathfrak{g} \longrightarrow 0,$$

and we find that $\chi(\Theta) = \chi(M) \cdot \dim(G)$. If M is a closed oriented 2–manifold we have that $\chi(M)$ is even, since the Euler characteristic of a manifold is a topological invariant and since $\chi(S^2) = 2$ and $\chi(g \cdot T^2) = 2 - 2g$, where $g \cdot T^2$ is the 2–sphere with g handles (the connected sum of g tori T^2), g being the genus of the surface.

The equation $\Omega = 0$ is the *Euler–Lagrange equation* for the Chern–Simons theory. Note that if $\Theta \in \mathcal{C}_X$ is flat every connection in the equivalens class $[\Theta] \in \overline{\mathcal{C}_X}$ is flat. We let $\mathcal{M}_X \subseteq \overline{\mathcal{C}_X}$ be the set of equivalens classes of flat connections. $\mathcal{M}_X$ is called the moduli space (of flat connections). We will now determine the stationary points to the Chern–Simons action, in the case where our compact oriented 3–manifold X has a nonempty boundary. The first problem to overcome is that S_X is no longer well defined as pointed out earlier. We will therefore calculate the stationary points under some boundary conditions. Let $\eta \in \mathcal{C}_{\partial X}$ be a fixed connection over the boundary. We now let $\mathcal{C}_X(\eta)$ be the subcategory of $\mathcal{C}_X$ whose objects are connections $\Theta \in \mathcal{C}_X$ with $\partial\Theta = \eta$. The morphisms in $\mathcal{C}_X(\eta)$ are required to be the identity over ∂X. We calculate the stationary points of the Chern–Simons action with respect to this category. Precisely we let $P \to X$ be a G bundle and let η be a connection on ∂P. Now let $\{\Theta_s\}_{s\in I}$ be a smooth curve in $\mathcal{A}_P(\eta)$, the connections on P which equal η on the boundary. Observe that the Chern–Simons action $e^{2\pi i S_X(\Theta_t)} \in L_\eta$ along the curve, where L_η is the Chern–Simons line over η. What we will actually calculate is the stationary points of the phase. For a section $p : X \to P$ we have that $e^{2\pi i S_X(\Theta_t)}(\partial p) = \exp(2\pi i S_X(p, \Theta_t))$. Now we are looking for connections $\Theta \in \mathcal{A}_P(\eta)$ such that

$$\frac{d}{dt}|_{t=0} S_X(p, \Theta_t) = 0 \tag{18}$$

for all curves Θ_t in $\mathcal{A}_P(\eta)$ with $\Theta_0 = \Theta$ and all sections $p : X \to P$. As the next result shows it turns out that the derivative on the left-hand side in (18) does not depend on the sections of P. If Θ is a solution to (18) for one hence for all sections of P, then Θ is really a stationary point for (the phase of) the Chern–Simons action on $\mathcal{C}_X(\eta)$. The proposition generalies Proposition 3.1.

Proposition 3.4. *Let Θ_t be a curve in $\mathcal{A}_P(\eta)$ and let $\Theta = \Theta_0$ and $\dot\Theta = \dot\Theta_0 = \frac{d}{dt}|_{t=0}\Theta_t$. Then*

$$\frac{d}{dt}|_{t=0} S_X(p, \Theta_t) = 2\int_X \langle \Omega \wedge \dot\Theta \rangle$$

for an arbitrary section $p : X \to P$, where Ω is the curvature of Θ.

The proof is similar to the proof of Proposition 3.1 except for two point: (i) the cylinder $I \times X$ needs to be smoothed at the corners and (ii) the boundary $\partial(I \times X)$ has an extra part $I \times \partial X$. However if $\tilde\Theta$ is the connection on $I \times P$ associated to the curve Θ_t we have that $\tilde\Theta = pr_2^*(\eta)$ on $I \times \partial P$ where $pr_2 : I \times \partial P \to \partial P$ is the projection. Now if $q : \partial X \to \partial P$ is a section

$$S_{I\times\partial X}(id \times q, pr_2^*(\eta)) = \int_{I\times\partial X} pr_2^*(q^*(\alpha(\eta))) = 0$$

since $q^*(\alpha(\eta))$ is a 3–form on ∂X. If $p : X \to P$ is a section we therefore get $(I = [0, t])$

$$2\int_0^t ds \int_X \langle \Omega_s \wedge \dot\Theta_s \rangle = S_{\partial(I\times X)}(\partial(id \times p), \partial\tilde\Theta) = S_X(p, \Theta_t) - S_X(p, \Theta)$$

and the result follows. As in the Corollary 3.2 we therefore have that the critical points to (the phase of) the Chern–Simons action, is the flat connections.

Proposition 3.5. *Let X be any smooth finite dimensional manifold. Choose a basepoint x_i in each component of X. Then the holonomy provides an identification*

$$\mathcal{M}_X \cong \prod_i \mathrm{Hom}(\pi_1(X, x_i), G)/G$$

whish is independent of the basepoints. Here G acts on $\mathrm{Hom}(\pi_1(X, x_i), G)$ by conjugation, so that $x \mapsto \rho(x)$ and $x \mapsto g\rho(x)g^{-1}$, $g \in G$ and $\rho \in \mathrm{Hom}(\pi_1(X, x_i), G)$, represent the same element in $\mathrm{Hom}(\pi_1(X, x_i), G)/G$.

Proof. Assume first that X is connected and fix a base point x. Let $P \xrightarrow{\pi} X$ be a G bundle and let Θ be a flat connection on P. Let $u \in \pi^{-1}(x)$ and let $[\gamma] \in \pi_1(X, x)$ and define a map $f_u : \pi_1(X, x) \to G$ by $f([\gamma]) = a$, where a is the element in the holonomy group $\Phi(u)$ given by the parallel transport along γ. Note that f_u is well defined independent of the choise of representative γ, since Θ is flat. f_u is clearly a group homomorphism. If $v \in \pi^{-1}(x)$ is another element in the fiber over x, we can choose $g \in G$ such that $v = u \cdot g$, and find that $f_v = g^{-1} f_u g$ so f_u and f_v represent the same element in $\mathrm{Hom}(\pi_1(X,x),G)/G$. It is easy to see that the class $[f_u]$ is independent of the choise of representative of $[\Theta] \in \mathcal{M}_X$, so we have a map

$$F_x : \mathcal{M}_X \to \mathrm{Hom}(\pi_1(X,x),G)/G.$$

Note that $\mathrm{Im}(f_u) = \Phi(u)$. To construct the map the other way we use, that the universal covering manifold $\tilde{X}$ in a natural way is a principal $\pi_1(X,x)$ bundle over X. Note here that since X is a manifold $\pi_1(X,x)$ is a countable discrete group, hence a 0-dimensional Lie group. Now let $\rho \in \mathrm{Hom}(\pi_1(X,x),G)$ and let $P_\rho = \tilde{X} \times_{\pi_1(X,x)} G$ be the associated bundle with respect to the action of $\pi_1(X,x)$ on G induced by ρ, i.e. P_ρ is the extension of the bundle $\tilde{X} \to X$ to G relative to ρ. The only connection on $\tilde{X} \to X$ is the trivial connection $\Theta = 0$, and the G extension of Θ is a flat connection Θ_ρ on P_ρ. Explicitly

$$(\Theta_\rho)_{q(u,g)}((dq)_{(u,g)}(v,w)) = Ad_{g^{-1}}\dot{\rho}\Theta_u(v) + \theta_g(w) = \theta_g(w),$$

where θ is the Maurer-Cartan form on G and $q : \tilde{X} \times G \to P_\rho$ the projection. Note that $\tilde{\Theta}_\rho = q^*(\Theta_\rho)$ is the flat connection $pr_2^*(\theta)$ where $pr_2 : \tilde{X} \times G \to G$. We have $q^*(\Omega_\rho) = \tilde{\Omega}_\rho = 0$ and then $\Omega_\rho = 0$, since q is a surjective submersion. We therefore get an element $[\Theta_\rho] \in \mathcal{M}_X$. If ρ_1 and ρ_2 represent the same class, we can choose $g \in G$ such that $\rho_2(y) = g\rho_1(y)g^{-1}$. We then have a G bundle morphism $\varphi : P_{\rho_1} \to P_{\rho_2}$ given by $\varphi(q_1(u,h)) = q_2(u,gh)$. One finds that $\varphi^*(\Theta_{\rho_2}) = \Theta_{\rho_1}$ so Θ_{ρ_1} and Θ_{ρ_2} represent the same class in $\mathcal{M}_X$, so we have established a map

$$K_x : \mathrm{Hom}(\pi_1(X,x),G)/G \to \mathcal{M}_X.$$

I leave to the reader to show that F_x and K_x are inverse to each other. By the natural identification of the groups $\pi_1(X,x)$ for different base points, the bijection defined above is independent of the base point.

Finally if X is nonconnected with components $\{X_i\}_{i\in I}$, we have a natural bijection $\mathcal{M}_X = \prod_{i\in I} \mathcal{M}_{X_i}$ given by $[\Theta] \mapsto ([\Theta|_{X_i}])_{i\in I}$. Now if we choose base points x_i in X_i, we have $\mathcal{M}_{X_i} \cong \mathrm{Hom}(\pi_1(X_i,x_i),G)/G$ and $\pi_1(X,x_i) \cong \pi_1(X_i,x_i)$. □

From now on we further assume that our Lie group G is simple. This has among other things the consequence, that the center of G is discrete. Let Θ be a connection on the G bundle $P \to X$. We will here assume that X is connected. The centralizer of Θ is by definition

$$Z(\Theta) := [Z(\Phi(u))] \in \mathrm{Sub}(G)/G,$$

where $\Phi(u)$ is the holonomy group with reference point $u \in P$, $Z(\Phi(u))$ is the centralizer in G, $\mathrm{Sub}(G)$ is the set of subgroups of G and $\mathrm{Sub}(G)/G$ are the conjugation classes. Since X is connected, $\Phi(u)$ and $\Phi(v)$ and hence their centralizers are conjugate for all $u, v \in P$, since $\Phi(u) = \Phi(v)$ if u and v can be joined by a horizontal curve, and $\Phi(u \cdot a) = a^{-1}\Phi(u)a$. If $\varphi : P' \to P$ is a bundle morphism and $\Theta' = \varphi^*(\Theta)$, then $\Phi_{\Theta'}(u') = \Phi_\Theta(\varphi(u))$. We therefore have that $Z(\Theta') = Z(\Theta)$ and get an induced centralizer $Z([\Theta])$ for $[\Theta] \in \overline{\mathcal{C}_X}$.

Definition 3.6. A connection Θ on P is called irreducible (or generic) if $Z(\Phi(u))$ is the center of G for one, hence for all $u \in P$.

It is commen to call a connection Θ for irreducible if $\dim Z(\Phi(u)) = 0$ for one, hence for all $u \in P$. However in the situation we are mostly interested in, namely $G = SU(n)$, the two definitions coincide. Let us give an argument for this in the case $G = SU(2)$. If H is a subgroup of G, the centralizer for H in G is either $\mathbb{Z}_2$, a maximal torus in $SU(2)$ or $SU(2)$. The maximal tori in $SU(2)$ are all isomorphic to $U(1)$ which has dimension 1. To see this let G be a connected compact Lie group. Then any two maximal tori are conjugate and every element of G is contained in a maximal torus. The center of G is the intersection of all maximal tori in G. For all these assertions, see [BD] Chap. IV. Now let H be a subgroup of G. Then $Z(H) = \cap_{g\in H} Z(g)$. If g is in the center of

G, $Z(g) = G$, so $Z(H) = Z(G)$ if $H \leq Z(G)$. Now assume that $H \setminus Z(G) \neq \emptyset$. If $Z(g)_1$ is the connected component of $Z(g)$ containing the unit of G, we have that $Z(g)_1$ is the union of the maximal tori in G containing g. Now assume that for all $g \in G \setminus Z(G)$, $Z(g)$ is connected and g is contained in exactly one maximal torus T_g. Then we have $Z(H) = \cap_{g \in H \setminus Z(G)} T_g$ and for any two $g_1, g_2 \in G \setminus Z(G)$, $T_{g_1} \cap T_{g_2} = Z(G)$ or $T_{g_1} = T_{g_2}$. It follows that $Z(H)$ is $Z(G)$ or a maximal torus in G. Note that if there exists a $g_0 \in G$ such that $Z(b)$ is connected for every $b \in T_{g_0} \setminus Z(G)$, then $Z(g)$ is connected for all $g \in G \setminus Z(G)$, since every maximal torus in G is conjugate to T_{g_0}. Now in the case of $G = SU(2)$, the center is $\mathbb{Z}_2$ and the maximal torus

$$U(1) = \left\{ \begin{pmatrix} c & 0 \\ 0 & \bar{c} \end{pmatrix} \mid |c|^2 = 1 \ , \ c \in \mathbb{C} \right\}$$

satisfies, that $Z(b) = U(1)$ for all $b \in U(1) \setminus \{\pm 1\}$ and every $g \in SU(2) \setminus \{\pm 1\}$ is contained in exactly one maximal torus.

Now let $P \xrightarrow{\pi} X$ be a fixed G bundle, let $\mathcal{A} = \mathcal{A}_P$, and let $\mathcal{G} = \mathcal{G}_P$. For a connection Θ on P we let

$$\mathrm{Aut}(\Theta) = \{\varphi \in \mathcal{G} \mid \varphi^*(\Theta) = \Theta\} = \mathcal{G}_\Theta$$

be the stabilizer of Θ under the right action of $\mathcal{G}$ on $\mathcal{A}$. The relation between $\mathrm{Aut}(\Theta)$ and $Z(\Theta)$ is the following:

Proposition 3.7. *Let $u_0 \in P$ and let $f = f_{u_0} : \mathrm{Aut}(\Theta) \to G$ be given by $f(\varphi) = g_\varphi(u_0)$. Then f is an injective group homomorphism with $\mathrm{Im}(f) = Z(\Phi(u_0))$.*

Proof. That f is a group homomorphism is trivial. Now let $\varphi \in \mathrm{Aut}(\Theta)$, $g = g_\varphi(u_0)$ and $a \in \Phi(u_0)$ and let $x_0 = \pi(u_0)$. Let $\gamma : [0,1] \to X$ be a loop with $\gamma(0) = \gamma(1) = x_0$ such that $\gamma^*(1) = u_0 \cdot a$, where $\gamma^* : [0,1] \to P$ is the horizontal lift of γ with $\gamma^*(0) = u_0$. Since $\Theta = \varphi^*(\Theta) = Ad_{g_\varphi^{-1}} \Theta + g_\varphi^*(\theta)$ it follows that $(dg_\varphi)_u(H_u) = 0$, where H_u is the horizontal subspace of T_uP. By this it follows that $\varphi \circ \gamma^*$ is the horizontal lift of γ, which start from $\varphi(u_0) = u_0 \cdot g$. But then $\varphi \circ \gamma^* = R_g \circ \gamma^*$, so that

$$u_0 \cdot (ag) = \gamma^*(1) \cdot g = \varphi(\gamma^*(1)) = \varphi(u_0 \cdot a) = \varphi(u_0) \cdot a = u_0 \cdot (ga)$$

so $ag = ga$. That is $\mathrm{Im}(f) \leq Z(\Phi(u_0))$.

To show the opposite inclusion, note that if $\varphi \in \mathrm{Aut}(\Theta)$ the associated map $g_\varphi : P \to G$ is constant on horizontal curves. Now let $g \in Z(\Phi(u_0))$. If $u \in P$ we let $\gamma : [0,1] \to X$ be a curve from x_0 to $x = \pi(u)$ and let $\gamma^* : [0,1] \to P$ be the horizontal lift of γ with $\gamma^*(0) = u_0$. Choose $a \in G$ such that $u = \gamma^*(1) \cdot a$ and define $h : P \to G$ by $h(u_0) = g$ and $h(u) = h(\gamma^*(1) \cdot a) = a^{-1} h(u_0) a = a^{-1} g a$. $h(u)$ is independent of the choise of γ since $g \in Z(\Phi(u_0))$ and we have that $h(u \cdot b) = b^{-1} h(u) b$ for $u \in P$ and $b \in G$. Now define $\varphi : P \to P$ by $\varphi(u) = u \cdot h(u)$. Then it is not hard to see, that φ is in $\mathrm{Aut}(\Theta)$.

