

Generalized bilinear Calderón-Zygmund operators and applications

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Outline of the talk

- Bilinear pseudodifferential operators
- Paraproducts
- Generalized bilinear Calderón-Zygmund singular integral operators

BILINEAR PSEUDODIFFERENTIAL OPERATORS

- Fourier transform

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- Multiplier operator

$$T_m f(x) = \int m(\xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

- Pseudodifferential operator

$$T_\sigma f(x) = \int \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$T_m f(x) = \int m(\xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

Theorem (Mihlin, 1956)

If $m : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$, such that

$$|\partial_\xi^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|},$$

for $|\alpha| \leq [n/2] + 1$, then $T_m : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$, $1 < p < \infty$.

A class of symbols

Let $m \in \mathbb{R}$ and $0 \leq \delta, \rho \leq 1$.

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A symbol $\sigma(x, \xi)$ belongs to Hörmander's class $S_{\rho, \delta}^m$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m + \delta|\alpha| - \rho|\beta|}, \quad x, \xi \in \mathbb{R}^n, \quad (1)$$

where $\alpha, \beta \in \mathbb{Z}^n$ and $|\alpha|, |\beta|$ depend on the context.

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where $\alpha, \beta \in \mathbb{Z}^n$ and $|\alpha|, |\beta|$ depend on the context.

$$T_\sigma f(x) = \int \sigma(x, \xi) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

Theorem (Calderón-Vaillancourt, 1971)

If $\sigma \in S_{0,0}^0$, then $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$.

On the smoothness of the symbols

Question (Nirenberg, circa 1969)

Suppose that the symbol σ verifies

$$|\partial_{\xi}^{\beta} \sigma(x, \xi)| \leq C_{\beta} (1 + |\xi|)^{-|\beta|}, \quad x, \xi \in \mathbb{R}^n, \quad (2)$$

for all $\beta \in \mathbb{Z}^n$, with no a priori regularity in the x -variable.

Does it follow that $T_{\sigma} : L^2 \rightarrow L^2$?

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Suppose that the symbol σ verifies

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for all $\beta \in \mathbb{Z}^n$, with no a priori regularity in the x -variable.

Does it follow that $T_{\sigma} : L^2 \rightarrow L^2$?

Answer: No. (C.-H. Ching, 1972).

Dini continuity in x

Let ω be a modulus of continuity and let Σ_ω be the class of all $\sigma(x, \xi)$ satisfying, for $x, \xi \in \mathbb{R}^n$,

$$|\partial_\xi^\beta \sigma(x, \xi)| \leq C_\beta (1 + |\xi|)^{-|\beta|},$$

$$|\partial_\xi^\beta \sigma(x + h, \xi) - \partial_\xi^\beta \sigma(x, \xi)| \leq C_\beta \omega(|h|) (1 + |\xi|)^{-|\beta|}.$$

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Theorem (Coifman and Meyer, 70's)

The following statements are equivalent:

- (i) $\int_0^1 \omega^2(t) \frac{dt}{t} < \infty$
- (ii) For all $\sigma \in \Sigma_\omega$, T_σ is bounded on $L^2(\mathbb{R}^n)$.
- (iii) For all $\sigma \in \Sigma_\omega$ and all $p \in (1, \infty)$, T_σ is bounded on $L^p(\mathbb{R}^n)$.
- (iv) For all $\sigma \in \Sigma_\omega$, T_σ is continuous from $H^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

For $x \in \mathbb{R}^n$, $f, g \in \mathcal{S}(\mathbb{R}^n)$ consider

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

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Bilinear Hörmander classes. $\sigma(x, \xi, \eta) \in BS_{\rho, \delta}^m$ if

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha, \beta, \gamma} (1 + |\xi| + |\eta|)^{m - \rho(|\beta| + |\gamma|) + \delta|\alpha|}, \quad x, \xi, \eta \in \mathbb{R}^n.$$

Bilinearization is often tricky

The natural bilinear versions of the Calderón-Vaillancourt and Mihlin theorems fail.

Theorem (Coifman-Meyer, 70's)

If $\int_0^1 \omega^2(t) \frac{dt}{t} < \infty$ and $\sigma(x, \xi, \eta)$ satisfies

$$\left| \partial_\xi^\alpha \partial_\eta^\beta \sigma(x, \xi, \eta) \right| \leq C_{\alpha, \beta} (1 + |\xi| + |\eta|)^{-(|\alpha| + |\beta|)},$$

$$\left| \partial_\xi^\alpha \partial_\eta^\beta (\sigma(x + h, \xi, \eta) - \sigma(x, \xi, \eta)) \right| \leq C_{\alpha, \beta} \frac{\omega(|h|)}{(1 + |\xi| + |\eta|)^{(|\alpha| + |\beta|)}},$$

then T_σ can be extended as a bounded operator from

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \text{ into } L^r(\mathbb{R}^n)$$

for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \in (0, 1)$.

