



# On the boundedness of bilinear operators on products of Besov and Lebesgue spaces

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## ABSTRACT

We prove mapping properties of the form  $T : \dot{B}_{p_1}^{\alpha_1, q_1} \times L^{p_2} \rightarrow \dot{B}_{p_3}^{\alpha_2, q_2}$  and  $T : \dot{B}_{p_1}^{\alpha_1, q_1} \times \dot{B}_{p_2}^{\alpha_2, q_2} \rightarrow L^{p_3}$ , for certain related indices  $p_1, p_2, p_3, q_1, q_2, \alpha_1, \alpha_2 \in \mathbb{R}$ , where  $T$  is a bilinear Hörmander–Mihlin multiplier or a molecular paraproduct. Applications to bilinear Littlewood–Paley theory are discussed.

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## 1. Introduction

Beginning from the classical works of R. Coifman and Y. Meyer [7–9] on bilinear pseudo-differential operators and J.-M. Bony [6] and H. Triebel [30,31] on bilinear paraproducts through the recent progress in the development of the bilinear Calderón–Zygmund theory [18–22], the bilinear Hilbert transform [24,25], and molecular paraproducts [13,14,29], bilinear operators continue to be object of intense study. Of particular interest are the recent bilinear estimates in the scales of Besov and Triebel–Lizorkin spaces of the form

$$T : \dot{B}_{p_1}^{\alpha_1, q_1} \times \dot{B}_{p_2}^{\alpha_2, q_2} \rightarrow \dot{B}_{p_3}^{\alpha_3, q_3} \quad \text{and} \quad T : \dot{F}_{p_1}^{\alpha_1, q_1} \times \dot{F}_{p_2}^{\alpha_2, q_2} \rightarrow \dot{F}_{p_3}^{\alpha_3, q_3},$$

for related indices  $p_1, p_2, p_3, q_1, q_2, q_3, \alpha_1, \alpha_2, \alpha_3$ , and families of bilinear operators  $T$  including bilinear multipliers, bilinear Calderón–Zygmund operators, molecular paraproducts, and bilinear pseudo-differential operators established in, for instance, [1–5,13,14,16–22,27,29,33,34].

The purpose of this article is to address Besov–Lebesgue boundedness properties of the form

$$T : \dot{B}_{p_1}^{\alpha_1, q_1} \times L^{p_2} \rightarrow \dot{B}_{p_3}^{\alpha_2, q_2} \quad \text{and} \quad T : \dot{B}_{p_1}^{\alpha_1, q_1} \times \dot{B}_{p_2}^{\alpha_2, q_2} \rightarrow L^{p_3} \tag{1.1}$$

(as well as its corresponding non-homogeneous versions) that complement the existing results in the literature on the subject. Our key tool is a lemma (Lemma 2.1 below), which, despite its simplicity, provides an insightful viewpoint into the nature of the bilinear estimates of the form (1.1). From this perspective, in Sections 3 and 4 we prove Besov–Lebesgue estimates (1.1) for bilinear multipliers of Hörmander–Mihlin type and for bilinear molecular paraproducts, respectively, without resorting to the usual tools of molecular decompositions of Besov spaces or reduced bilinear symbols. In Section 5 we introduce a vector-valued interpretation of the present ideas, along with its applications to bilinear Littlewood–Paley theory.

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## 2. The basic lemma

We first fix some notation that will be used throughout the paper.

The class of Schwartz functions in  $\mathbb{R}^n$  will be denoted by  $\mathcal{S}(\mathbb{R}^n)$  and we set  $\mathcal{S}_0(\mathbb{R}^n) := \{f \in \mathcal{S}(\mathbb{R}^n) : \partial^\gamma \hat{f}(0) = 0, \text{ for all } \gamma\}$ . We write  $\psi \in \Psi$  if  $\psi \in \mathcal{S}(\mathbb{R}^n)$ ,  $\text{supp}(\hat{\psi}) \subset \{\xi : 1/2 \leq |\xi| \leq 2\}$  and  $\hat{\psi} \equiv 1$  in  $\{\xi : 3/5 \leq |\xi| \leq 5/3\}$ . For  $\alpha \in \mathbb{R}$ ,  $0 < p, q \leq \infty$ , and  $f \in \mathcal{S}(\mathbb{R}^n)'$  we define