The injectivity of f follows from the fact, that an element in $\mathrm{Aut}(\Theta)$ is determined by its value in a single point, since X is connected. □

In particular we see that if $G = SU(2)$ and Θ is an irreducible connection, then the stabilizer $\mathcal{G}_\Theta = \{\pm 1\}$. In the light of the above proposition we also denote the stabilizer of Θ by $Z(\Theta)$.

Now let Y be a closed oriented 2–manifold and let $Q \to Y$ be a G bundle. Recall from Remark 2.16 that the gauge group $\mathcal{G}_Q$ acts on $\mathcal{A}_Q$ by pullback, and that this action lifts to L_Q, where $L_Q \to \mathcal{A}_Q$ is the smooth hermitian line bundle, whose fibers are the Chern–Simons lines. We then have the following result from [F].

Proposition 3.8. *The Chern–Simons action defines a unitary connection on* $L_Q \xrightarrow{\pi} \mathcal{A}_Q$*. The curvature of this connection times* $\frac{i}{2\pi}$ *is the pullback* $\pi^*(\omega)$ *of the 2–form*

$$\omega(\dot\eta_1, \dot\eta_2) = -2 \int_Y \langle \dot\eta_1 \wedge \dot\eta_2 \rangle \quad , \dot\eta_1, \dot\eta_2 \in T\mathcal{A}_Q. \tag{19}$$

If $\langle \cdot \rangle$ *is nondegenerate, then* ω *is a symplectic form. The action of* $\mathcal{G}_Q$ *on* $\mathcal{A}_Q$ *is then symplectic, and the lifted action to* L_Q *preserves the metric and connection. There is an induced moment map* $\mu : \mathcal{A}_Q \to \Omega^0(Y, \mathfrak{g}_Q)^*$ *given by*

$$\mu(\eta)(\xi) = 2 \int_Y \langle \Omega(\eta) \wedge \xi \rangle,$$

where $\Omega(\eta)$ *is the curvature of* η*. If we let* $\mathcal{A}_F^s \subseteq \mathcal{A}_Q$ *be the subset of irreducible flat connections and* $\mathcal{M}_Q^s = \mathcal{A}_F^s / \mathcal{G}_Q \subseteq \mathcal{M}_Y$*, we get an induced hermitian line bundle* $L_Q^s \to \mathcal{M}_Q^s$ *with unitary connection and an induced symplectic form on* $\mathcal{M}_Q^s$*.* $\frac{i}{2\pi}$ *times the curvature of this induced connection is the pullback of the induced symplectic form on* $\mathcal{M}_Q^s$*.*

Remark 3.9. We will not give a proof for this proposition, since a rather detailed proof is given in [F]. Note that $\langle \cdot \rangle$ is automatically nondegenerate, since we have assumed that G is simple, see Remark 3.3.

Let us make some comments on the Lie algebra of the gauge group (for precise statements about these infinite dimensional objects, the reader is referred to [AB]). We have a bundle $P \times_G G$ where G acts by conjugation on itself, and there is a natural group structure on the set of sections $\Omega^0(P \times_G G)$. By the remarks around (2) it easily follows that the gauge group $\mathcal{G}_Q$ is group isomorphic with the group af sections on $P \times_G G$. Now the Lie bracket on $\mathfrak{g}$ induces a (fiberwise) Lie bracket on the adjoint bundle $\mathfrak{g}_Q$ and hence a Lie algebra structure on the vector space $\Omega^0(Y, \mathfrak{g}_Q)$ of sections. This space then plays the role of the Lie algebra of the gauge group, and a moment map takes values in $\Omega^0(Y, \mathfrak{g}_Q)^*$. Note that under the dual paring

$$\Omega^2(Y, \mathfrak{g}_Q) \otimes \Omega^0(Y, \mathfrak{g}_Q) \to \mathbb{R}$$
$$\varphi \otimes \xi \mapsto \int_Y (\varphi \wedge \xi)_Q$$

see (16) in Remark 3.3, the momont map $\mu : \mathcal{A}_Q \to \Omega^0(Y, \mathfrak{g}_Q)^* \cong \Omega^2(Y, \mathfrak{g}_Q)$ is simply the curvature map, and $\mu^{-1}(0)$ is exactly the flat connections on Q.

To ensure that the quotient $L_Q^s = (L_Q|_{\mathcal{A}_F^s})/\mathcal{G}_Q$ is a smooth hermitian line bundle we have to check, that the stabilizer of an irreducibel connection acts trivially on the fibers of L_Q. But this follows from [F, Proposition (3.26)]. Recall here that the stabilizer of a connection $\eta \in \mathcal{A}_Q$ corresponds to the centralizer $Z(\eta)$, by Proposition 3.7.

The symplectic manifold $\mathcal{M}_Q^s$ is actually finite dimensional, and this dimension is therefore even. Let us give an explicit formula for this dimension. Let η be a flat connection on Q and consider the complex

$$0 \longrightarrow \Omega^0(Y, \mathfrak{g}_Q) \xrightarrow{d_\eta} \Omega^1(Y, \mathfrak{g}_Q) \xrightarrow{d_\eta} \Omega^2(Y, \mathfrak{g}_Q) \longrightarrow 0$$

from (17). Under the identification of $\Omega^0(Y, \mathfrak{g}_Q)$ by the Lie algebra of the gauge group $\mathcal{G}_Q$, $H^0(Y, d_\eta)$ can be identified with the Lie algebra of the stabilizer $Z(\eta) \leq \mathcal{G}_Q$. The paring (16) induces a nonsingular paring of cohomology groups, so $H^2(Y, d_\eta) \cong H^0(Y, d_\eta)$ (alternatively use Poincaré duality and identification of homology with cohomology). If η is irreducible we therefore have that these two cohomology groups are 0-dimensional. Futhermore we have by [AB] an identification $T_\eta M_Q^s \cong H^1(Y, d_\eta)$. But then we get from Remark 3.3 that

$$\dim \mathcal{M}_Q^s = -\chi(\eta) = -\chi(Y) \dim G$$

which is even by the same remark.

We close this section by showing, that the moduli space of a connected compact manifold X has finitely many connected components, and that the Chern–Simons action is constant on these components in the case where $\partial X = \emptyset$. For the first assertion use that the fundamental group of X has a presentation $\pi_1(X) = \langle a_1, \dots, a_n \mid r_1, \dots r_m \rangle$ with finitely many generators and relations (use that an arbitrary triangulation of X has only finitely many cells since X is compact). Now if $\rho \in \mathrm{Hom}(\pi_1(X), G)$, ρ is uniquely determined by the values $\rho(a_1), \dots, \rho(a_n)$, and satisfies $\rho(r_1) = \dots = \rho(r_m) = e$. Write $r_j = a_{i_{j1}} \cdots a_{i_{jn_j}}$ and let $f : G^n \to G^m$ be defined by

$$f(g_1, \dots, g_n) = (g_{i_{11}} \cdots g_{i_{1n_1}}, \dots, g_{i_{m1}} \cdots g_{i_{1n_m}}).$$

Then $\mathrm{Hom}(\pi_1(X), G)$ is homeomorphic with $f^{-1}(e, \dots, e)$, which is a closed set in the compact G^n. But then $\mathcal{M}_X \cong \mathrm{Hom}(\pi_1(X), G)/G$ is compact so has finitely many components, since the components of $\mathcal{M}_X$ are open. For the second assertion we let $\mathcal{M}_X(\eta)$ be the equivalence classes of flat connections in $\mathcal{C}_X(\eta)$, where η is a connection over the boundary ∂X. Recall here that the morphisms in $\mathcal{C}_X(\eta)$ are required to be the identity over ∂X. If X is closed $\mathcal{M}_X(\eta)$ is simply $\mathcal{M}_X$ and if η is not flat $\mathcal{M}_X(\eta) = \emptyset$. Now assume the dimension of X is ≤ 3. Then after fixing a G bundle $P \to X$ with $\eta \in \mathcal{A}_{\partial P}$ we can identify $\mathcal{M}_X(\eta)$ with $\mathcal{A}_P^F(\eta)/\mathcal{G}_P^o$ where $\mathcal{A}_P^F(\eta)$ are the flat connections on P which equal η on the boundary, and $\mathcal{G}_P^o$ is the elements in the gauge group which are the identity over ∂X. Note that by Proposition 2.8 and Lemma 2.10, we have an induced Chern–Simons action on the qoutient $\mathcal{A}_P^F(\eta)/\mathcal{G}_P^o$, which we also denote $e^{2\pi i S_X(-)}$. Now we can formulate our result, which is slightly more general than the second assertion above.

Proposition 3.10. *Let X be a compact 3–manifold. Then the Chern–Simons action is constant on the connected components of $\mathcal{M}_X(\eta)$.*

Proof. Let $P \to X$ be as above and let $\{\Theta_t\}_{t \in [0,1]}$ be a smooth curve in $\mathcal{A}_P^F(\eta)$. By the remarks after Proposition 3.4 we then have

$$S_X(p, \Theta_1) - S_X(p, \Theta_0) = 2 \int_0^1 ds \int_X \langle \Omega_s \wedge \dot{\Theta}_s \rangle = 0$$

for all sections $p : X \to P$, since $\Omega_s = 0$ for all $s \in [0,1]$. That is $e^{2\pi S_X(\Theta_1)} = e^{2\pi S_X(\Theta_0)} \in L_\eta$ so the Chern–Simons action is constant on the connected components of the space $\mathcal{A}_P^F(\eta)$. Now let $\pi : \mathcal{A}_P^F(\eta) \to \mathcal{A}_P^F(\eta)/\mathcal{G}_P^o$ be the projection onto the orbit space. Since $\Theta \mapsto \Theta \cdot \varphi = \varphi^*(\Theta)$ is continuous for all $\varphi \in \mathcal{G}_P^o$, it follows that π is an open mapping. Since $\mathcal{A}_P^F(\eta)$ and $\mathcal{A}_P^F(\eta)/\mathcal{G}_P^o$ are locally connected, they have open connected components. Now if D is a component of $\mathcal{A}_P^F(\eta)/\mathcal{G}_P^o$, we have that $D = \cup_{i \in I} \pi(C_i)$, where the C_i's are connected components of $\mathcal{A}_P^F(\eta)$. Since $e^{2\pi i S_X(-)}$ is constant on $\pi(C_i)$ for all $i \in I$ and since the $\pi(C_i)$ are open in D, it follows that $e^{2\pi i S_X(-)}$ is locally constant, hence constant on D. □

4. Topological quantum field theories

We start this section by giving a rough description of what is meant by an axiomatic topological quantum field theory denoted TQFT in the following. Thereafter we will say a few words about field theories and explain how the physicists obtain (topological) quantum field theories from these field theories. I will emphasize here, that the physicists approach rely heavily on the use of the Feynman Path integral in situations where this "integral" is not (yet) defined rigorously from a mathematical point of view.

We are mainly interested in the 2 + 1–dimensional Chern–Simons TQFT's. This theory was studied by Witten in [W], and on physical grounds he showed how this field theory leads naturally to invariants of compact oriented 3–manifolds and of links in these manifolds. Partially inspired by the ideas of Witten, Reshetikhin and Turaev [RT] constructed invariants of compact oriented 3–manifolds and of framed links in these manifolds by combinatorial means. They conjectured that their invariants are the same as the invariants introduced by Witten. However as long as the path integral is a nonrigorous object this conjecture can of course not be proved (or rejected).

An axiomatic $d+1$–dimensional TQFT (over $\mathbb{C}$) can be described by the following data:

D1. *There is a functor Z from the category of oriented closed d-dimensional manifolds and orientation preserving diffeomorphisms to the category of complex finite dimensional vector spaces and linear isomorphisms.*

D2. *To each oriented compact $d+1$–dimensional manifold X there is associated an element $Z(X) \in Z(\partial X)$.*

Futhermore it is assumed that the following axioms are satisfied:

A0. *If $F : X \to X'$ is an orientation preserving diffeomorphism then $Z(\partial F)Z(X) = Z(X')$.*

A1. *(Orientation) If $-Y$ is the manifold Y with the opposite orientation then $Z(-Y) = Z(Y)^*$, where $Z(Y)^*$ is the dual space of $Z(Y)$.*

A2. *(Multiplicativity) For a disjoint union $Y = Y_1 \sqcup Y_2$ we have*

$$Z(Y) = Z(Y_1) \otimes Z(Y_2).$$

A3. *(Gluing) Suppose $Y \hookrightarrow X$ is a closed oriented codimension one submanifold and X^{cut} is the manifold obtained by cutting X along Y. Then $\partial X^{cut} = \partial X \sqcup Y \sqcup -Y$ and*

$$Z(X) = Tr_Y(Z(X^{cut})),$$

where Tr_Y is the contraction

$$Tr_Y : Z(\partial X^{cut}) \cong Z(\partial X) \otimes Z(Y) \otimes Z(Y)^* \to Z(\partial X)$$

induced by the usual trace $Z(Y)^* \otimes Z(Y) \to \mathbb{C}$.

A4. *We have $Z(\emptyset) = \mathbb{C}$ for the empty d-manifold.*

A5. *We have $Z(I \times Y) = Id_{Z(Y)}$, where I is the unit interval.*

For a comprehensive introduction to axiomatic TQFT's the reader is referred to [FU]. We will let the axioms talk for themselves here. Quite often there is extra structure on the vector spaces $Z(Y)$ such as a hermitian metric. $Z(X)$ is called the partition function of X and $Z(Y)$ the state space. When $Z(Y)$ is endowed with a hermitian metric it will also be termed the Hilbert space.

The physicists produce (topological) quantum field theories by quantizing field theories. A $d+1$–dimensional field theory is described by the following data:

1) *An oriented compact smooth $d+1$–dimensional manifold X.*

2) *A space of fields Υ, which can be smooth functions on X, connections on smooth principal bundles over X etc.*

3) *A functional $\mathcal{L} : \Upsilon \to \mathbb{R}$ on the space of fields called the Lagrangian.*

4) *Possible a set of observables $\mathcal{O} : \Upsilon \to \mathbb{R}$.*

The physicists then proceed by using the path integral to "quantize" the field theory by defining:

5) *The partition function $Z(X) = \int_\Upsilon \exp(i\mathcal{L}(\phi))\mathcal{D}\phi$.*

6) *The expectation values of observables $\mathcal{O}$: $\langle \mathcal{O} \rangle = \int_\Upsilon \mathcal{O}(\phi) \exp(i\mathcal{L}(\phi))\mathcal{D}\phi$.*

The quantities $Z(X) = \langle 1 \rangle$ and $\langle \mathcal{O} \rangle$ are are here complex numbers. We shall need to generalize the above picture a little bit in that we will allow the action $\exp(i\mathcal{L}(-))$ to take values in a vector space with metric. If the expectation values do not depend on a choise of metric on the underlying manifold X, then the physicists say that the above data defines a $d+1$–dimensional topological (quantum) field theory. The space, the expectation values belong to, is called the state space or the Hilbert space.

It is usually natural to introduce a parameter k (the coupling constant) multiplying the Lagrangian, so that our expectation values become functions of k:

$$\langle \mathcal{O} \rangle(k) = \int_\Upsilon \mathcal{O}(\phi) e^{ikL(\phi)} \mathcal{D}\phi.$$

It is known that in many cases one cannot define a measure with the invariance properties normally assumed for the path integral "measure" $\mathcal{D}\phi$. However the physicists have a set of rules, which tell them how to "calculate" the path integrals. These calculations have often lead to predictions which subsequently have been proved rigorously. One standard technique for treating path integrals is to evaluate them as power series in $\frac{1}{k}$, using stationary phase approximation around the critical points of the Lagrangian. Roughly speaking they handle the integrand as if the above path integrals are well defined integrals over finite dimensional Euclidian space.