- Reduced symbols
- Almost orthogonality principles
- $r > 1$

Theorem

In the linear case, $S_{1,1}^0$ is the maximal class of symbols such that T_σ has a Calderón-Zygmund kernel.

That is,

$$T_\sigma(f)(x) = \int K(x, y)f(y) dy,$$

where $K(x, y)$ satisfies

$$|\partial^\alpha K(x, y)| \leq \frac{C_\alpha}{|x - y|^{n+|\alpha|}}, \quad x, y \in \mathbb{R}^n.$$

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Questions (for the bilinear case)

- When is T_σ associated to a “bilinear Calderón-Zygmund kernel”?
- When is T_σ bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$?
- Can r go down to $\frac{1}{2}$?

For $b > 0$, we write $\omega \in Dini(b)$ if $\omega : [0, \infty) \rightarrow [0, \infty)$, ω is non-decreasing, concave, and

$$\int_0^1 \omega^b(t) \frac{dt}{t} < \infty.$$

We will consider $a \in (0, 1)$, $\omega \in Dini(a/2)$, and $\Omega : [0, \infty) \rightarrow [0, \infty)$ non-decreasing such that

$$\sup_{0 < t < 1} \omega^{1-a}(t) \Omega(1/t) < \infty.$$

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$$\sup_{0 < t < 1} \omega^{1-a}(t) \Omega(1/t) < \infty.$$

Example: $\omega(t) = t^\alpha$, $\Omega(t) = t^\beta$, $0 < \beta < \alpha \leq 1$. (M. Nagase, 1977, linear case.)

Theorem (D. M, V.Naibo)

Let a, ω, Ω be as in the previous slide, and $1 \leq p, q \leq \infty$ with $\frac{1}{2} \leq r < \infty, \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Assume that the bilinear symbol σ verifies

$$\left| \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} \sigma(x, \xi, \eta) \right| \leq C_{\alpha, \beta} (1 + |\xi| + |\eta|)^{-(|\alpha| + |\beta|)},$$

$$\left| \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} (\sigma(x + h, \xi, \eta) - \sigma(x, \xi, \eta)) \right| \leq C_{\alpha, \beta} \frac{\omega(|h|) \Omega(|\xi| + |\eta|)}{(1 + |\xi| + |\eta|)^{(|\alpha| + |\beta|)}},$$

for $|\alpha| + |\beta| \leq 4n + 4$. Then, the associated bilinear pseudo-differential operator T_{σ} has the following boundedness properties:

Theorem (continued)

- If $p, q > 1$,

$$\|T_\sigma(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

- If $p = 1$ or $q = 1$,

$$\|T_\sigma(f, g)\|_{L^{r,\infty}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

- $\|T_\sigma(f, g)\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}.$

Theorem (continued)

- If $1 < p, q < \infty$, and $w \in A_{\min(p,q)}$,

$$\|T_\sigma(f, g)\|_{L_w^r(\mathbb{R}^n)} \leq C \|f\|_{L_w^p(\mathbb{R}^n)} \|g\|_{L_w^q(\mathbb{R}^n)}.$$

- If $w \in A_1$,

$$\|T_\sigma(f, g)\|_{L_w^{1/2, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L_w^1(\mathbb{R}^n)} \|g\|_{L_w^1(\mathbb{R}^n)},$$

$$\|T_\sigma(f, g)\|_{L_w^{1/2}(\mathbb{R}^n)} \leq C \|f\|_{H_w^1(\mathbb{R}^n)} \|g\|_{H_w^1(\mathbb{R}^n)}.$$

Theorem (continued)

- If $1 < p, q < \infty$, $1 < s_1, s_2 < \infty$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_{\min(s_1, s_2)}(\mathbb{Z}^n)$,

$$\|T_\sigma(f, g)\|_{(L^r, l_w^{s_3})} \leq C \|f\|_{(L^p, l_w^{s_1})} \|g\|_{(L^q, l_w^{s_2})}.$$

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Here

$$\|f\|_{(L^p, l_w^q)} := \left(\sum_{z \in \mathbb{Z}^n} \|f\|_{L^p(Q_z)}^q w_z \right)^{1/q},$$

where $Q_z = z + [-1/2, 1/2]^n$ and $w = \{w_z\}_{z \in \mathbb{Z}^n}$ is in the discrete Muckenhoupt class $A_p(\mathbb{Z}^n)$.