$$\|f\|_{\dot{B}_p^{\alpha,q}} := \left( \sum_{v \in \mathbb{Z}} 2^{v\alpha q} \|\Delta_v(f)\|_{L^p}^q \right)^{1/q}, \tag{2.1}$$

where  $\Delta_v(f) = \psi_v * f$  and  $\psi_v(x) = 2^{vn} \psi(2^v x)$ , with the usual interpretation when  $q = \infty$ . The homogeneous Besov spaces  $\dot{B}_p^{\alpha,q}$  is the set of tempered distributions  $f$ , modulo polynomials, such that  $\|f\|_{\dot{B}_p^{\alpha,q}}$  is finite. The definition is independent of the choice of  $\psi$  and the dual of  $\dot{B}_p^{\alpha,q}$  is  $\dot{B}_{p'}^{-\alpha,q'}$ , where  $p'$  and  $q'$  denote the conjugate pairs of  $p$  and  $q$ , respectively. The scale of homogeneous Triebel–Lizorkin spaces  $\dot{F}_p^{\alpha,q}$  is defined similarly, for  $q < \infty$ , with the sum and the integral in (2.1) taken in reverse order, see [12] and [32] for more details.

Finally,  $C$  will denote a constant that may depend only on the parameters involved, and that may change from line to line.

Our starting point is the following lemma, which is essentially based on a bilinear Schur-type inequality and Calderón’s reproducing formula.

**Lemma 2.1.** *Let  $T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)'$  be a continuous bilinear operator and denote by  $T^{*2}$  its second adjoint. Let  $1 \leq p, q, r, s \leq \infty$ , with  $1/p + 1/q = 1/r$ , and suppose that there exist  $l > 0$  and  $C > 0$  such that for all  $j, k \in \mathbb{Z}$  and  $h = 1, 2, 3$ ,*

$$\sup_{x_h \in \mathbb{R}^n} \iint |T^{*2}(\psi_j^{(1)}(\cdot - x_1), \psi_k^{(2)}(x_2 - \cdot))(x_3)| \prod_{\substack{m=1 \\ m \neq h}}^3 dx_m \leq C 2^{-l|j-k|}, \tag{2.2}$$

for some  $\psi^{(1)}, \psi^{(2)} \in \Psi$ , such that Calderón’s formula holds true for  $\psi^{(2)}$ , i.e.,

$$h = \sum_{j \in \mathbb{Z}} \Delta_j^{(2)} \Delta_j^{(2)}(h), \quad h \in \mathcal{S}_0(\mathbb{R}^n). \tag{2.3}$$

Then, for all  $\alpha \in \mathbb{R}$  with  $|\alpha| < l$  there exists a constant  $C_1 > 0$ , depending on  $C, n$ , and  $\alpha$ , such that

$$\|T(f, g)\|_{\dot{B}_r^{\alpha,s}} \leq C_1 \|f\|_{\dot{B}_p^{\alpha,s}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

**Proof.** Let  $K(x, y, z)$  denote the Schwartz distributional kernel of  $T$ . We express this by formally writing

$$T(f, g)(x) = \iint K(x, y, z) f(y) g(z) dy dz.$$

In this sense, the second adjoint operator  $T^{*2}$  has kernel  $K^{*2}(x, y, z) = K(z, y, x)$ , see Section 2 in [20], for more details. For  $k \in \mathbb{Z}$  and  $\psi^{(1)} \in \Psi$ , define the bilinear operators  $\Theta_k$  as  $\Theta_k(f, g) := \Delta_k^{(1)} T(f, g) = \psi_k^{(1)} * T(f, g)$ , that is,

$$\Theta_k(f, g)(x) = \iint \left( \int \psi_k^{(1)}(x - u) K(u, y, z) du \right) f(y) g(z) dy dz.$$

Consequently, the bilinear operator  $(f, g) \mapsto \Theta_k(\Delta_j^{(2)} f, g)$  has kernel

$$\begin{aligned} K_{jk}^T(x, y, z) &:= \iint \psi_k^{(1)}(x - w) K(w, u, z) \psi_j^{(2)}(u - y) du dw \\ &= T^{*2}(\psi_j^{(2)}(\cdot - y), \psi_k^{(1)}(x - \cdot))(z). \end{aligned}$$