We will now specialize to the 2+1 dimensional theories introduced by Witten and Reshetikhin–Turaev. Witten follows the physicists path integral approach. Starting by a compact oriented 3–manifold X, the fields Υ are the space $\overline{\mathcal{C}_X}$ of equivalence classes of connections on G bundles over X, where $G = SU(N)$. As all G bundles over X are trivializable, $\Upsilon \cong \mathcal{A}_P/\mathcal{G}_P$ after fixing a G bundle $P \to X$. The Lagrangian is the Chern–Simons action $S_X(-)$ defined in section 2. As we have seen this is only a well defined functional on Υ if X is closed. Let us take a look on this case first. The partition function for X at level k is here

$$Z_k(X) = \int_\Upsilon e^{2\pi i S_X(\phi)} \mathcal{D}\phi,$$

where k is a positive integer. No measure $\mathcal{D}\phi$ has been constructed so far, so we must as usual regard the above as a formal expression. Thinking of the path integral as a proper integral we see that $Z_k(X) \in \mathbb{C}$.

Now if $\partial X \neq \emptyset$ we have the usual problem that $S_X(-)$ is no longer a well defined object. In the last section we saw that the solution is to work with the spaces $\Upsilon(\eta) = \mathcal{A}_P(\eta)/\mathcal{G}_P^o$ as the spaces of fields. The partition function at level k is a map depending on η

$$Z_k(X)(\eta) = \int_{\Upsilon(\eta)} e^{2\pi i k S_X(\phi)} \mathcal{D}\phi \in L_\eta^k.$$

Here L^k_η is the fiber over $\eta \in \mathcal{A}_{\partial P}$ of the hermitian line bundle $L^k_{\partial P} = \otimes^k L_{\partial P} \to \mathcal{A}_{\partial P}$ where $L_{\partial P} \to \mathcal{A}_{\partial P}$ is the hermitian Chern Simons line bundle modelled on ∂P. The action $e^{2\pi i k S_X(-)} = \otimes^k e^{2\pi i S_X(-)}$ is well defined on $\Upsilon(\eta)$ as noted before Proposition 3.10. We have thereby defined a section of the bundle $L^k_{\partial P} \to \mathcal{A}_{\partial P}$. The action of $\mathcal{G}_{\partial P}$ on $\mathcal{A}_{\partial P}$ lifts to an action on $L^k_{\partial P}$ by $(v_1 \otimes \ldots \otimes v_k) \cdot \psi = \psi^*(v_1) \otimes \ldots \psi^*(v_k)$, and we have that $Z_k(X)$ is an equivariant section under this action. To see this let $\psi \in \mathcal{G}_{\partial P}$ and let $\varphi \in \mathcal{G}_P$ be an extension of ψ and let $\Theta \in \mathcal{A}_P$. Then

$$(e^{2\pi i k S_X(\Theta)}) \cdot \psi = \otimes^k \psi^*(e^{2\pi i S_X(\Theta)}) = \otimes^k e^{2\pi i S_X(\varphi^*(\Theta))}$$

and $\partial\varphi^*(\Theta) = (\partial\varphi)^*(\partial\Theta)$. Futhermore $e^{2\pi i S_X(\varphi^*(\Theta))}$ is independent of the choise of extension φ by Proposition 2.8 and Lemma 2.10. It follows that we formally have $Z_k(X)(\eta \cdot \psi) = Z_k(X)(\eta) \cdot \psi$. In particular $Z_k(X)$ restricts to an equivariant section on the flat irreducible connections, and hence induces a section of the bundle $(L^s_{\partial P})^k \to \mathcal{M}^s_{\partial P}$, where $L^s_{\partial P} \to \mathcal{M}^s_{\partial P}$ is the hermitian line bundle from Proposition 3.8. At a first glance then the path integral at level k associate to every closed oriented 3–manifold a complex number and to every compact oriented 3–manifold with nonempty boundary a section in a certain hermitian line bundle.

A more elaborate analysis (which on a physical level of rigor is done by Witten in [W]) suggests that the Hilbert space $Z_k(\partial X)$ we are looking after, is a certain space of sections on a vector bundle over the Teichmüller space of complex structures on ∂X. The fibers in this bundle are constructed from the line bundle $(L^s_{\partial P})^k \to \mathcal{M}^s_{\partial P}$ in a certain way which I will indicate now.

Let Y be a closed oriented 2–manifold and let $Q \to Y$ be a G bundle. Let $L^s_Q \to \mathcal{M}^s_Q$ be the smooth hermitian line bundle with unitary connection ∇ and symplectic form ω on $\mathcal{M}^s_Q$ from Proposition 3.8. Write $M = \mathcal{M}^s_Q$ and $L = L^s_Q$, and let T be the Teichmüller space of complex structures on Y. (A Riemannian metric on the oriented 2–manifold Y defines a complex structure on Y and makes Y a Riemann surface. The Teichmüller space singles out the distinct complex structures on Y. There is a one to one corespondence between the complex structures and conformal equivalence classes of metrics.) Now let Y_σ be the Riemann surface (Y, σ) for $\sigma \in T$. We then have the Hodge $*$-operator $* : \Omega^1(Y_\sigma, \mathbb{C}) \to \Omega^1(Y_\sigma, \mathbb{C})$ defined by

$$*\tau := i(\bar{\tau}_1 - \bar{\tau}_2), \tag{20}$$

where we write $\tau = \tau_1 + \tau_2$ according to the usual decomposition $\Omega^1(Y_\sigma, \mathbb{C}) = \Omega^{1,0}(Y_\sigma) \oplus \Omega^{0,1}(Y_\sigma)$. (The complex structure σ corresponds as mentioned above to a conformal equivalence class of metrics on Y. Now if R_σ is a representative for this class, we have a star operator on $\Omega^*(Y)$ induced by this Riemannian metric and the orientation on Y. It can be shown that this star operator restricted to $\Omega^1(Y)$ is independent of the chosen representatative R_σ and coincide with the $*$-operator defined in (20).) $*$ induces an operator $* : \Omega^1(Y, \mathfrak{g}_Q) \to \Omega^1(Y, \mathfrak{g}_Q)$ by tensoring with the identity on $\Omega^0(\mathfrak{g}_Q)$. By Hodge theory we have an isomorphism

$$H^1(Y, d_\eta) \cong \mathcal{H}^1(Y, d_\eta) := \{\varphi \in \Omega^1(Y, \mathfrak{g}_Q) \mid d_\eta\varphi = 0,\ d^*_\eta\varphi = 0\}$$

for $\eta \in \mathcal{A}_Q$, where $d^*_\eta = - * d_\eta *$ is the adjoint to d_η with respect to an L^2–inner product on $\Omega(\Lambda^*(T^*Y) \otimes \mathfrak{g}_Q)$, see Remark 3.3. $\mathcal{H}^1(Y, d_\eta)$ are the so-called harmonic forms. By [AB] we have an identification

$$T_\eta M \cong H^1(Y, d_\eta).$$

The $*$–operator is an automorphism on the harmonic forms and $*^2 = -1$, so by the above identifications we have then obtained an almost complex structure J_σ on M. We denote M with this almost complex structure by M_σ. J_σ is compatible with our symplectic form ω on M. This follows since the pullback of ω to the space of irreducible flat connections on Q is given by the 2–form (19) in Proposition 3.8, which we denote by $\tilde{\omega}$ here. Now by Remark 3.3 this is given by

$$\tilde{\omega}(\varphi, \psi) = -2 \int_Y (\varphi \wedge \psi)_Q = 2(\varphi, *\psi) \quad , \varphi, \psi \in \Omega^1(Y, \mathfrak{g}_Q),$$

where $(\cdot,\cdot)$ is a symmetric form on $\Omega^1(Y,\mathfrak{g}_Q)$. But then it follows that

$$\tilde{\omega}(*\varphi,*\psi) = -2(*\varphi,\psi) = -2(\psi,*\varphi) = -\tilde{\omega}(\psi,\varphi) = \tilde{\omega}(\varphi,\psi)$$

which shows that J_σ is compatible with ω. It can also be shown that J_σ is integrabel. M_σ is an open subvariety of a closed projective algebraic variety $\tilde{M}_\sigma$. Futhermore the space of holomorphic sections $H^0(M_\sigma,L)$ is isomorphic with $H^0(\tilde{M}_\sigma,L)$ and $\dim(H^0(\tilde{M}_\sigma,L)) < \infty$. Now there is a vector bundle $H \to T$ over the Teichmüller space with fiber $H_\sigma = H^0(M_\sigma,L)$. A theorem states that there on this bundle is a so-called projectively flat connection ∇_H. If the genus g of Y is ≥ 2, T is topologically an open cell of dimension $6g-6$ an is therefore contractible. If ∇_H was flat we would have a trivialization of $H \to T$ and could define the Hilbert space $Z(Y)$, we a looking after, to be the covariant constant sections in the bundle $H \to T$ with respect to ∇_H. One obtains a flat vector bundle over T by tensoring with a certain line bundle L'_D constructed from the determinant line bundle $L_D \to T$ over the Teichmüller space with fiber $L_{D,\sigma} = \det H^1(Y_\sigma,\mathcal{O})$ (for more details, see [An]).

Theorem 4.1. *There exists a flat bundle $Z \to T$ with fiber $Z_\sigma = H^0(M_\sigma,L) \otimes L'_{D,\sigma}$, where L'_D is a certain line bundle constructed from L_D.*

We then define $Z(Y)$ to be the covariant constant sections of this bundle. The story above can be repeated with the bundle $L \to M$ replaced by the bundle $L^{\otimes k} \to M$. Note that the bundle L'_D in Theorem 4.1 depends on this k. We denote the resulting Hilbert space by $Z_k(Y)$. It is then believed that our partition function $Z_k(X)$ calculated via the path integral takes values in $Z_k(\partial X)$ and that this pair constitute our Chern–Simons TQFT, see [W] and [RSW]. Actually if one uses the formal properties of the path integral and the properties of the Chern–Simons action described in Theorem 2.15 one ends up with a TQFT satisfying the axioms A0-A6. Note here that the Chern–Simons action $e^{2\pi i S_X(-)}$ does not depend on a choise of metric on X. (However as pointed out in [W] this is not enough to secure that the Chern–Simons quantum field theory is a TQFT. Actually it turns out that one needs to choose a framing of X, see next section, to obtain a topological invariant. Thus one has an invariant for each positive integer k and each choise of framing.)

On the other hand Reshetikhin and Turaev's combinatorial and mathematical rigorous approach also produces a TQFT satisfying these axioms. Let us denote Witten's and Reshetikhin and Turaev's invariants at level k respectively by $\mathrm{W}_k(-)$ and $\mathrm{RT}_k(-)$ as in the introduction. From now on we assume that our 3–manifold X is closed. One main difference between the two approaches is then that in Witten's theory, the formal stationary phase approximation of the path integral predicts a formula for the asymptotic behaviour of $\mathrm{W}_k(X)$ in the large k limit. If the moduli space $\mathcal{M}_X$ of flat connections is discrete, this formula is a sum over the points in $\mathcal{M}_X$ according to [W], [FG], [J] and [Ro], see below. What is really interesting about this asymptotic expression is that it involves such quantities as Chern–Simons invariants, Reidemeister (or Ray–Singer analytic) torsions, spectral flows and dimensions of certain cohomology groups, quantities which do not enter into the theory of Reshetikhin and Turaev in any obvious way. Moreover, the theory of Reshetikhin and Turaev does not at all produce such an asymptotic formula. Their invariants are expressed as polynomials in a root of unity, whose order is some power of K ($K = k+n$ for the $\mathrm{SU}(n)$-theory). Thus the asymptotic behaviour of $\mathrm{RT}_k(X)$ as $k \to \infty$ is far from obvious. Nevertheless, if $\mathrm{RT}_k(X) = \mathrm{W}_k(X)$ for all k, their should be such an asymptotic formula for $\mathrm{RT}_k(X)$. In [FG] Freed and Gompf investigate the large k limit numerically for lens spaces and Briskorn spheres, and their results confirm the asymptotic formula for $\mathrm{RT}_k(-)$ predicted by the path integral in these cases. (In these computer tests they work with well-defined explicit expressions of the invariants of the spaces under investigation.) It should be said that the asymptotics of Witten and also the asymptotics investigated by Freed and Gompf are the leading order large k asymptotics (also called the semiclassical approximation). In [J] Jeffrey finds the exact asymptotic expansion to all order of $\mathrm{RT}_k(-)$ in the case of lens spaces for $G = \mathrm{SU}(2)$. She also determine these asymptotics for certain torus bundles over S^1 for any compact simply laced simply connected simple Lie group G. Her results also agree with the asymptotic formula predicted by the path integral. Inspired by Jeffrey's work, Rozansky studies in [Ro] the case of 3–fibered and more generally n–fibered Seifert manifolds with base S^2. However, several points need to be clarified in that paper. In the next section we take up a careful examination of Rozansky's results.

It is a major challenge to find a relation between the asymptotic formula predicted by the path integral and the RT–invariants for an arbitrary closed 3–manifold X. This could be done either by finding a mathematical rigorous way to define the path integral and then prove the asymptotic expression predicted by that path integral and the equality $\mathrm{W}_k(X) = \mathrm{RT}_k(X)$, or by finding an asymptotic formula for $\mathrm{RT}_k(X)$ in the limit of large k and then show, that this is equivalent to the asymptotic formula predicted by the path integral. Both approaches are difficult, but probably the second approach is the most likely to succeed in the near future.

We end this section by conjecturing an asymptotic formula for Witten–Reshetikhin–Turaev's invariant, for an arbitrary closed 3–manifold X. As mentioned above Witten calculates in [W] the semiclassical approximation of the Witten invariant in the case where the moduli space is discrete, hence finite. In [FG] Freed and Gompf make some minor corrections to this formula and conjecture a more general expression. Their result is

$$\begin{aligned} Z_k(X) \quad &\sim \quad \frac{1}{2} e^{3\pi i(1+b^1(X))/4} \sum_i \tau_X(A_i)^{1/2} e^{-2\pi i(I_{A_i}/4 + (\dim H^0_{A_i} + \dim H^1_{A_i})/8)} \\ &\times \quad e^{2\pi i(k+2)S_X(A_i)} (k+2)^{(\dim H^1_{A_i} - \dim H^0_{A_i})/2}, \end{aligned}$$

where $H^*_A = H^*(X, d_A)$ and $S_X(A)$ is the Chern–Simons action defined in Section 2. Here $\sim$ indicates that the right-hand side is an asymptotic expansion of the left hand side, meaning that we could replace $\sim$ by an equality if we at the same time add a remainder term on the right-hand side which is small compared to the shown expression on the right-hand side for large k. The number $b^1(X)$ is the first Betti number of X as usual and $\tau_X(A)$ is the Reidemeister torsion of the complex $(\Omega^*(X, \mathfrak{g}_P), d_A)$, where we have fixed a $SU(2)$ bundle $P \to X$. Finally I_A is the spectral flow associated to the Atiyah–Patodi–Singer operator $D_A = \begin{bmatrix} *d_A & -d_A* \\ d_A* & 0 \end{bmatrix} \in \mathrm{End}(\Omega^1(X, \mathfrak{g}_P) \oplus \Omega^1(X, \mathfrak{g}_P))$ as we run along a path from the product connection (recall that P is trivializable) to A. Like for 2–manifolds $H^0(X, d_A)$ can be identified with the Lie algebra of the stabilizer of A in the gauge group. If A is irreducible this cohomology group therefore vanishes. For an irreducible flat connection A we also have an identification $T_{[A]}\mathcal{M}^s_X \cong H^1(X, d_A)$ where $\mathcal{M}^s_X$ is the manifold of flat irreducible connections (to be precise here $H^1(X, d_A)$ is the so-called Zariski tangent space of the moduli space $\mathcal{M}_X$). If the moduli space is discrete and if A is irreducible, we then have that $H^*(X, d_A) = 0$ by Poincaré duality. In this case the Reidemeister torsion is a well defined nonnegative real number. If the moduli space is no longer discrete, one can interpret $\tau_X(A)^{1/2}$ as an element in $\det H^1(X, d_A)^*$, see [J]. It is then believed that these "square roots" in a certain sense define density functions on the connected components of the moduli space, that is they define integration measures on these components.