PARAPRODUCTS

Elements of the Littlewood-Paley theory

$\Phi \in \mathcal{S}(\mathbb{R}^n)$, real, radially symmetric, and

$$\chi_{\{|\xi| \leq 1/2\}} \leq \hat{\Phi} \leq \chi_{\{|\xi| \leq 1\}}$$

Define $\Psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\Psi}(\xi) := \hat{\Phi}(\xi/2) - \hat{\Phi}(\xi), \quad \xi \in \mathbb{R}^n.$$

For $j \in \mathbb{Z}$, set

$$\widehat{\Psi}_j(\xi) := \hat{\Psi}(\xi/2^j) \quad \text{and} \quad \widehat{\Phi}_j(\xi) := \hat{\Phi}(\xi/2^j)$$

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Hence, $\text{supp}(\widehat{\Psi}_j) \subset \{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+1}\}$, for $j \in \mathbb{Z}$, and

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

Define the operators S_j and Δ_j by

$$S_j(f) = \Phi_j * f,$$

and

$$\Delta_j(f) = \Psi_j * f.$$

In particular,

$$S_j(f) = \sum_{k \leq j} \Delta_k(f) \quad \text{and} \quad f = \sum_{j \in \mathbb{Z}} \Delta_j(f),$$

Product vs. Paraproduct

Given $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$fg = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \Delta_j(f) \Delta_k(g)$$

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Definition

The paraproduct of f and g is defined as

$$\begin{aligned} \Pi(f, g) &:= \sum_{j \in \mathbb{Z}} \sum_{k \leq j} \Delta_j(f) \Delta_k(g) \\ &= \sum_{j \in \mathbb{Z}} \Delta_j(f) S_j(g) \end{aligned}$$

We have

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Moreover, for a fixed $b \in BMO$, the mapping

$$f \mapsto \Pi(b, f)$$

is a Calderón-Zygmund singular integral.

We write $Q \in \mathcal{D}$ if Q is a **dyadic cube**.

That is, for some $k \in \mathbb{Z}^n$, $\nu \in \mathbb{Z}$

$$Q = \{x \in \mathbb{R}^n : k_i \leq 2^\nu x_i \leq k_i + 1; i = 1, \dots, n\}$$

Set $x_Q = 2^{-\nu}k$ and also write $Q = Q_{\nu k}$.

Molecules of mild regularity

Let ω be a modulus of continuity. An ω -molecule associated to a dyadic cube $P = P_{\nu k}$ is a function

$$\phi_P = \phi_{\nu k} : \mathbb{R}^n \rightarrow \mathbb{C}$$

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such that for some $A > 0$ and $N > n$, it satisfies the decay condition

$$|\phi_{\nu k}(x)| \leq \frac{A2^{\nu n/2}}{(1 + 2^\nu |x - x_P|)^N}, \quad x \in \mathbb{R}^n, \quad (3)$$

and...

... the following mild regularity condition

$$|\phi_P(x) - \phi_P(y)| \leq A2^{\nu n/2} \omega(2^\nu |x - y|) \left[\frac{1}{(1 + 2^\nu |x - x_P|)^N} + \frac{1}{(1 + 2^\nu |y - x_P|)^N} \right]$$

for all $x, y \in \mathbb{R}^n$.

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for all $x, y \in \mathbb{R}^n$.

Ultimate example: $\phi_{\nu k}(x) = 2^{\nu n/2} \phi(2^\nu x - k)$, for $\phi \in \mathcal{S}(\mathbb{R}^n)$.

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Notice: No cancelation required (yet).

Given three families of ω -molecules $\{\phi_Q^j\}_{Q \in \mathcal{D}}$, $j = 1, 2, 3$, the **paraproduct** $\Pi(f, g)$ associated to these families is defined by

$$\Pi(f, g) = \sum_{Q \in \mathcal{D}} |Q|^{-1/2} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^2 \rangle \phi_Q^3, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

Theorem (D. M., V. Naibo)

Consider $\omega \in \text{Dini}(1/2)$ and let $\{\phi_Q^j\}_{Q \in \mathcal{D}}$, $j = 1, 2, 3$ be three families of ω -molecules with decay $N > 5n + 5$ and such that at least two of them, say $j = 1, 2$, enjoy the cancelation property

$$\int_{\mathbb{R}^n} \phi_Q^j(x) dx = 0, \quad Q \in \mathcal{D}, j = 1, 2.$$

Let $1 \leq p, q \leq \infty$ with $\frac{1}{2} \leq r < \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then, the paraproduct Π verifies the following