Set  $f_1 = f, f_2 = g, p_1 = p, p_2 = q$  and  $p_3 = r'$ . For  $f_3 \in L^{r'}(\mathbb{R}^n)$ , Hölder’s inequality and inequality (2.2) yield the following bilinear Schur-type inequalities

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \Theta_k(\Delta_j^{(2)} f, g)(x_3) f_3(x_3) dx_3 \right| &\leq \iiint |K_{jk}^T(x_1, x_2, x_3)| \prod_{m=1}^3 |f_m(x_m)| dx_m \\ &= \iiint \prod_{m=1}^3 |K_{jk}^T(x_1, x_2, x_3)|^{1/p_m} |f_m(x_m)| dx_m \end{aligned}$$

$$\begin{aligned} &\leq \prod_{m=1}^3 \left( \iiint |K_{jk}^T(x_1, x_2, x_3)| |f_m(x_m)|^{p_m} dx_1 dx_2 dx_3 \right)^{1/p_m} \\ &\leq C 2^{-l|j-k|} \prod_{m=1}^3 \|f_m\|_{L^{p_m}}. \end{aligned}$$

Thus,

$$\|\Theta_k(\Delta_j^{(2)} f, g)\|_{L^r} \leq C 2^{-l|j-k|} \|f\|_{L^p} \|g\|_{L^q}. \tag{2.4}$$

Next, let  $\{G_k\}_{k \in \mathbb{Z}}$  be a sequence of functions such that  $\sum_{k \in \mathbb{Z}} 2^{-\alpha ks'} \|G_k\|_{L^{r'}}^{s'} \leq 1$ . By using Hölder's inequality, (2.3), (2.4), and choosing  $\lambda$  such that  $0 < \lambda < l - |\alpha|$ , we obtain

$$\begin{aligned} &\left| \sum_{k \in \mathbb{Z}} \int \Theta_k(f, g)(x) G_k(x) dx \right| \\ &= \left| \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \int \Theta_k(\Delta_j^{(2)} \Delta_j^{(2)} f, g)(x) G_k(x) dx \right| \\ &\leq \left( \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{-\lambda|j-k|s'} 2^{-k\alpha s'} \|G_k\|_{L^{r'}(\mathbb{R}^n)}^{s'} \right)^{1/s'} \left( \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{\lambda|j-k|s} 2^{k\alpha s} \|\Theta_k(\Delta_j^{(2)} \Delta_j^{(2)} f, g)\|_{L^r(\mathbb{R}^n)}^s \right)^{1/s} \\ &\leq C \left( \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{(\lambda-l)|j-k|s} 2^{k\alpha s} \|\Delta_j^{(2)} f\|_{L^p(\mathbb{R}^n)}^s \|g\|_{L^q(\mathbb{R}^n)}^s \right)^{1/s} \\ &= C \left( \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^{(k-j)\alpha s} 2^{(\lambda-l)|j-k|s} 2^{j\alpha s} \|\Delta_j^{(2)} f\|_{L^p(\mathbb{R}^n)}^s \right)^{1/s} \|g\|_{L^q(\mathbb{R}^n)} \\ &\leq C \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} 2^{(\lambda-l+|\alpha|)|j-k|s} \right) 2^{j\alpha s} \|\Delta_j^{(2)} f\|_{L^p(\mathbb{R}^n)}^s \right)^{1/s} \|g\|_{L^q(\mathbb{R}^n)} \\ &\leq C_1 \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{L^q(\mathbb{R}^n)}, \end{aligned}$$

and the lemma follows by duality.  $\square$

**Remark 1.** By duality, if the condition (2.2) in Lemma 2.1 holds with  $T$  or  $T^{*1}$  instead of  $T^{*2}$ , then the resulting bounds are of the form

$$\|T(f, g)\|_{L^r} \leq C_1 \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{\dot{B}_q^{-\alpha, s'}}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

or

$$\|T(f, g)\|_{\dot{B}_r^{\alpha, s}} \leq C_1 \|f\|_{L^p} \|g\|_{\dot{B}_q^{\alpha, s}}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

respectively.