In general when physicists make a stationary phase approximation of the path integral $\int_\Upsilon e^{2\pi i k \mathcal{L}(\phi)} \mathcal{D}\phi$, each stationary point ϕ_i of the Lagrangian $\mathcal{L} : \Upsilon \to \mathbb{R}$ contributes a classical exponential $e^{2\pi i k \mathcal{L}(\phi_i)}$ times an asymptotic series in $1/k$:

$$\int_\Upsilon e^{2\pi i k \mathcal{L}(\phi)} \mathcal{D}\phi \sim \sum_i e^{2\pi i k \mathcal{L}(\phi_i)} \left(\sum_{j=0}^{\infty} c^i_j k_j \right)$$

We assume here that the stationary points are isolated in the space of fields. (If this is not the case, the physicists use a special trick called the Faddeev-Popov procedure to get an asymptotic expression. For more on this subject, see [BN, Chap. 1].)

We have seen that the Chern–Simons action is constant on the connected components of the moduli space $\mathcal{M}_X$ and that $\mathcal{M}_X$ has only finitely many components, see the end of Section 3. In the light of all these observations we speculate that the full asymptotic expansion of the RT–invariants of an arbitrary closed 3–manifold X is given by

$$Z_k(X) \sim \sum_{l=1}^{n} e^{2\pi i k \alpha_l} k^{d_l} \sum_{j=0}^{\infty} c_j^l k^{-j},$$

where the sum has to be thougt of as a sum over the components $\mathcal{M}_X^l$ of the moduli space. Here $e^{2\pi i k\alpha_l}$ is the constant value of the Chern–Simons action on the component $\mathcal{M}_X^l$, and one should think of d_l as $d_l = \max_{A\in\mathcal{M}_X^l} 1/2(\dim H^1(X,d_A) - \dim H^0(X,d_A))$, see however below about these d_l. Finally it is believed that the c_j^l are complex numbers given by certain integrals of functions on the components of the moduli space with respect to the integration measures, which generalize the square root of the Reidemeister torsion, that is $c_j^l = \int_{A\in\mathcal{M}_X^l} f_j^l(A)\tau_X^{1/2}(A)$. If we let $b_l = c_0^l$ we get (if $c_0^l \neq 0$ which we assume here)

$$Z_k(X) \sim \sum_{l=1}^{n} b_l e^{2\pi i k\alpha_l} k^{d_l} \left(1 + \sum_{j=1}^{\infty} a_j^l k^{-j}\right). \tag{21}$$

One interesting question which arise in connection with the expression (21) is the following: If we can show that this asymptotic expression is a topological invariant is there then topological information hidden in the numbers α_l, b_l, d_l and a_j^l which are functions of the 3–manifolds? Is any or maybe all of these functions topological invariants?

It is not at the moment clear what is the right interpretation of the d_l. One can define a generic max $d_l^{gen} = \max_{A\in U_l} d(A)$ where $d(A) = 1/2(\dim H^1(X,d_A) - \dim H^0(X,d_A))$. Here U_l is a certain open dense subset of the component $\mathcal{M}_X^l$ of the projective algebraic variety $\mathcal{M}_X$. As above we can also define the total max $d_l^{\max} = \max_{A\in\mathcal{M}_X^l} d(A)$. (An instructive way of thinking of $\mathcal{M}_X$ here is as a union of manifolds (maybe with different dimensions) such that $\mathcal{M}_X$ is locally diffeomorphic with euclidian spaces (or half-spaces) except in some (isolated) points called the singularities (where the different manifold parts intersect nonsmoothly). One can then think of the manifolds in topdimension as partitioned into (finitely many) open subsets each on which $d(-)$ is constant. Then d_l^{gen} is the maximum of these constant values, and d_l^{max} is the maximum of these values and the values of $d(-)$ in the singularities belonging to the component $\mathcal{M}_X^l$.) It is then believed that one shall use a d_l such that $d_l^{gen} \leq d_l \leq d_l^{max}$. Their are situations where $d_l^{gen} < d_l^{max}$, and in these situations it is not yet known what is the correct choice of the d_l.

A question which arise is the following: If the asymptotic expansion is itself a topological invariant, how strong is this invariant compared to the original invariant $Z_k(X)$? Do their exist two closed 3–manifolds X_1 and X_2 with the same asymptotic expansion but with $Z_k(X_1) \neq Z_k(X_2)$ for some $k \in \mathbb{N}$? It is my intension to investigate all these matters in the futere.

5. Large k asymptotics of Witten–Reshetikhin–Turaev's invariants of 3–fibered Seifert manifolds

The first main part of this section is devoted to the task of finding the exact asymptotic expansion in the limit of large k of the Reshetikhin–Turaev invariants of 3–fibered Seifert manifolds. We will concentrate on the $SU(2)$–case, and in all what follows $K = k + 2$.

Let $\mathcal{H}_{T^2}^{(k)}$ be the Hilbert space at level k for the standard torus $T^2 = S^1 \times S^1$ for Witten's path integral TQFT. This is a $k+1$ dimensional complex vector space with a hermitian metric and a destinguished orthonormal basis $\mathbf{1},\mathbf{2},\ldots,\mathbf{k+1}$. $\mathcal{H}_{T^2}^{(k)}$ caries a unitary representation $\mathcal{R}$ of the mapping class group of the torus, $SL(2,\mathbb{Z})$. The standard generators for $SL(2,\mathbb{Z})$ are

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We write $\tilde{M}$ for the matrix of $\mathcal{R}(M)$ in the basis $\mathbf{1},\mathbf{2},\ldots,\mathbf{k+1}$. Then

$$\tilde{S}_{jl} = \sqrt{\frac{2}{K}} \sin\left(\frac{jl\pi}{K}\right), \quad \tilde{T}_{jl} = e^{-\frac{i\pi}{4}} \exp\left(\frac{i\pi}{2K} j^2\right) \delta_{jl}. \tag{22}$$

Actually $\mathcal{R}$ is a representation of the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$, so $\mathcal{R}(S^2) = \mathcal{R}((ST)^3) = 1$. Now let $p_1/q_1, \ldots, p_n/q_n$ be rational numbers in lowest terms. The n–fibered Seifert manifold

$$X = X(p_1/q_1, \ldots, p_n/q_n) \tag{23}$$

is obtained from $S^2 \times S^1$ by cutting out n disjoint solid tori $D_i \times S^1$ ($D_1, \ldots, D_n$ disjoint disks in S^2) and gluing them back after twisting the boundary of $D_i \times S^1$ by a matrix $M_i = M(p_i, q_i) \in SL(2,\mathbb{Z})$, where $\begin{pmatrix} p_i \\ q_i \end{pmatrix}$ is the first column in M_i, i.e. $M_i = \begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix}$, where $p_i s_i - q_i r_i = 1$. The particular choise of the r_i and s_i affects the framing (see below), but not the diffeomorphism type of the resulting manifold. By [FG, Formula (1.17)] we have

$$Z_k(X) = \sum_{\alpha_1,\cdots,\alpha_n=1}^{K-1} (\tilde{M}_1)_{\alpha_1 1} \ldots (\tilde{M}_n)_{\alpha_n 1} N_{\alpha_1 \ldots \alpha_n}, \tag{24}$$

where $N_{\alpha_1 \ldots \alpha_n}$ are the so-called Verlinde numbers given by

$$N_{\alpha_1 \ldots \alpha_n} = \sum_{\beta=1}^{K-1} \frac{\tilde{S}_{\alpha_1 \beta} \tilde{S}_{\alpha_2 \beta} \ldots \tilde{S}_{\alpha_n \beta}}{(\tilde{S}_{1\beta})^{n-2}} \tag{25}$$

for $n \geq 2$. For $n = 1$ we have $N_\alpha = \delta_{\alpha,1}$. Note that $N_{\alpha_1 \alpha_2} = \delta_{\alpha_1,\alpha_2}$, since $\tilde{S}$ is unitary with $\tilde{S}^2 = I$ and $\tilde{S}$ is real. $X(q/p) = L(p,q)$ are the so-called lens spaces, and we see that $Z_k(L(p,q)) = (\tilde{M}(q,p))_{11}$.

In [W] the Verlinde number $N_{\alpha_1 \ldots \alpha_n}$ is identified with the invariant of $S^2 \times S^1$ containing a n–component link, where the α_i are certain numbers attached to the components. Actually the above expression (24) for $Z_k(X)$ follows from arguments given in [W]. However since these arguments are nonrigorous, it is important to note here that the right-hand side in (24) equals Reshetikhin–Turaev's invariant of the Seifert manifold X at level k. This follows from a result in [KM]. Therefore we also just write $Z_k(X)$ instead of $W_k(X)$ and $RT_k(X)$ for a Seifert manifold X.

As mentioned in Sect. 4, Witten's invariant is only "well-defined" after a choise of framing of the 3–manifold. When we compare our results with the asymptotic formula predicted by the path integral, we therefore need to keep track of which framing the path integral invariants are calculated with respect to. Let X be a closed oriented 3–manifold. A 2–framing (or just a framing) for X is a homotopy equivalence class of trivializations of $TX \oplus TX$ viewed as a $Spin(6)$ bundle. According to Atiyah [A1] there is a *canonical* framing for X. The possible framings correspond to $\mathbb{Z}$. A change of framing by one unit multiplies the path integral $Z_k(X)$ by $\exp\left(-\frac{i\pi c}{12}\right)$, where $c = \frac{3k}{k+2}$ for the $SU(2)$-theory, so we see that the numerical value of $Z_k(X)$ is not affected by the choise of framing. We adopt the convention that the canonical framing on X corresponds to the zero element in $\mathbb{Z}$. For $m \in \mathbb{Z}$ we let $Z_k(X, m)$ be $Z_k(X)$ calculated with respect to the framing on X corresponding to m. One way of calculating $Z_k(X)$ is by using the way the path integral is acting under surgery (see the gluing law for a TQFT, axiom A3 in Sect. 4). Every closed oriented 3–manifold can be obtained by rational surgery on some link $L = \sqcup L_i$ in S^3 (or on some link in another fixed closed oriented 3–manifold). Now let X_L be a closed oriented 3–manifold obtained by doing surgery on the link L in S^3. Suppose we can calculate the path integral $Z_k(X_L)$ from our knowledge of the path integral $Z_k(S^3, 0)$ and the given surgery data. Then the question is which framing $Z_k(X_L)$ is calculated with respect to? This problem is studied carefully in [FG]. Lets review their results here. A specific surgery along L is carried out in the following way. For each component L_i we cut out a tubular neighborhood of L_i (a solid torus) and glue it back after twisting the boundary by a $SL(2, \mathbb{Z})$-matrix $M_i = \begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix}$. The diffeomorphism type of the resulting manifold is independent of the r_i and s_i. However the framing is sensitive to our choise of these numbers. Let us first look on integer surgery (where the q_i are all equal to 1). Then we will use the convention that $M_i = \begin{pmatrix} p_i & -1 \\ 1 & 0 \end{pmatrix} = T^{p_i} S$. If S^3 is given the canonical framing, then there is a specific framing $\varphi_L \in \mathbb{Z}$ on X_L determined uniquely by the above surgery data. If one calculate $Z_k(X_L)$ from the gluing law by using the surgery description and $Z_k(S^3, 0)$, then one actually ends up with $Z_k(X_L, \varphi_L)$. The integer φ_L is determined in [FG] Theorem 2.3. To get $Z_k(X_L, 0)$ one therefore needs to multiply $Z_k(X, \varphi_L)$ by $\exp\left(i\frac{\pi c}{12}\varphi_L\right)$ (according to the sign convention in [FG]). Now if we do rational surgery, there is a way to reduce it to integer surgery. If $M \in SL(2, \mathbb{Z})$ we can always write $M = T^{a_t} S \ldots T^{a_1} S$ and according to [J, Proposition 2.5], the a_i form a continued fraction expansion of p/q. That is,

$$p/q = a_t - \cfrac{1}{a_{t-1} - \frac{1}{\cdots \frac{1}{a_1}}}, \qquad a_i \in \mathbb{Z},$$

which we abbreviate $p/q = (a_1, \ldots, a_t)$. Now let $M_i = T^{a^{(i)}_{t_i}} S \ldots T^{a^{(i)}_1} S$. Then the rational surgery (M_i, L_i) can be replaced by integer surgery on a t_i-component link with matrices $M_i = \begin{pmatrix} a^{(i)}_j & -1 \\ 1 & 0 \end{pmatrix} = T^{a^{(i)}_j} S$ without changing the diffeomorphism type of the resulting manifold X_L, see Fig. 1, [FG] and [R, Chap. 9]. According to the above there is a specific framing on X_L determined uniquely by these integer surgery data.

Figure 1

Now let X be the n–fibered Seifert manifold (23), and let M_i be as above $i = 1, \dots, n$, such that $p_i/q_i = (a_1^{(i)}, \dots, a_{t_i}^{(i)})$. Then the invariant at level k with respect to the canonical framing on X is

$$Z_k(X) = e^{i\phi_{fr}} \sum_{\alpha_1,\dots,\alpha_n=1}^{K-1} (\tilde{M}_1)_{\alpha_1 1} \dots (\tilde{M}_n)_{\alpha_n 1} N_{\alpha_1 \dots \alpha_n}. \tag{26}$$

Here

$$\phi_{fr} = \frac{\pi c}{12}\left[-3\sigma + \sum_{i=1}^{n} \sum_{j=1}^{t_i} a_j^{(i)} \right], \tag{27}$$

where $c = \frac{3k}{k+2}$ and $\sigma = -\text{sign}\left(\sum_{i=1}^n \frac{q_i}{p_i}\right) + \sum_{i=1}^n \text{sign}\left(\frac{p_i}{q_i}\right) + \sum_{i=1}^n \sum_{j=1}^{t_i-1} \text{sign}(a_j^{(i)})$. Actually, Freed and Gompf conjecture in [FG, p. 105] that the Reshetikhin–Turaev invariants (if they equal Witten's invariants) are calculated with respect to the canonical framing. In the following we will not write any framing factor, but use (24). We can always at the end adjust our result to the canonical framing by multiplying with $e^{i\phi_{fr}}$. If $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2,\mathbb{Z})$ we have by [J, Proposition 2.7 (a)] that

$$\begin{aligned} \tilde{M}_{\alpha\beta} &= -i\frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(M)} \exp\left(\frac{i\pi}{2Kq}s\beta^2\right) \\ &\times \sum_{\substack{\gamma \pmod{2Kq} \\ \gamma = \alpha \pmod{2K}}} \exp\left(\frac{i\pi}{2Kq}p\gamma^2\right)\left\{\exp\left(\frac{i\pi}{Kq}\gamma\beta\right) - \exp\left(-\frac{i\pi}{Kq}s\beta^2\right)\right\} \\ &= i\frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(M)} \sum_{\mu=\pm1} \sum_{\substack{\gamma \pmod{2Kq} \\ \gamma = \alpha \pmod{2K}}} \mu \exp\left(\frac{i\pi}{2Kq}[p\gamma^2 - 2\mu\gamma\beta + s\beta^2]\right) \\ &= i\frac{\text{sign}(q)}{\sqrt{2K|q|}} e^{-\frac{i\pi}{4}\Phi(M)} \sum_{\mu=\pm1} \sum_{n=0}^{|q|-1} \mu \exp\left(\frac{i\pi}{2Kq}[p(\alpha+2Kn)^2 - 2\mu\beta(\alpha+2Kn) + s\beta^2]\right), \end{aligned}$$

where Φ is the Rademacher phi function defined by

$$\Phi\begin{bmatrix} p & r \\ q & s \end{bmatrix} = \begin{cases} \frac{p+s}{q} - 12(\text{sign}(q))s(s,|q|) & ,q \neq 0 \\ \frac{r}{s} & ,q = 0. \end{cases} \tag{28}$$

Here, for $q > 0$ the Dedekind sum $s(s,q)$ is defined by

$$s(s,q) = \frac{1}{4q}\sum_{j=1}^{q-1} \cot\frac{\pi j}{q} \cot\frac{\pi s j}{q}. \tag{29}$$

If $M_i = \begin{pmatrix} p_i & r_i \\ q_i & s_i \end{pmatrix} \in SL(2,\mathbb{Z})$ such that $M_3 = M_1 M_2$ we have

$$\Phi(M_3) = \Phi(M_1) + \Phi(M_2) - 3\text{sign}(q_1 q_2 q_3). \tag{30}$$

Since the representation $\mathcal{R}$ of $SL(2,\mathbb{Z})$ is unitary we have $\mathcal{R}(M^{-1}) = \mathcal{R}(M)^*$, so $\tilde{M}_{\alpha\beta} = \overline{(\widetilde{M^{-1}})_{\beta\alpha}}$. Now since $M \in SL(2,\mathbb{Z})$, $M^{-1} = \begin{pmatrix} s & -r \\ -q & p \end{pmatrix}$, and (30) implies that $\Phi(M^{-1}) = -\Phi(M)$. All in all this implies that

$$
\begin{aligned}
\tilde{M}_{\alpha\beta} &= i\frac{\operatorname{sign}(q)}{\sqrt{2K|q|}}e^{-\frac{i\pi}{4}\Phi(M)} \\
&\times \sum_{\mu=\pm1}\sum_{n=0}^{|q|-1}\mu\exp\left(\frac{i\pi}{2Kq}[s(\beta+2Kn)^2-2\mu\alpha(\beta+2Kn)+p\alpha^2]\right), \qquad (31)
\end{aligned}
$$

which is identical with [Ro, Formula (2.25)]. Before going any futher I will show some technical lemmas. They are necessary to secure that the method we will use to rewrite the partition function for the 3–fibered Seifert manifolds is exact. We will need Poisson's summation formula

$$
\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n}\varphi(k_1,\dots,k_n)=\sum_{(m_1,\dots,m_n)\in\mathbb{Z}^n}\int_{\mathbb{R}^n}e^{2\pi i(m_1x_1+\cdots+m_nx_n)}\varphi(x_1,\dots,x_n)dx_1\dots dx_n. \qquad (32)
$$

This is valid for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space of smooth functions, that are rapidly decreasing at infinity), see [Ho, p. 178].