- If $1 < p, q < \infty$

$$\|\Pi(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

- If $p = 1$ or $q = 1$,

$$\|\Pi(f, g)\|_{L^{r, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Theorem (continued)

- $\|\Pi(f, g)\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}$.
- If $1 < p, q < \infty$, and $w \in A_{\min(p,q)}$,

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- If $1 < p, q < \infty$, $1 < s_1, s_2 < \infty$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_{\min(s_1, s_2)}(\mathbb{Z}^n)$,

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GENERALIZED BILINEAR CALDERÓN-ZYGMUND SINGULAR INTEGRAL OPERATORS

$K(x, y, z)$ is called a

bilinear Calderón-Zygmund kernel of type $\omega(t)$ if

$$|K(x, y, z)| \leq \frac{C_K}{(|x - y| + |x - z|)^{2n}},$$

$$|K(x + h, y, z) - K(x, y, z)| \leq \frac{\omega\left(\frac{|h|}{|x - y| + |x - z|}\right)}{(|x - y| + |x - z|)^{2n}},$$

for $|h| \leq \frac{1}{2} \max(|x - y|, |x - z|)$, and similarly for y and z .

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The linear case was first introduced by K. Yabuta, 1985.

Bilinear Calderón-Zygmund operator of type $\omega(t)$

A bilinear operator $T : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}'$ is said to be associated to a bilinear Calderón-Zygmund kernel of type $\omega(t)$, $K(x, y, z)$, if

$$T(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y, z) f(y) g(z) dy dz$$

whenever $x \notin \text{supp}(f) \cap \text{supp}(g)$ and $f, g \in C_0^\infty(\mathbb{R}^n)$.

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whenever $x \notin \text{supp}(f) \cap \text{supp}(g)$ and $f, g \in C_0^\infty(\mathbb{R}^n)$.

If, besides, T maps

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^{r, \infty}(\mathbb{R}^n),$$

for some $1 < p, q < \infty$ and $r > 1$ with $1/p + 1/q = 1/r$; or

$$L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n),$$

for some $1 < p, q < \infty$ with $1/p + 1/q = 1$, T is called a
bilinear Calderón-Zygmund operator of type $\omega(t)$.

Theorem (D. M., V. Naibo)

A bilinear Calderón-Zygmund operator of type ω verifies

- If $p, q > 1$,

$$\|T_\sigma(f, g)\|_{L^r(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

- If $p = 1$ or $q = 1$,

$$\|T_\sigma(f, g)\|_{L^{r, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

- $\|T_\sigma(f, g)\|_{BMO(\mathbb{R}^n)} \leq C \|f\|_{L^\infty(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)}.$

Theorem (continued)

- If $1 < p, q < \infty$, and $w \in A_{\min(p,q)}$,

$$\|T_\sigma(f, g)\|_{L_w^r(\mathbb{R}^n)} \leq C \|f\|_{L_w^p(\mathbb{R}^n)} \|g\|_{L_w^q(\mathbb{R}^n)}.$$

- If $w \in A_1$,

$$\|T_\sigma(f, g)\|_{L_w^{1/2, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L_w^1(\mathbb{R}^n)} \|g\|_{L_w^1(\mathbb{R}^n)},$$

$$\|T_\sigma(f, g)\|_{L_w^{1/2}(\mathbb{R}^n)} \leq C \|f\|_{H_w^1(\mathbb{R}^n)} \|g\|_{H_w^1(\mathbb{R}^n)}.$$

Theorem (continued)

- If $1 < p, q < \infty$, $1 < s_1, s_2 < \infty$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_{\min(s_1, s_2)}(\mathbb{Z}^n)$,

$$\|T_\sigma(f, g)\|_{(L^r, l_w^{s_3})} \leq C \|f\|_{(L^p, l_w^{s_1})} \|g\|_{(L^q, l_w^{s_2})}.$$

Theorem (continued)

- If $1 < p, q < \infty$, $1 < s_1, s_2 < \infty$, $1/s_3 = 1/s_1 + 1/s_2$, and $w \in A_{\min(s_1, s_2)}(\mathbb{Z}^n)$,

$$\|T_\sigma(f, g)\|_{(L^r, l_w^{s_3})} \leq C \|f\|_{(L^p, l_w^{s_1})} \|g\|_{(L^q, l_w^{s_2})}.$$