### 3. Boundedness of bilinear Hörmander–Mihlin multipliers

In this section we consider bilinear multipliers of the form

$$T_\sigma(f, g)(x) = \iint \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta,$$

where  $\sigma(\xi, \eta)$  is an infinitely differentiable function defined on  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$  verifying the Hörmander–Mihlin condition, namely,

$$|\partial_\xi^\gamma \partial_\eta^\beta \sigma(\xi, \eta)| \leq C_{\gamma, \beta} (|\xi| + |\eta|)^{-|\gamma| - |\beta|}, \tag{3.1}$$

for all  $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0, 0)\}$  and all multiindices  $\gamma$  and  $\beta$ . Here  $|\gamma| = \gamma_1 + \dots + \gamma_n$  if  $\gamma = (\gamma_1, \dots, \gamma_n)$ , and similarly for  $|\beta|$ .

In [18], L. Grafakos and R. Torres used the molecular decomposition of homogeneous Besov spaces to study mapping properties of  $T_\sigma$  in the diagonal Besov cases of the form  $T_\sigma : \dot{B}_p^{\alpha_1, p} \times \dot{B}_q^{\alpha_2, q} \rightarrow \dot{B}_r^{\alpha_1 + \alpha_2, r}$ ,  $\alpha_1, \alpha_2 > 0$ ,  $1 < p, q, r < \infty$ ,  $1/p + 1/q = 1/r$ , under the following cancelation conditions on  $\sigma(\xi, \eta)$

$$\partial_\xi^\rho \sigma(0, \eta) = 0, \quad \text{for all } \eta \neq 0, \tag{3.2}$$

$$\partial_\xi^\rho \sigma(\eta, -\eta) = 0, \quad \text{for all } \eta \neq 0, \tag{3.3}$$

$$\partial_\eta^\rho \sigma(\xi, 0) = 0, \quad \text{for all } \xi \neq 0, \tag{3.4}$$

for suitably many multiindices  $\rho$ , see [18, Theorem 3]. Under the same cancellation hypotheses, mapping properties of  $T_\sigma : \dot{F}_{p_1}^{\alpha_1, q_1} \times \dot{F}_{p_2}^{\alpha_2, q_2} \rightarrow \dot{F}_{p_3}^{\alpha_3, q_3}$  for the scale of homogeneous Triebel–Lizorkin spaces  $\dot{F}_p^{\alpha, q}$  have been addressed by Á. Bényi in [1, Proposition 3] and L. Grafakos and R. Torres in [19, Theorem 7].

For  $s \in \mathbb{R}$ , let  $[s]$  denote the largest integer smaller than  $s$ . We will use Lemma 2.1 and only two of the cancellation hypotheses above to prove

**Theorem 3.1.** Consider  $1 \leq p, q, r \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ , and  $\alpha \in \mathbb{R}^n$ . Let  $\sigma(\xi, \eta)$  be an infinitely differentiable function satisfying (3.1) for all  $|\gamma|, |\beta| \leq n + 1$  and the cancellation conditions (3.2) and (3.3) for all multiindices  $\rho$  satisfying  $|\rho| \leq [|\alpha|] + n + 1$ . Then

$$\|T_\sigma(f, g)\|_{\dot{B}_r^{\alpha, s}} \leq C \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{L^q}. \tag{3.5}$$

**Proof.** Let  $\psi \in \Psi$  such that Calderón’s formula holds true for  $\psi$ . For  $k, j \in \mathbb{Z}$  define

$$\begin{aligned} K_{jk}(x, y, z) &:= T_\sigma^{*2}(\psi_j(\cdot - y), \psi_k(x - \cdot))(z) \\ &= \iint \sigma(\xi, \eta) \hat{\psi}(2^{-j}\xi) \hat{\psi}(2^{-k}(\xi + \eta)) e^{i\xi(x-y)} e^{i\eta(x-z)} d\xi d\eta. \end{aligned} \tag{3.6}$$

By Lemma 2.1 it will be enough to prove that  $K_{jk}$  satisfy the conditions (2.2) for some  $l > |\alpha|$ . We consider three different cases given by  $j - k \leq -3, j - k \geq 3$ , and  $-2 \leq j - k \leq 2$ .