Lemma 5.1. *Let $F:\mathbb{Z}^n\to\mathbb{C}$ be a function periodic in all variables with a period of N_i in the ith variabel. Let $P_i(x)=A_ix^2+B_ix+C_i$ be polynomials with real coefficients such that $A_i>0$ for all i. Then*

$$
\begin{aligned}
&\sum_{k_1=0}^{N_1-1}\dots\sum_{k_n=0}^{N_n-1}F(k_1,\dots,k_n) \\
&= \left(\prod_{i=1}^n\sqrt{A_i}N_i\right)\lim_{\epsilon\to0_+}\epsilon^{n/2}\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n}e^{-\pi\epsilon f(\epsilon)(P_1(k_1)+\cdots+P_n(k_n))}F(k_1,\dots,k_n)
\end{aligned}
$$

for all functions $f:]0,a]\to]0,\infty[$ with $\lim_{\epsilon\to0_+}f(\epsilon)=1$.

Proof. The series $\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n}e^{-\pi\epsilon f(\epsilon)(P_1(k_1)+\cdots+P_n(k_n))}F(k_1,\dots,k_n)$ is absolutely convergent since F is periodic in all variables, and hence

$$
\begin{aligned}
&\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n}e^{-\pi\epsilon f(\epsilon)(P_1(k_1)+\cdots+P_n(k_n))}F(k_1,\dots,k_n) \\
&= \sum_{j_1=0}^{N_1-1}\dots\sum_{j_n=0}^{N_n-1}F(j_1,\dots,j_n)\sum_{(k_1,\dots,k_n)\in\mathbb{Z}^n}e^{-\pi\epsilon f(\epsilon)(P_1(j_1+k_1N_1)+\cdots+P_n(j_n+k_nN_n))}.
\end{aligned}
$$

We therefore just have to show that

$$
\lim_{\epsilon\to0_+}\epsilon^{1/2}\sqrt{A}N\sum_{k\in\mathbb{Z}}e^{-\pi\epsilon f(\epsilon)P(j+kN)}=1
$$

for a polynomial $P(x)=Ax^2+Bx+C$ with real coefficients and $A>0$. It is enough to prove this for $P(x)=(d+Lx)^2$ where $L>0$ and $d\in\mathbb{R}$. Let $j\in\{0,1,\dots,N-1\}$ be fixed. By Poisson's formula we have

$$
Q(\epsilon):=\sum_{k\in\mathbb{Z}}e^{-\pi\epsilon f(\epsilon)(d+L(j+kN))^2}=\sum_{m\in\mathbb{Z}}\int_{-\infty}^{\infty}e^{2\pi imx}e^{-\pi\epsilon f(\epsilon)(d+L(j+xN))^2}dx.
$$

Now let $\epsilon \in]0,a]$ and let $k(\epsilon)=\epsilon f(\epsilon)$, $c=d+Lj$, $M=NL$ and $y=k(\epsilon)^{1/2}(c+xM)$. Then

$$\begin{aligned} Q(\epsilon) &= \frac{1}{LNk(\epsilon)^{1/2}}\sum_{m\in\mathbb{Z}} e^{-\frac{2\pi imc}{M}}\int_{-\infty}^{\infty}\exp\left(2\pi im\frac{y}{Mk(\epsilon)^{1/2}}\right)e^{-\pi y^2}dy \\ &= \frac{1}{LNk(\epsilon)^{1/2}}\sum_{m\in\mathbb{Z}} e^{-\frac{2\pi imc}{M}}\exp\left(-\frac{\pi m^2}{M^2k(\epsilon)}\right) \end{aligned}$$

since $\int_{-\infty}^{\infty} e^{by}e^{-ay^2}dy=\left(\frac{\pi}{a}\right)^{1/2}e^{b^2/4a}$ for $a,b\in\mathbb{C}$ with $Re(a)>0$. But then

$$LN\lim_{\epsilon\to 0_+}\epsilon^{1/2}Q(\epsilon)=\lim_{\epsilon\to 0_+}\sum_{m\in\mathbb{Z}} e^{-\frac{2\pi imc}{M}}\exp\left(-\frac{\pi m^2}{M^2k(\epsilon)}\right)$$

since $\lim_{\epsilon\to 0_+} f(\epsilon)=1$. Now

$$\sum_{m\in\mathbb{Z}} e^{-\frac{2\pi imc}{M}}\exp\left(-\frac{\pi m^2}{M^2k(\epsilon)}\right)=1+2\sum_{m=1}^{\infty}\cos\left(\frac{2\pi imc}{M}\right)\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}m^2\right)^2$$

and since $k(\epsilon)>0$ we have

$$\left|\sum_{m=1}^{\infty}\cos\left(\frac{2\pi imc}{M}\right)\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}m^2\right)^2\right|\le\left(\sum_{m=1}^{\infty}\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}m^2\right)\right)\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}\right).$$

Now choose $D>0$ so $f(\epsilon)\le D$ for all sufficiently small $\epsilon\in]0,a]$. Then $k(\epsilon)=\epsilon f(\epsilon)\le aD$ so

$$\sum_{m=1}^{\infty}\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}m^2\right)\le\sum_{m=1}^{\infty}\exp\left(-\frac{1}{2}\frac{\pi}{M^2aD}m^2\right)=C<\infty.$$

But then

$$\left(\sum_{m=1}^{\infty}\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}m^2\right)\right)\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}\right)\le C\exp\left(-\frac{1}{2}\frac{\pi}{M^2k(\epsilon)}\right)\to 0$$

when $\epsilon\to 0_+$. □

Lemma 5.2. *Let $F:\mathbb{Z}\to\mathbb{C}$ be a bounded function and let $f:]0,a]\to]0,\infty[$ and $h:]0,a]\to\mathbb{R}$ be functions such that $f(\epsilon)\to 1$ and $g(\epsilon)/\epsilon\to 0$ when $\epsilon\to 0_+$, where $g(\epsilon)=h(\epsilon)-1$. Let $P_\nu(x)=A_\nu x^2+B_\nu x+C_\nu$ be polynomials with real coefficients such that $A_1>0$. Then*

$$\lim_{\epsilon\to 0_+}\epsilon^{1/2}\sum_{n\in\mathbb{Z}}e^{-\epsilon f(\epsilon)P_1(n)}\exp(ih(\epsilon)P_2(n))F(n)=\lim_{\epsilon\to 0_+}\epsilon^{1/2}\sum_{n\in\mathbb{Z}}e^{-\epsilon f(\epsilon)P_1(n)}\exp(iP_2(n))F(n)$$

in the sense that if one of the two limits exists then does the other and they are equal.

Proof. We can assume that $P_1(n)=(c+Ln)^2$ where $L>0$ and $c\in\mathbb{R}$. Let $M\in]0,\infty[$ such that $|F(n)|\le M$ for all $n\in\mathbb{Z}$. We then have

$$\begin{aligned} Q(\epsilon) &:= \left|\epsilon^{1/2}\sum_{n\in Z}e^{-\epsilon f(\epsilon)(c+Ln)^2}\exp(ih(\epsilon)P_2(n))F(n)-\epsilon^{1/2}\sum_{n\in Z}e^{-\epsilon f(\epsilon)(c+Ln)^2}\exp(iP_2(n))F(n)\right| \\ &\le M\epsilon^{1/2}\sum_{n\in Z}e^{-\epsilon f(\epsilon)(c+Ln)^2}|\exp(ig(\epsilon)P_2(n))-1|. \end{aligned}$$

Now use that $|\exp(i\theta)-1| = 2|\sin(\theta/2)| \leq 2|\theta/2| = |\theta|$ for all $\theta \in \mathbb{R}$ so

$$Q(\epsilon) \leq M\epsilon^{1/2}|g(\epsilon)| \sum_{n\in\mathbb{Z}} e^{-\epsilon f(\epsilon)(c+Ln)^2}|P_2(n)|.$$

If we write $P_2(n) = b(d+Ln)^2 + u$ we get

$$\begin{aligned} Q(\epsilon) &\leq M\epsilon^{1/2}|g(\epsilon)||u| \sum_{n\in\mathbb{Z}} e^{-\epsilon f(\epsilon)(c+Ln)^2} + M\epsilon^{1/2}|g(\epsilon)||b| \sum_{n\in\mathbb{Z}} (c+Ln)^2 e^{-\epsilon f(\epsilon)(c+Ln)^2} \\ &+ M\epsilon^{1/2}|g(\epsilon)||b|(d-c)^2 \sum_{n\in\mathbb{Z}} e^{-\epsilon f(\epsilon)(c+Ln)^2} \\ &+ M\epsilon^{1/2}|g(\epsilon)||b|2(d-c) \sum_{n\in\mathbb{Z}} (c+Ln) e^{-\epsilon f(\epsilon)(c+Ln)^2}. \end{aligned}$$

We compare the sums $s(\lambda) = \sum_{n\in\mathbb{Z}}(c+Ln)^\lambda e^{-\epsilon f(\epsilon)(c+Ln)^2}$ with the integrals $I(\lambda) = \int_0^\infty (Lx)^\lambda e^{-\epsilon f(\epsilon)(Lx)^2}\,dx$, $\lambda = 0,1,2$. Let $G(x) = Lxe^{-\epsilon f(\epsilon)L^2x^2}$ and $H(x) = L^2x^2e^{-\epsilon f(\epsilon)L^2x^2}$. Then an easy computation shows that G is increasing in $[0,a_\epsilon]$ and decreasing in $[a_\epsilon,\infty[$ and H is increasing in $[0,b_\epsilon]$ and decreasing in $[b_\epsilon,\infty[$, where $a_\epsilon = (2\epsilon f(\epsilon)L^2)^{-1/2}$ and $b_\epsilon = (\epsilon f(\epsilon)L^2)^{-1/2}$. Now $G(a_\epsilon) = (2e\epsilon f(\epsilon))^{-1/2}$ and $H(b_\epsilon) = (e\epsilon f(\epsilon))^{-1}$ and $I(0) = (2L)^{-1}\sqrt{\pi/(\epsilon f(\epsilon))}$, $I(1) = (2L\epsilon f(\epsilon))^{-1}$ and $I(2) = \sqrt{\pi}/(4L(\epsilon f(\epsilon)^{3/2})$. We see that $\lim_{\epsilon\to 0_+}\epsilon^{1/2}|g(\epsilon)|I(\lambda) = 0$ for $\lambda = 0,1,2$ and $\lim_{\epsilon\to 0_+}\epsilon^{1/2}|g(\epsilon)|G(a_\epsilon) = 0$ and $\lim_{\epsilon\to 0_+}\epsilon^{1/2}|g(\epsilon)|H(b_\epsilon) = 0$, so the result follows since

$$|s(0)| \leq 2 + 2I(0), \qquad |s(1)| \leq 4G(a_\epsilon) + 2I(1), \qquad |s(2)| \leq 4H(b_\epsilon) + 2I(2).$$

□

We will use the above lemmas to evaluate expressions of the form

$$I = \lim_{\epsilon\to 0_+} \epsilon^{1/2} \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi\epsilon P(x)} \exp(i\pi Q_n(x))\,dx,$$

where $P(x) = (\sqrt{A_1}x + B_1)^2$ and $Q_n(x) = A_2x^2 + B_2(n)x + C_2(n)$ have real coefficients and $A_1 > 0$, $A_2 \neq 0$, $B_2(n) = B_2n + D_2$. It is a fact that the integrals in I can be calculated exact by stationary phase approximation (SPA), if one uses the stationary points of $i\epsilon P(x) + Q_n(x)$. However, we will show that to calculate I exact by SPA, it is enough to consider the stationary points of the Q_n in the situations we will meet. This will follow from the proof of the following proposition.

Proposition 5.3. *Let $Q_n(x) = A_2x^2 + B_2(n)x + C_2(n)$ be polynomials with real coefficients such that $A_2 \neq 0$ and $B_2(n) = B_2n + D_2$. Let $A_1 > 0$ and let $B_1 \in \mathbb{R}$ and assume that $F : \mathbb{Z} \to \mathbb{C}$ given by $F(n) = \exp(i\pi C_2(n))exp(-i\pi B_2(n)^2/(4A_2))$ is periodic in n with a period of N. Finally let $G : \mathbb{Z} \to \mathbb{C}$ be periodic with the same period N. Then*

$$\begin{aligned} &\lim_{\epsilon\to 0_+} \epsilon^{1/2} \sum_{n\in\mathbb{Z}} G(n) \int_{-\infty}^{\infty} e^{-\pi\epsilon(\sqrt{A_1}x+B_1)^2} \exp(i\pi Q_n(x))dx \\ &= \frac{2}{N} e^{i\frac{\pi}{4}sign(A_2)} \frac{1}{|B_2|} \sqrt{\frac{|A_2|}{A_1}} \sum_{n=0}^{N-1} G(n)F(n). \end{aligned}$$

Proof. By change of variables we can assume that $B_1 = 0$. Note first that

$$I = \lim_{\epsilon\to 0_+} \epsilon^{1/2} \sum_{n\in\mathbb{Z}} G(n) \exp(i\pi C_2(n)) \int_{-\infty}^{\infty} e^{-\pi\epsilon A_1 x^2} \exp(i\pi A_2 x^2 + B_2(n)x))dx.$$

Here

$$\int_{-\infty}^{\infty} e^{-\pi\epsilon A_1 x^2}\exp(i\pi A_2 x^2 + B_2(n)x))dx = \int_{-\infty}^{\infty} e^{-a_\epsilon x^2}\exp(b(n)x)dx = \left(\frac{\pi}{a_\epsilon}\right)^{1/2}\exp\left(\frac{b(n)^2}{4a_\epsilon}\right),$$

where $a_\epsilon = \pi\epsilon A_1 - i\pi A_2$ and $b(n) = i\pi B_2(n) = i\pi(B_2 n + D_2)$. Note that $Re(a_\epsilon) > 0$. We use here and elsewhere the branch of the square root, which is positive on positive real numbers and defined on $\mathbb{C}\setminus\{x\in\mathbb{R}|x<0\}$. Now

$$\frac{b(n)^2}{4a_\epsilon} = -\frac{\pi B_2(n)^2}{4A_2^2} f(\epsilon)(\epsilon A_1 + iA_2),$$

where $f(\epsilon) = \left(1+\left(\frac{A_1\epsilon}{A_2}\right)^2\right)^{-1}$, so

$$I = \lim_{\epsilon\to 0_+}(\pi/a_\epsilon)^{1/2}\lim_{\epsilon\to 0_+}\epsilon^{1/2}\sum_{n\in\mathbb{Z}} G(n)\exp(i\pi C_2(n)) \times\exp\left(-\pi\epsilon f(\epsilon)\frac{A_1 B_2(n)^2}{4A_2^2}\right)\exp\left(-i\pi f(\epsilon)\frac{B_2(n)^2}{4A_2}\right).$$