Case  $j - k \leq -3$ . Let  $l = [|\alpha|] + 1$ . We will prove that there exists  $C_\alpha$  such that

$$|K_{jk}(x, y, z)| \leq C_\alpha 2^{-l|j-k|} \frac{2^{kn}}{(1 + 2^k|x - z|)^{n+1}} \frac{2^{jn}}{(1 + 2^j|x - y|)^{n+1}}, \tag{3.7}$$

for any  $k, j \in \mathbb{Z}, j - k \leq -2$ . Then conditions (2.2) follow for this range of  $j$  and  $k$ .

We make the following change of variables  $(\xi, \eta) \rightarrow (2^j\xi, 2^k\eta)$ . Then

$$K_{jk}(x, y, z) = 2^{kn} 2^{jn} \iint \sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta) e^{i2^j\xi(x-y)} e^{i2^k\eta(x-z)} d\xi d\eta.$$

Without loss of generality assume that  $|x - y| \sim |x_1 - y_1|$  and  $|x - z| \sim |x_1 - z_1|$ . Choose multiindices  $\gamma$  and  $\beta$  so that  $\gamma_1 \geq 0$  and  $\gamma_m = 0$  for  $m = 2, \dots, n, \beta_1 \geq 0$  and  $\beta_m = 0, m = 2, \dots, n$ , and  $|\gamma| = |\beta| = 0$ , or  $|\gamma| = |\beta| = n + 1$ , or  $|\gamma| = n + 1$  and  $|\beta| = 0$ , or  $|\gamma| = 0$  and  $|\beta| = n + 1$ , according to whether  $2^j|x - y|$  and  $2^k|x - z|$  are smaller or larger than 1. Noticing

that  $e^{i2^j\xi(x-y)} = \frac{\partial_\xi^\gamma e^{i2^j\xi(x-y)}}{(i2^j(x_1 - y_1))^{|\gamma|}}$  and  $e^{i2^k\eta(x-z)} = \frac{\partial_\eta^\beta e^{i2^k\eta(x-z)}}{(i2^k(x_1 - z_1))^{|\beta|}}$  and integrating by parts when  $\gamma \neq 0$  or  $\beta \neq 0$  we get

$$K_{jk}(x, y, z) = C \frac{2^{jn} 2^{kn}}{(2^j(x_1 - y_1))^{|\gamma|} (2^k(x_1 - z_1))^{|\beta|}} F(x, y, z),$$

where

$$F(x, y, z) = \iint \partial_\xi^\gamma \partial_\eta^\beta (\sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)) e^{i2^j\xi(x-y)} e^{i2^k\eta(x-z)} d\xi d\eta.$$

Then (3.7) will follow if we show that  $|F(x, y, z)| \leq C 2^{-l|j-k|}$ . Using Leibniz rule we obtain

$$\begin{aligned} &\partial_\xi^\gamma \partial_\eta^\beta (\sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)) \\ &= \sum_{\mu \leq \gamma, \nu \leq \beta} c_{\nu, \mu} 2^{j|\mu|} 2^{k|\nu|} (\partial_\xi^\mu \partial_\eta^\nu \sigma)(2^j\xi, 2^k\eta) \partial_\xi^{\gamma-\mu} \partial_\eta^{\beta-\nu} (\hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)). \end{aligned}$$

Using (3.2), the mean value theorem repeatedly, and condition (3.1) we obtain

$$\begin{aligned} |2^{j|\mu|} 2^{k|\nu|} (\partial_\xi^\mu \partial_\eta^\nu \sigma)(2^j\xi, 2^k\eta)| &\leq C |(\partial_\xi^{\mu+\rho} \partial_\eta^\nu \sigma)(\tau, 2^k\eta)| (2^j|\xi|)^{|\rho|} 2^{j|\mu|} 2^{k|\nu|} \\ &\leq C 2^{(j-k)(|\rho|+|\mu|)} \frac{|\xi|^{|\rho|}}{|\eta|^{|\mu|+|\nu|+|\rho|}}, \end{aligned}$$

where  $\tau \in \mathbb{R}^n$  is in the segment joining  $0 \in \mathbb{R}^n$  and  $2^j\xi$  and the multiindex  $\rho$  is chosen appropriately and such that  $|\rho| = l$ . We now have

$$|F(x, y, z)| \leq 2^{-|j-k|} \int_{\substack{\frac{1}{2} \leq |\xi| \leq 2 \\ \frac{1}{4} \leq |\eta| \leq \frac{9}{4}}} \sum_{\substack{\mu \leq \gamma \\ \nu \leq \beta}} c_{\nu, \mu} 2^{(j-k)|\mu|} |\partial_\xi^\gamma \partial_\eta^{\beta-\nu} (\hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta))| d\xi d\eta.$$

This yields (3.7) since the last integral is uniformly bounded as  $j - k < 0$ .