Here $\lim_{\epsilon\to 0_+}(\pi/a_\epsilon)^{1/2} = (i/A_2)^{1/2}$. Let us next evaluate I by SPA where we only use the stationary points of the Q_n. We find immediately

$$\int_{-\infty}^{\infty} e^{-\pi\epsilon A_1 x^2}\exp(i\pi(A_2 x^2 + B_2(n)x))dx \sim (2\pi/\pi)^{1/2} e^{i\frac{\pi}{4}sign(A_2)}\frac{1}{\sqrt{2|A_2|}} e^{-\pi\epsilon A_1 x_{st}^2}\exp(i\pi(A_2 x_{st}^2 + B_2(n)x_{st})),$$

where $x_{st} = -B_2(n)/(2A_2)$ is the sationary point of $Q_n(x)$ (so depends on n). Here $(2\pi/\pi)^{1/2}e^{i\frac{\pi}{4}sign(A_2)}/\sqrt{2|A_2|} = (i/A_2)^{1/2}$ and $A_2 x_{st}^2 + B_2(n)x_{st} = -B_2(n)^2/(4A_2)$ so

$$\int_{-\infty}^{\infty} e^{-\pi\epsilon A_1 x^2}\exp(i\pi(A_2 x^2 + B_2(n)x))dx \sim \sqrt{\frac{i}{A_2}}\exp\left(-\pi\epsilon\frac{A_1 B_2(n)^2}{4A_2^2}\right)\exp\left(-i\pi\frac{B_2(n)^2}{4A_2}\right)$$

and we see that $I \sim I_{spa}$, where

$$I_{spa} = \sqrt{\frac{i}{A_2}}\lim_{\epsilon\to 0_+}\epsilon^{1/2}\sum_{n\in\mathbb{Z}} G(n)\exp(i\pi C_2(n))\exp\left(-\pi\epsilon\frac{A_1 B_2(n)^2}{4A_2^2}\right)\exp\left(-i\pi\frac{B_2(n)^2}{4A_2}\right).$$

Now, since $\exp(i\pi C_2(n))$ has unit norm and $B_2(n)$ is linear in n, we get by Lemma 5.2 (the involved limits exists by Lemma 5.1 and the periodicity assumption in the proposition) that

$$I = \sqrt{\frac{i}{A_2}}\lim_{\epsilon\to 0_+}\epsilon^{1/2}\sum_{n\in\mathbb{Z}} G(n)\exp(i\pi C_2(n))\exp\left(-\pi\epsilon f(\epsilon)\frac{A_1 B_2(n)^2}{4A_2^2}\right)\exp\left(-i\pi\frac{B_2(n)^2}{4A_2}\right).$$

Finally, since $\exp(i\pi C_2(n))\exp(-i\pi B_2(n)^2/(4A_2))G(n)$ is periodic in n, it follows from Lemma 5.1 that $I = I_{spa}$. To end the proof simply note that the factor in front of n^2 in $\frac{A_1 B_2(n)^2}{4A_2^2}$ is $A_1 B_2^2/(4A_2^2)$ and use Lemma 5.1. □

We will now start analysing the invariants of the 3–fibered Seifert manifolds. According to (24) we need to know the Verlinde numbers $N_{\alpha_1,\alpha_2,\alpha_3}$ for $(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{Z}^3$ inside the fundamental cube

(33) $$0 < \alpha_1,\alpha_2,\alpha_3 < K.$$

By (25) we get immediately $N_{\alpha_1,\alpha_2,\alpha_3} = \delta_{\alpha_j,\alpha_k}$ if $\alpha_i = 1$ where $\{j,k\} = \{1,2,3\} \setminus \{i\}$. Actually we have according to [Ro] that $N_{\alpha_1,\alpha_2,\alpha_3} = 1$ if $\alpha_1+\alpha_2+\alpha_3$ is odd and

$$\begin{aligned} \alpha_1+\alpha_2-\alpha_3 &> 0, \\ \alpha_1-\alpha_2+\alpha_3 &> 0, \\ -\alpha_1+\alpha_2+\alpha_3 &> 0, \\ \alpha_1+\alpha_2+\alpha_3 &< 2K. \end{aligned} \tag{34}$$

For all other $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}^3$ inside the fundamental cube, $N_{\alpha_1,\alpha_2,\alpha_3} = 0$. Now

$$\sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} = \frac{1}{2}(1-(-1)^{\alpha_1+\alpha_2+\alpha_3}) = \begin{cases} 1 & ,\alpha_1+\alpha_2+\alpha_3 \text{ odd} \\ 0 & ,\alpha_1+\alpha_2+\alpha_3 \text{ even} \end{cases}$$

so we can drop the restriction of the parity of $\alpha_1+\alpha_2+\alpha_3$ if we change the formula (24) into

$$Z_k(X) = \sum_{0<\alpha_1,\alpha_2,\alpha_3<K} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} (\tilde{M}_1)_{\alpha_1 1} (\tilde{M}_2)_{\alpha_2 1} (\tilde{M}_3)_{\alpha_3 1} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}, \tag{35}$$

where $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} = 1$ for all $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}^3$ satisfying (34). Here X is our 3–fibered Seifert manifold given in (23) for $n = 3$. Note that the region in $\mathbb{R}^3$ defined by the inequalities (34) is an open tetrahedron inside the fundamental cube (33). We will change this formula, so that we get a sum over a set, which does not depend on K. We first change it into a sum over all of $\mathbb{Z}^3$. To this end we use the symmetries of the matrices $\tilde{M}_{\alpha\beta}$ given in (31). By using this expression we can actually extend $\tilde{M}_{\alpha\beta}$ to be defined for all $\alpha,\beta \in \mathbb{Z}$. Now if $M \in SL(2,\mathbb{Z})$ we can write $M = T^{a_t}S \dots T^{a_1}S$. Then for $0 < \alpha, \beta < K$

$$\tilde{M}_{\alpha\beta} = \sum_{j_1,\dots,j_{t-1}=1}^{K-1} \tilde{T}_{\alpha}^{a_t} \tilde{S}_{\alpha j_{t-1}} \tilde{T}_{j_{t-1}}^{a_{t-1}} \tilde{S}_{j_{t-1}j_{t-2}} \dots \tilde{T}_{j_1}^{a_1} \tilde{S}_{j_1\beta} \tag{36}$$

since $\tilde{T}$ is diagonal according to (22). Here $\tilde{T}_j = \tilde{T}_{jj}$. We can use (22) to define $\tilde{S}_{jl}$ and $\tilde{T}_{jl}$ for all $j,l \in \mathbb{Z}$, and we can then use the above formula (36) to define $\tilde{M}_{\alpha\beta}$ for all $\alpha,\beta \in \mathbb{Z}$. Note that this extended $\tilde{M}_{\alpha\beta}$ satisfies (31) for all $\alpha,\beta \in \mathbb{Z}$, so the two ways of extending $\tilde{M}_{\alpha\beta}$ coincide. Now by (22) and (36) we easily get

$$\tilde{M}_{-\alpha,\beta} = -\tilde{M}_{\alpha,\beta}, \qquad \tilde{M}_{\alpha+2K,\beta} = \tilde{M}_{\alpha,\beta}. \tag{37}$$

We also have to extend $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$. For later use we extend it to all of $\mathbb{R}^3$. We simply define it to be zero on the boundary of the fundamental cube (33) and let $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} \equiv 1$ on the tetrahedron (34) and finally let $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} = \text{sign}(\alpha_1\alpha_2\alpha_3)\tilde{N}_{|\alpha_1|,|\alpha_2|,|\alpha_3|}$ for $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{R}^3$ in the cube $-K \le \alpha_1,\alpha_2,\alpha_3 \le K$. The mapping $(\alpha_1,\alpha_2,\alpha_3) \mapsto \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$ is now extended to all off $\mathbb{R}^3$ by requiring it to be a periodic function of its indicis with a period of $2K$ in all variables. Note that $e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} = e^{2\pi i\lambda(1-|\alpha_1|-|\alpha_2|-|\alpha_3|)}$ for all $(\alpha_1,\alpha_2,\alpha_3) \in \mathbb{Z}^3$ and $\lambda = 0, \frac{1}{2}$. With these preparations we have

$$Z_k(X) = \frac{1}{8} \sum_{-K\le\alpha_1,\alpha_2,\alpha_3<K} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} (\tilde{M}_1)_{\alpha_1 1} (\tilde{M}_2)_{\alpha_2 1} (\tilde{M}_3)_{\alpha_3 1} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}. \tag{38}$$

The next step is to extend this sum to all of $\mathbb{Z}^3$. For a fixed $\lambda \in \{0, \frac{1}{2}\}$ the function $F : \mathbb{Z}^3 \to \mathbb{C}$ given by

$$F(\alpha_1,\alpha_2,\alpha_3) = \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)} (\tilde{M}_1)_{\alpha_1 1} (\tilde{M}_2)_{\alpha_2 1} (\tilde{M}_3)_{\alpha_3 1} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$$

is periodic in all its variables with a period of $2K$ according to (37). By Lemma 5.1 it then follows that

$$Z_k(X) = \lim_{\epsilon\to 0_+} \frac{(2K\epsilon^{1/2})^3}{8} \sum_{\alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda(1-\alpha_1-\alpha_2-\alpha_3)-\pi\epsilon(\alpha_1^2+\alpha_2^2+\alpha_3^2)}$$

$$\times(\tilde{M}_1)_{\alpha_1 1}(\tilde{M}_2)_{\alpha_2 1}(\tilde{M}_3)_{\alpha_3 1}\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}. \tag{39}$$

Note that the functions $\tilde{M}_{\alpha\beta}$, $M \in SL(2,\mathbb{Z})$, can be extended to all of $\mathbb{R}^2$ in exactly the same way, as we extended them to $\mathbb{Z}^2$, that is by using (31) or (36). (However, only the first relation in (37) is valid for all $\alpha,\beta\in\mathbb{R}$. The second is only valid for $\alpha\in\frac{1}{2}\mathbb{Z}$ and $\beta\in\mathbb{R}$. Note also here that [Ro, Formula (3.7)] is only valid for $\alpha\in\frac{1}{2}\mathbb{Z}$ and $\beta\in\mathbb{R}$.) Now the functions in the sum in (39) are defined for all $(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3$ and belong to $\mathcal{S}(\mathbb{R}^3)$, so we can use Poisson's formula (32) to get

$$Z_k(X) = \lim_{\epsilon\to 0_+} \frac{(2K\epsilon^{1/2})^3}{8} \sum_{\lambda=0,\frac{1}{2}} \sum_{(m_1,m_2,m_3)\in\mathbb{Z}^3} \frac{1}{2} e^{2\pi i\lambda} \int_{\mathbb{R}^3} \exp(-\pi\epsilon\sum_{j=1}^3 \alpha_j^2)$$

$$\times\exp(2\pi i\sum_{j=1}^3 (m_j-\lambda)\alpha_j)(\tilde{M}_1)_{\alpha_1 1}(\tilde{M}_2)_{\alpha_2 1}(\tilde{M}_3)_{\alpha_3 1}\tilde{N}_{\alpha_1,\alpha_2,\alpha_3} d\alpha_1 d\alpha_2 d\alpha_3.$$

Now let

$$Z_1 = i^3 \mathrm{sign}(q_1q_2q_3)\exp\left[-\frac{i\pi}{4}\sum_{j=1}^3 \Phi(M_j)\right]. \tag{40}$$

Then we immediately get from (31) that

$$Z_k(X) = Z_1 Z_2, \tag{41}$$

where

$$Z_2 = \lim_{\epsilon\to 0_+} \frac{(2K\epsilon^{1/2})^3}{8} \sum_{\lambda=0,\frac{1}{2}} \frac{1}{2} e^{2\pi i\lambda} \sum_{(m_1,m_2,m_3)\in\mathbb{Z}^3} \int_{\mathbb{R}^3} \tilde{N}_{\alpha_1,\alpha_2,\alpha_3} \prod_{j=1}^3 \frac{d\alpha_j}{\sqrt{2K|q_j|}}$$

$$\times L^j_{\alpha_j}\exp(2\pi i(m_j-\lambda)\alpha_j - \pi\epsilon\alpha_j^2).$$

Here

$$L^j_\alpha = \sum_{\mu=\pm 1}\sum_{n=0}^{|q_j|-1} \mu\exp\left(\frac{i\pi}{2Kq_j}[p_j\alpha^2 - 2\mu\alpha(1+2Kn) + s_j(1+2Kn)^2]\right).$$

By using $\exp\left(\frac{i\pi}{2Kq_j}s_j(1+2Kn)^2\right) = \exp\left(\frac{i\pi}{2Kq_j}s_j(\mu(1+2Kn)+2Kmq_j)^2\right)$ for $m\in Z$ one finds that

$$\exp(2\pi im\alpha)L^j_\alpha = \sum_{\mu=\pm 1}\sum_{n=0}^{|q_j|-1} \mu\exp\left(\frac{i\pi}{2Kq_j}[p_j\alpha^2 - 2\alpha(2K(n\mu+mq_j)+\mu)\right.$$

$$\left.+s_j(2K(n\mu+mq_j)+\mu)^2]\right),$$

so

$$\sum_{m\in Z}\exp(2\pi im\alpha)L^j_\alpha = \sum_{\mu=\pm 1}\sum_{n\in\mathbb{Z}} \mu\exp\left(\frac{i\pi}{2Kq_j}[p_j\alpha^2 - 2\alpha(2Kn+\mu) + s_j(2Kn+\mu)^2]\right).$$

We therefore have

$$Z_2 = \lim_{\epsilon\to 0_+} \frac{(2K\epsilon^{1/2})^3}{8}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\int_{\mathbb{R}^3}\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}\prod_{j=1}^{3}\frac{d\alpha_j}{\sqrt{2K|q_j|}}\sum_{\mu_j=\pm 1}\sum_{n_j\in\mathbb{Z}}\mu_j e^{-\pi\epsilon\alpha_j^2}$$

$$\times\exp\left(\frac{i\pi}{2Kq_j}[p_j\alpha_j^2 - 2\alpha_j(2K(n_j+q_j\lambda)+\mu_j) + s_j(2Kn_j+\mu_j)^2]\right). \tag{42}$$

Note that (40)-(42) corresponds to [Ro, Formulas (3.8)–(3.10)].

The next task is to get rid of the function $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$, so we can get in a position, where we can use Proposition 5.3. We will need some notation. For $\nu = (\nu_1,\nu_2,\nu_3)\in\{\pm 1\}^3$ and $l\in\mathbb{Z}$ we let $H(\nu;l)$ be the half-space $\sum_{i=1}^3 \nu_i\alpha_i + 2Kl > 0$. For $m = (m_1,m_2,m_3)\in\mathbb{Z}^3$ we let $T(m)$ be the open tetrahedron inside the cube $\prod_i^3[m_iK,(m_i+1)K]$ on which $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}\neq 0$. Let us give $T(m)$ a sign corresponding to the sign of $\tilde{N}_{\alpha_1,\alpha_2,\alpha_3}$ on $T(m)$. We first observe that the sides of $T(m)$ all lies inside plans of the form

$$-\alpha_1+\nu_2\alpha_2+\nu_3\alpha_3+2Kl = 0,\qquad \nu_2,\nu_3=\pm 1,\quad l\in\mathbb{Z}. \tag{43}$$

This is certainly true for the fundamental +1 tetrahedron $T(0)$ given by (34). As the fundamental -1 tetrahedron we choose $T(-1,0,0)$. This is given by the inequalities

$$\begin{aligned} -\alpha_1+\alpha_2-\alpha_3 &> 0,\\ -\alpha_1-\alpha_2+\alpha_3 &> 0,\\ -\alpha_1-\alpha_2-\alpha_3 &< 0,\\ -\alpha_1+\alpha_2+\alpha_3 &< 2K, \end{aligned} \tag{44}$$

and the assertion is then also true for this tetrahedron. Now the other six tetrahedrons in the cube $[-K,K]^3$ are translations of these two fundamental tetrahedrons, by translations of the form $x\mapsto x\pm K(e_i-e_j)$, $i\neq j$, where e_1,e_2,e_3 is the standard basis in $\mathbb{R}^3$, and these translations take plans of the form (43) to plans of the same form. Now if $m\in\mathbb{Z}^3$, $T(m)$ is the translation of one of the above eight tetrahedrons by a translation of the form $x\mapsto x+2Kn$, $n\in\mathbb{Z}^3$, so the assertion above is true. Now let P be the plan $-\alpha_1+\nu_1\alpha_2+\nu_3\alpha_3+2Kl=0$ and let $H = H(-1,\nu_2,\nu_3;l)$. Let $m=(m_1,m_2,m_3)\in\mathbb{Z}^3$ so that $T=T(m)$ has a side in P. Then we have

$$\begin{aligned} T\subseteq H &\Leftrightarrow \nu_2 v_3 = \lambda,\\ T\cap H=\emptyset &\Leftrightarrow \nu_2 v_3 = -\lambda, \end{aligned} \tag{45}$$

where λ is the sign of T. Again this follows by first observing that it is satisfied for the fundamental tetrahedrons $T(0,0,0)$ and $T(-1,0,0)$ according to (34) and (44). Now by the translation description above one realizes that it is true for the remaining six tetrahedrons inside the cube $[-K,K]^3$, and finally that it is true for an arbitrary $T(m)$. The main lemma is now

Lemma 5.4. *For all $f\in L^1(\mathbb{R}^3)$ we have*

$$\int_{\mathbb{R}^3} f\tilde{N}\,d\alpha_1 d\alpha_2 d\alpha_3 = \sum_{l\in\mathbb{Z}}\sum_{\nu_2,\nu_3=\pm 1}\nu_2\nu_3\int_{H(-1,\nu_2,\nu_3;l)} f\,d\alpha_1 d\alpha_2 d\alpha_3.$$

We emphazise that the summation order on the right-hand side is crucial.

Proof. Let us investigate what happens in the plan $a_3 = c$ where $0<c<K$, see Fig. 2. The dark regions indicate the tetrahedrons and the signs indicate the signs of the tetrahedrons.

Let $\gamma_1, \ldots, \gamma_4$ be the lines

$$\begin{aligned} \gamma_1: &\quad -\alpha_1 + \alpha_2 - c = 0, \\ \gamma_2: &\quad -\alpha_1 + \alpha_2 + c = 0, \\ \gamma_3: &\quad -\alpha_1 - \alpha_2 + c = 0, \\ \gamma_4: &\quad -\alpha_1 - \alpha_2 - c = 0, \end{aligned}$$

see Fig. 2. The lines parallel to γ_1 and γ_2 in Fig. 2 have equations of the form $-\alpha_1+\alpha_2+\nu c+2Kl = 0$, $\nu = \pm 1$, $l \in Z$ and these lines move in the direction $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ for increasing l. The lines parallel to γ_3 and γ_4 in Fig. 2 have equations of the form $-\alpha_1 - \alpha_2 + \nu c + 2Kl = 0$, $\nu = \pm 1$, $l \in Z$ and these lines move in the direction $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for increasing l.

Figure 2

Now let Ω be a subset of the band $0 < \alpha_3 < K$. We assume that Ω is contained either in one of the tetrahedrons $T = T(m_1, m_2, 0)$ inside the band or in one of the regions between these tetrahedrons. Let

$$\begin{aligned} -\alpha_1 + \alpha_2 - \alpha_3 + 2Kl_1 &= 0, \\ -\alpha_1 + \alpha_2 + \alpha_3 + 2Kl_2 &= 0, \\ -\alpha_1 - \alpha_2 + \alpha_3 + 2Kl_3 &= 0, \\ -\alpha_1 - \alpha_2 - \alpha_3 + 2Kl_4 &= 0 \end{aligned}$$

be the plans bounding T or the region and let χ_Ω be the characteristic function of Ω and note that

$$\sum_{\nu=\pm 1} \int_{H(-1,-1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3 = 0, \qquad l < \min\{l_3, l_4\} \text{ or } l > \max\{l_3, l_4\}$$
$$\sum_{\nu=\pm 1} \int_{H(-1,1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3 = 0, \qquad l < \min\{l_1, l_2\} \text{ or } l > \max\{l_1, l_2\}.$$

Observe that $|l_1 - l_2| \leq 1$ and $|l_3 - l_4| \leq 1$ and get

$$\sum_{l\in\mathbb{Z}} \sum_{\nu_2,\nu_3=\pm 1} \nu_2\nu_3 \int_{H(-1,\nu_2,\nu_3;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3$$
$$= \sum_{l=l_1,l_2} \sum_{\nu=\pm 1} \nu \int_{H(-1,1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3 - \sum_{l=l_3,l_4} \sum_{\nu=\pm 1} \nu \int_{H(-1,-1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3.$$

Now consider Fig. 3 where $l_a = \min\{l_1, l_2\}$ and $l_b = \min\{l_3, l_4\}$. The normal to the plan $-\alpha_1 + \nu_2\alpha_2 + \nu_3\alpha_3 + 2Kl = 0$ pointing into the half-space $H(-1, \nu_2, \nu_3; l)$ is $(-1, \nu_2, \nu_3)$. The arrows in Fig. 3 indicate the orthogonal projection onto the plan $\alpha_3 = c$ of these normals. The signs after the l indicate the signs of $\nu_2\nu_3$ in the equations for the lines (e.g. $(l_b + 1)-$ is the line $-\alpha_1 - \alpha_2 + c + 2K(l_b + 1) = 0$). These signs follow from (45).

Figure 3

Now the result follows immediately. If e.g. Ω is contained in the $+$ tetrahedron indicated by T on Fig. 3, then we have that $l_1 = l_2 = l_a$, $l_3 = l_b$ and $l_4 = l_b + 1$ and

$$\sum_{l=l_1,l_2} \sum_{\nu=\pm 1} \nu \int_{H(-1,1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3 - \sum_{l=l_3,l_4} \sum_{\nu=\pm 1} \nu \int_{H(-1,-1,\nu;l)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3$$
$$= \int_{H(-1,1,1;l_a)} \chi_\Omega f \, d\alpha_1 d\alpha_2 d\alpha_3 = \int_{\mathbb{R}^3} \chi_\Omega f \tilde{N} \, d\alpha_1 d\alpha_2 d\alpha_3$$

since $\tilde{N} \equiv 1$ on T. The other bands $mK < \alpha_3 < (m+1)K$ are treated similarly. The picture is the same as in Fig. 2 for $2mK < \alpha_3 < (2m+1)K$, and the signs on the tetrahedrons has to be changed for the bands $(2m-1)K < \alpha_3 < 2mK$. Now $\mathbb{R}^3$ can be partitioned into subsets like Ω and the result follows. $\square$

Now let $f^{\mu}_{n,\lambda} : \mathbb{R}^3 \to \mathbb{C}$ be given by

$$\begin{aligned} f^{\mu}_{n,\lambda}(\alpha_1,\alpha_2,\alpha_3) &= \prod_{i=1}^{3} \frac{1}{\sqrt{2K|q_i|}} e^{-\pi\epsilon\alpha_i^2} \\ &\times \exp\left(\frac{i\pi}{2Kq_i}[p_i\alpha_i^2 - 2\alpha_i(2K(n_i+q_i\lambda)+\mu_i) + s_i(2Kn_i+\mu_i)^2] \right). \end{aligned} \tag{46}$$

Then we have by Lemma 5.4 and (40)-(42)

Proposition 5.5. *The Witten–Reshetikhin–Turaev invariant at level k of the 3–fibered Seifert manifold $X = X(p_1/q_1, p_2/q_2, p_3/q_3)$ is given by $Z_k(X) = Z_1Z_2$ where Z_1 is given in (40) and*

$$\begin{aligned} Z_2 &= \lim_{\epsilon\to 0_+} \frac{(2K\epsilon^{1/2})^3}{8} \frac{1}{2} \sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda} \sum_{n\in\mathbb{Z}^3} \sum_{\mu_1,\mu_2,\mu_3=\pm 1} \\ &\times \quad \mu_1\mu_2\mu_3 \sum_{l\in\mathbb{Z}} \sum_{\nu_2,\nu_3=\pm 1} \nu_2\nu_3 \int_{H(-1,\nu_2,\nu_3;l)} f^{\mu}_{n,\lambda}\, d\alpha. \end{aligned} \tag{47}$$

$\square$

We will evaluate the integrals $\int_{H(-1,\nu_2,\nu_3;l)} f^{\mu}_{n,\lambda}\, d\alpha$ by stationary phase approximation (SPA). Inspired by Proposition 5.3 we will use the stationary points of the phase funtions

$$\phi^{\mu}_{n,\lambda}(\alpha) = \sum_{i=1}^{3} \frac{1}{2Kq_i}[p_i\alpha_i^2 - 2\alpha_i(2K(n_i+q_i\lambda)+\mu_i)]. \tag{48}$$

The stationary point of $\phi^{\mu}_{n,\lambda}(\alpha)$ is given by

$$\alpha_i^{st} = \frac{2K}{p_i}\hat{n}_i, \tag{49}$$

where $\hat{n}_i = \tilde{n}_i + \frac{\mu_i}{2K} = n_i + q_i\lambda + \frac{\mu_i}{2K}$. Since the integrals are over half-spaces, we also need to consider the stationary points on the boundaries of these halfspaces, see [Ho, Theorem 7.7.17]. That is, we have to calculate the stationary point of $\phi^{\mu}_{n,\lambda}$ under the condition $-\alpha_1 + \nu_2\alpha_2 + \nu_3\alpha_3 + 2Kl = 0$ (there is one conditional stationary point (cst) for $\phi^{\mu}_{n,\lambda}$ on every boundary $\partial H(-1,\nu_2,\nu_3;l)$). By a rather long but elementary calculation one finds that the cst on $\partial H(-1,\nu_2,\nu_3;l)$ is

$$\alpha_i^{cst} = \frac{2K}{p_i}\nu_i(\nu_i\hat{n}_i - q_i\hat{c}_0), \tag{50}$$

where $\hat{c}_0 = \frac{P}{H}\left(\sum_{j=1}^{3} \frac{\nu_j\hat{n}_j}{p_j} + l\right)$ and $\nu_1 = -1$ here ((50) is also valid for $\nu_1 = 1$). Here $P = p_1p_2p_3$ and

$$H = P\sum_{j=1}^{3} \frac{q_j}{p_j} = p_1p_2q_3 + p_1q_2p_3 + q_1p_2p_3. \tag{51}$$

Note that $|H|$ is the order of the first homology group $H_1(X;\mathbb{Z})$ of the Seifert manifold $X = X(p_1/q_1, p_2/q_2, p_3/q_3)$. The above formulas for the stationary and conditional stationary points correct [Ro, Formulas (3.15) and (3.17)]. Consider now one of the integrals $\int_{H(-1,\nu_2,\nu_3;l)} f^{\mu}_{n,\lambda}\, d\alpha$. If the point (49) does not belong to the half-space $H(-1,\nu_2,\nu_3;l)$, then the integral is to first order given by the contribution of the conditional stationary point (50). If, however, the point (49) is within the half-space $H(-1,\nu_2,\nu_3;l)$, then we use the relation

$$\int_{H(-1,\nu_2,\nu_3;l)} f^{\mu}_{n,\lambda}\,d\alpha = \int_{\mathbb{R}^3} f^{\mu}_{n,\lambda}\,d\alpha - \int_{\bar{H}(-1,\nu_2,\nu_3;l)^c} f^{\mu}_{n,\lambda}\,d\alpha, \tag{52}$$

where $\bar{H}(-1,\nu_2,\nu_3;l)^c$ is the complement of the closure of $H(-1,\nu_2,\nu_3;l)$, that is the half-space $-\alpha_1+\nu_2\alpha_2+\nu_3\alpha_3+2Kl<0$. The second integral on the right-hand side is again determined by the cst in (50). The first integral is determined by the point (49). Note from this that the contribution of the point (49) to the integral $\int_H f^{\mu}_{n,\lambda}\,d\alpha$ is either zero or a quantity, which does not depend on the half-space to which it belongs. If α^{st} given by (49) belongs to $\partial H(-1,\nu_2,\nu_3;l)$, then α^{st} is equal to α^{cst} given by (50). This simply follows from (49)-(50). If namely $\alpha^{st}\in\partial H(\nu;l)$ then

$$0 = 2Kl + \sum_{i=1}^{3}\nu_i\alpha_i^{st} = 2K\left(l+\sum_{i=1}^{3}\frac{\nu_i\hat{n}_i}{p_i}\right) = 2K\frac{H}{P}\hat{c}_0$$

so $\hat{c}_0=0$. But then $\alpha_i^{cst}=\frac{2K}{p_i}\hat{n}_i=\alpha_i^{st}$. In the next lemma we determine the total contribution to Z_2 of the points α^{st}, which lie inside the half-spaces $H(-1,\nu_1,\nu_3;l)$.

Lemma 5.6. *The total contribution to Z_2 of the stationary points* (49)*, which do not lie on the boundary plans* (43)*, is given by*

$$Z_2^{st} = \lim_{\epsilon\to 0_+}\frac{(2K\epsilon^{1/2})^3}{8}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\sum_{\mu_1,\mu_2,\mu_3=\pm 1}\mu_1\mu_2\mu_3\sum_{n\in\mathbb{Z}^3}\delta^{\mu}_{n,\lambda}\int_{\mathbb{R}^3} f^{\mu}_{n,\lambda}\,d\alpha,$$

where $\delta^{n}_{n,\lambda}=\beta\in\{\pm1\}$ if the stationary point (49) *of $f^{\mu}_{n,\lambda}$ lies inside a β-tetrahedron, and $\delta^{n}_{n,\lambda}=0$ if this stationary point lies outside the tetrahedrons.*

Proof. Assume that the stationary point α^{st} of $f^{\mu}_{n,\lambda}$ lies inside W, where W is one of the open tetrahedrons $T(m)$ or one of the open regions between the closures of these tetrahedrons. Note that $\alpha^{st}\in H(-1,\nu_2,\nu_3;l)$ if and only if $W\subseteq H(-1,\nu_2,\nu_3;l)$. Now recall that for every half-space $H(-1,\nu_2,\nu_3;l)$ containing α^{st}, α^{st} contributes by $\nu_2\nu_3\int_{\mathbb{R}^3} f^{\mu}_{n,\lambda}\,d\alpha$ to the integral $\int_{\mathbb{R}^3}\tilde{N} f^{\mu}_{n,\lambda}\,d\alpha$ according to Lemma 5.4, (52) and the remarks to (52). But then the total contribution of α^{st} to this integral is

$$\sum_{l,\nu_2,\nu_3:\,\alpha^{st}\in H(-1,\nu_2,\nu_3;l)}\nu_2\nu_3\int_{\mathbb{R}^3} f^{\mu}_{n,\lambda}\,d\alpha = \sum_{l,\nu_2,\nu_3:\,W\subseteq H(-1,\nu_2,\nu_3;l)}\nu_2\nu_3\int_{\mathbb{R}^3} f^{\mu}_{n,\lambda}\,d\alpha$$

and by the same counting method as used in the proof of Lemma 5.4 we see that this is precisely $\delta^{\mu}_{n,\lambda}\int_{\mathbb{R}^3} f^{\nu}_{n,\lambda}\,d\alpha$. □

The following proposition brings us to the final formula for the contribution of the stationary points (49) to the invariants $Z_k(X)$. The proposition determines exact the first sum $\sum_{\lambda=0,\frac{1}{2}}\sum_{(n_1,n_2,n_3)\in St} Z_{st}^{(n_1,n_2,n_3;\lambda)}$ in [Ro, Formula (3.22)].

Proposition 5.7. *The total contribution to $Z_k(X)$ of the stationary points* (49)*, which do not lie on the boundary plans* (43)*, is given by $Z_1 Z_2^{st}$ where Z_1 is given in* (40) *and*

$$Z_2^{st} = \frac{Z_3}{8}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda} \sum_{\mu_1,\mu_2,\mu_3=\pm 1}\sum_{n_1=0}^{|p_1|-1}\sum_{n_2=0}^{|p_2|-1}\sum_{n_3=0}^{|p_3|-1}\delta^{\mu}_{n,\lambda}$$
$$\times \prod_{i=1}^{3}\frac{1}{\sqrt{|p_i|}}\mu_i \exp\left(\mu_i 2\pi i\left[\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right]\right)\exp\left(2\pi i K\left[\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i\lambda^2\right]\right).$$

Here

$$Z_3 = \prod_{i=1}^{3} e^{i\frac{\pi}{4}\operatorname{sign}(p_i q_i)}\exp\left(\frac{i\pi}{2K}\frac{r_i}{p_i}\right). \tag{53}$$

Proof. According to Lemma 5.6 we shall evaluate an expression of the type

$$\lim_{\epsilon\to 0_+}\epsilon^{3/2}\sum_{n\in\mathbb{Z}^3}\int_{\mathbb{R}^3}\delta^{\mu}_{n,\lambda} f^{\mu}_{n,\lambda}\, d\alpha.$$

Let $\alpha^{st}(n,\mu,\lambda)$ be the stationary point of $f^{\mu}_{n,\lambda}$. Moreover, let $n^i = n + p_i e_i$, where e_1, e_2, e_3 is the standard basis in $\mathbb{R}^3$, and note that

$$\begin{aligned}\alpha_i^{st}(n^i,\mu,\lambda) &= \frac{2K}{p_i}(n_i + p_i + q_i\lambda + \frac{\mu_i}{2K}) = 2K + \alpha_i^{st}(n,\mu,\lambda),\\ \alpha_j^{st}(n^i,\mu,\lambda) &= \alpha_j^{st}(n,\mu,\lambda), \quad j\neq i.\end{aligned}$$

That is, if $\alpha^{st}(n,\mu,\lambda)\in T(m)$ then $\alpha^{st}(n^i,\mu,\lambda)\in T(m+2e_i)$, and we see that $\delta^{\mu}_{n^i,\lambda} = \delta^{\mu}_{n,\lambda}$, i.e. $\delta^{\mu}_{n,\lambda}$ is periodic with a period of $|p_i|$ in n_i. Now according to Proposition 5.3 and (46) we let $A_1 = 1$, $A_2 = \frac{p_i}{2Kq_i}$, $B_2(n_i) = -\frac{2}{q_i}\hat{n}_i = -\frac{2}{q_i}n_i + (-2\lambda - \frac{2\mu_i}{q_i K})$ and $C_2(n_i) = \frac{1}{2Kq_i}s_i(2Kn_i + \mu_i)^2 = \frac{1}{2Kq_i}s_i(2K\hat{n}_i - 2Kq_i\lambda)^2$. Note that $B_2 = -2/q_i$ and that $-i\pi\frac{B_2(n_i)^2}{4A_2} = -\frac{i\pi}{2Kq_i}\frac{1}{p_i}(2K\hat{n}_i)^2$. We therefore get

$$\begin{aligned}F(n_i) &= \exp(i\pi C_2(n_i))\exp\left(-i\pi\frac{B_2(n_i)^2}{4A_2}\right)\\ &= \exp\left(\frac{i\pi}{2Kq_i}[-\frac{1}{p_i}(2K\hat{n}_i)^2 + s_i(2K(\hat{n}_i - q_i\lambda))^2]\right)\\ &= \exp\left(2\pi i K[\frac{r_i}{p_i}\hat{n}_i^2 + s_i q_i\lambda^2 - 2s_i\lambda\hat{n}_i]\right),\end{aligned}$$

where we use that $\frac{s_i}{q_i} = \frac{1}{p_i q_i} + \frac{r_i}{p_i}$ since $M_i = \begin{pmatrix} p_i & r_i\\ q_i & s_i\end{pmatrix}\in SL(2,\mathbb{Z})$. Now $\hat{n}_i = \tilde{n}_i + \frac{\mu_i}{2K} = n_i + q_i\lambda + \frac{\mu_i}{2K}$ so

$$\begin{aligned}F(n_i) &= \exp\left(\frac{i\pi}{2K}\frac{r_i}{p_i}\right)\exp\left(2\pi i K[\frac{r_i}{p_i}\tilde{n}_i^2 + s_i q_i\lambda^2]\right)\\ &\quad\times\exp\left(\mu_i 2\pi i[\frac{r_i}{p_i}\tilde{n}_i - s_i\lambda]\right)\exp(-2\pi i K s_i(2\lambda)\tilde{n}_i).\end{aligned}$$

Finally we note that $\exp(-2\pi i K s_i(2\lambda)\tilde{n}_i) = \exp(2\pi i K(-2s_iq_i\lambda^2))$ and $\exp\left(\mu_i 2\pi i[\frac{r_i}{p_i}\tilde{n}_i - s_i\lambda)]\right) = \exp\left(\mu_i 2\pi i[\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda)]\right)$ so

$$F(n_i) = \exp\left(\frac{i\pi}{2K}\frac{r_i}{p_i}\right)\exp\left(2\pi i K[\frac{r_i}{p_i}\tilde{n}_i^2 - s_iq_i\lambda^2]\right)\exp\left(\mu_i 2\pi i[\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda)]\right),$$

which has a period of $|p_i|$. Now

$$\frac{2}{|p_i|}e^{i\frac{\pi}{4}\mathrm{sign}(A_2)}\frac{1}{|B_2|}\sqrt{\frac{|A_2|}{A_1}} = e^{i\frac{\pi}{4}\mathrm{sign}(p_iq_i)}\sqrt{\frac{|q_i|}{2K|p_i|}}$$

so the result follows from Proposition 5.3. □

Remark 5.8. For large K, $\alpha_i^{st} \approx \frac{2K}{p_i}(n_i + q_i\lambda)$, so in the limet of large K, $\delta_{n,\lambda}^{\mu}$ become independent of μ. In the limit $K \to \infty$ we therefore get

$$\begin{aligned} Z_2^{st} \quad &\sim \quad \frac{iZ_3}{8}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\sum_{n_1=0}^{|p_1|-1}\sum_{n_2=0}^{|p_2|-1}\sum_{n_3=0}^{|p_3|-1}\delta_{n,\lambda} \\ &\times \prod_{i=1}^{3}\frac{1}{\sqrt{|p_i|}}\sin\left(2\pi\left[\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right]\right)\exp\left(2\pi i K\left[\frac{r_i}{p_i}\tilde{n}_i^2 - q_is_i\lambda^2\right]\right) \end{aligned}$$

Note that this formula coincide with [Ro, Formula (3.29)].

We now start an investigation of the conditional stationary points contribution to $Z_k(X)$. From Proposition 5.5 and (52) and the remarks to (52) we immediately have

Proposition 5.9. *The total contribution to $Z_k(X)$ of the conditional stationary points (50) is given by $Z_1Z_2^{cst}$, where Z_1 is given in (40) and*

$$\begin{aligned} Z_2^{cst} \quad &= \quad \lim_{\epsilon\to 0_+}\frac{(2K\epsilon^{1/2})^3}{8}\frac{1}{2}\sum_{\lambda=0,\frac{1}{2}} e^{2\pi i\lambda}\sum_{n\in\mathbb{Z}^3}\sum_{\mu_1,\mu_2,\mu_3=\pm 1} \\ &\times \quad \mu_1\mu_2\mu_3\sum_{l\in\mathbb{Z}}\sum_{\nu_2,\nu_3=\pm 1}\nu_2\nu_3\int_{[\alpha^{cst}]} f_{n,\lambda}^{\mu}\, d\alpha, \end{aligned}$$

where $\int_{[\alpha^{cst}]} f_{n,\lambda}^{\mu}\, d\alpha$ denotes the contribution of α^{cst} to the integral $\int_{H(-1,\nu_2,\nu_3;l)} f_{n,\lambda}^{\mu}(\alpha)\, d\alpha$. To first order we then have

$$Z_k(X) \sim Z_1(Z_2^{st} + Z_2^{cst}),$$

where Z_2^{st} is given in Remark 5.8. □

Let us take a look on the integrals $\int_{H(-1,\nu_2,\nu_3;l)} f_{n,\lambda}^{\mu}(\alpha)\, d\alpha$. By a change of variables we get

$$\int_{H(-1,\nu_2,\nu_3;l)} f_{n,\lambda}^{\mu}(\alpha)\, d\alpha = \int_0^{\infty} d\alpha_1\int_{\mathbb{R}^2} d\alpha_2 d\alpha_3 f_{n,\lambda}^{\mu}(\sigma(\alpha))$$

where $\sigma(\alpha) = Ag(\alpha)$, $g(\alpha) = (\alpha_1 + \frac{2Kl}{\sqrt{3}}, \alpha_2, \alpha_3)$ and $A \in O(3)$ is given by

$$A = \begin{pmatrix} \nu_1/\sqrt{3} & -\nu_1\nu_2/\sqrt{6} & -\nu_1\nu_3/\sqrt{2} \\ \nu_2/\sqrt{3} & 2/\sqrt{6} & 0 \\ \nu_3/\sqrt{3} & -\nu_2\nu_3/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}.$$

Now let $F : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be the Fouriertransform and $F^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ the inverse transform, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. Using that

$$\int_0^\infty u(x)\,dx = \int_0^\infty (F^*Fu)(x)\,dx$$

for $u \in \mathcal{S}(\mathbb{R})$ one finds that

$$\int_0^\infty u(\alpha_1)\,d\alpha_1 = \int_0^\infty dx \int_{-\infty}^\infty dc \int_{-\infty}^\infty d\alpha_1 e^{2\pi i c(x-\alpha_1)} u(\alpha_1).$$

Use this for $u(\alpha_1) = \int_{\mathbb{R}^2} d\alpha_2 d\alpha_3 f^\mu_{n,\lambda}(\sigma(\alpha)) \in \mathcal{S}(\mathbb{R})$ and get

$$\int_0^\infty d\alpha_1 \int_{\mathbb{R}^2} d\alpha_2 d\alpha_3 f^\mu_{n,\lambda}(\sigma(\alpha)) = \int_0^\infty dx \int_{-\infty}^\infty dc \int_{\mathbb{R}^3} d\alpha_1 d\alpha_2 d\alpha_3 e^{2\pi i c(x-\alpha_1)} f^\mu_{n,\lambda}(\sigma(\alpha)).$$

Now since $\sigma \in O(3)$ we have

$$\int_{H(-1,\nu_2,\nu_3;l)} f^\mu_{n,\lambda}(\alpha)\,d\alpha = \int_0^\infty dx \int_{-\infty}^\infty dc \int_{\mathbb{R}^3} d\alpha e^{2\pi i c(x-(\sum_{j=1}^3 \nu_j\alpha_j + 2Kl))} f^\mu_{n,\lambda}(\alpha),$$

where $\nu_1 = -1$. Inspired by Proposition 5.3 we will calculate the α–integral by SPA by only using the stationary point of the phase-function

$$\alpha \mapsto 2c(x - \sum_{j=1}^3 \nu_j\alpha_j + 2Kl) + \phi^\mu_{n,\lambda}(\alpha),$$

where $\phi^\mu_{n,\lambda}$ is given by (48). This is given by

$$\alpha_{i,c} = \frac{2K}{p_i}(\hat{n}_i + \nu_i q_i c) = \alpha_i^{st} + \frac{2K}{p_i}\nu_i q_i c,$$

and one finds by a rather long computation

$$\begin{aligned}
&\int_{H(-1,\nu_2,\nu_3;l)} f^\mu_{n,\lambda}(\alpha)\,d\alpha \sim Z_3 \left\{ \prod_{i=1}^3 \sum_{\mu_i=\pm1} \frac{1}{\sqrt{|p_i|}} \mu_i \exp\left(\mu_i 2\pi i \left[\frac{r_i}{p_i}\tilde{n}_i - s_i\lambda\right]\right) \right. \\
\times\quad & \left. \exp\left(2\pi i K\left[\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i \lambda^2\right]\right)\right\} \int_0^\infty dx \int_{-\infty}^\infty dc \exp\left(-\pi\epsilon \sum_{i=1}^3 \alpha_{i,c}^2\right) \\
\times\quad & \exp(2\pi i c(x - 2Kl)) \exp\left(-2\pi i K\left[\frac{H}{P}c^2 + 2\left(\frac{H}{P}\hat{c}_0 - l\right)c\right]\right),
\end{aligned}$$

where Z_3 is given in (53). Calculating the c–integral in the same way one finds that

$$\begin{aligned}
&\int_{H(-1,\nu_2,\nu_3;l)} f^\mu_{n,\lambda}(\alpha)\,d\alpha \sim Z_3 \frac{e^{-i\frac{\pi}{4}\operatorname{sign}(H/P)}}{\sqrt{2K|H|}} \exp\left(-\pi\epsilon \sum_{i=1}^3 \left(\frac{2K}{p_i}\hat{n}_i\right)^2\right) \\
\times\quad & \prod_{i=1}^3 \mu_i \exp\left(\mu_i 2\pi i\left[\frac{r_i}{p_i}\tilde{n}_i + s_i\lambda\right]\right) \exp\left(2\pi i K\left[\frac{r_i}{p_i}\tilde{n}_i^2 - q_i s_i\lambda^2 + \frac{H}{P}\hat{c}_0^2\right]\right) \\
\times\quad & \int_0^\infty dx \exp(-\pi\epsilon 4K^2(\beta c_{st}^2 + 2\gamma c_{st})) \exp\left(i\pi\left[\frac{P}{2H}\frac{1}{K}x^2 - 2\hat{c}_0 x\right]\right),
\end{aligned}$$

where $c_{st} = -\hat{c}_0 + \frac{P}{H}\frac{x}{2K}$ and $\beta = \sum_{i=1}^3 \frac{q_i^2}{p_i^2}$ and $\gamma = \sum_{i=1}^3 \frac{\nu_i q_i \hat{n}_i^2}{p_i^2}$. Finally, if one let I be the integral over x in this expression one finds

$$I = exp(-\pi\epsilon 4K^2(\beta\hat{c}_0^2 - 2\gamma\hat{c}_0))J,$$

where

$$\begin{aligned} J &= \int_0^\infty \exp(-\pi\epsilon(b_1x^2+b_2x))\exp\left(i\pi\left[\frac{P}{2H}\frac{1}{K}x^2-2\hat{c}_0x\right]\right)dx \\ &= \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{i\pi}{2K}\frac{P}{H}\right)^j\int_0^\infty \exp(-\pi\epsilon(b_1x^2+b_2x))\exp(-2\pi i\hat{c}_0x)x^{2j}\,dx. \end{aligned}$$

Here $b_1 = 4K\beta\left(\frac{P}{2H}\right)^2 > 0$ and $b_2 = 4K\frac{P}{H}(\gamma-\beta\hat{c}_0)$. We now have to bring all these calculations together and use possible symmetries to reduce the expressions. One way of reducing the calculations is by observing that the point α^{cst} in (50) does not change under a transformation

$$(n_1,n_2,n_3)\mapsto(n_1,n_2,n_3)+(\nu_1q_1,\nu_2q_2,\nu_3q_3)m, \qquad m\in\mathbb{Z}, \tag{54}$$

where ν has to be the one which determine α^{cst}. Using this and possible other symmetries one can maybe reduce the expression

$$\sum_{l\in\mathbb{Z}}\sum_{\nu_2,\nu_3=\pm1}\nu_2\nu_3\int_{[\alpha^{cst}]}f^{\mu}_{n,\lambda}\,d\alpha$$

to a single expression of the kind

$$\sum_{m\in\mathbb{Z}}\int_{[\alpha_0^{cst}]}\prod_{i=1}^3\frac{1}{\sqrt{2K|q_i}}e^{-\pi\epsilon\alpha_i^2}\exp\left(\frac{i\pi}{2Kq_i}[p_i\alpha_i^2-2\alpha_i(2K(n_i-q_im)+\mu_i)+s_i(2Kn_i+\mu_i)^2]\right),$$

where α_0^{cst} is the conditional stationary point on the plane $\partial H(-1,-1,-1;0)$. Note here that $\exp\left(\frac{i\pi}{2Kq_i}s_i(2K(n_i-q_im)+\mu_i)^2\right) = \exp\left(\frac{i\pi}{2Kq_i}s_i(2Kn_i+\mu_i)^2\right)$. This last sum of integrals is considered in [Ro]. It is then the hope that one can find a formula for Z_2^{cst} which is a sum over an indexset which is independent of K (like the expression for Z_2^{st} in Remark 5.8). If this succeed we have an asymptotic formula for $Z_k(X)$, which we then can compare with the formula predicted by the path integral in a way illustrated in [Ro, Sect. 4].

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