Case  $j - k \geq 3$ . In this case we make the change of variables  $(\xi, \eta) \rightarrow (\xi + \eta, -\eta)$  in (3.6) obtaining

$$K_{jk}(x, y, z) = \iint \sigma(\xi + \eta, -\eta) \hat{\psi}(2^{-j}(\xi + \eta)) \hat{\psi}(2^{-k}\xi) e^{i\eta(z-y)} e^{i\xi(x-y)} d\xi d\eta.$$

Define  $\tilde{\sigma}(\xi, \eta) = \sigma(\xi + \eta, -\eta)$  and note that  $\tilde{\sigma}$  satisfies conditions (3.1) and (3.2) since  $\sigma$  satisfies (3.1) and (3.3). Now, the change of variables  $(\xi, \eta) \rightarrow (2^k\xi, 2^j\eta)$  gives

$$K_{jk}(x, y, z) = 2^{jn} 2^{kn} \iint \tilde{\sigma}(2^k\xi, 2^j\eta) \hat{\psi}(2^{k-j}\xi + \eta) \hat{\psi}(\xi) e^{i2^j\eta(z-y)} e^{i2^k\xi(x-y)} d\xi d\eta.$$

We are now in the exact same situation as in the previous case. Therefore,

$$|K_{jk}(x, y, z)| \leq C 2^{-l|j-k|} \frac{2^{kn}}{(1 + 2^k|x - y|)^{n+1}} \frac{2^{jn}}{(1 + 2^j|z - y|)^{n+1}},$$

for  $l = \lfloor |\alpha| \rfloor + 1$ .

Case  $|j - k| \leq 2$ . As in the first case ( $j - k \leq -3$ ), we obtain

$$K_{jk}(x, y, z) = C \frac{2^{jn} 2^{kn}}{(2^j(x_1 - y_1))^{\lfloor \gamma \rfloor} (2^k(x_1 - z_1))^{\lfloor \beta \rfloor}} F(x, y, z),$$

where

$$F(x, y, z) = \iint \partial_\xi^\gamma \partial_\eta^\beta (\sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)) e^{i2^j\xi(x-y)} e^{i2^k\eta(x-z)} d\xi d\eta$$

and we assume, without loss of generality, that  $|x - y| \sim |x_1 - y_1|$  and  $|x - z| \sim |x_1 - z_1|$ . We take  $|\gamma| = |\beta| = 0$ , or  $|\gamma| = |\beta| = n + 1$ , or  $|\gamma| = n + 1$  and  $|\beta| = 0$ , or  $|\gamma| = 0$  and  $|\beta| = n + 1$ , according to whether  $2^j|x - y|$  and  $2^k|x - z|$  are smaller or larger than 1. Since  $|j - k| \leq 2$  the desired result will follow if we prove that  $|F(x, y, z)|$  is bounded as a function of  $x, y, z, j, k, |j - k| \leq 2$ , for any values of  $\alpha$  and  $\beta$ . From Leibniz rule,

$$\begin{aligned} & \partial_\xi^\gamma \partial_\eta^\beta (\sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)) \\ &= \sum_{\mu \leq \gamma, \nu \leq \beta} c_{\nu, \mu} 2^{j|\mu|} 2^{k|\nu|} (\partial_\xi^\mu \partial_\eta^\nu \sigma)(2^j\xi, 2^k\eta) \partial_\xi^{\gamma-\mu} \partial_\eta^{\beta-\nu} (\hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta)), \end{aligned}$$

and using the condition (3.1),

$$\begin{aligned} & |\partial_\xi^\gamma \partial_\eta^\beta (\sigma(2^j\xi, 2^k\eta) \hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta))| \\ & \leq \sum_{\mu \leq \gamma, \nu \leq \beta} \tilde{c}_{\nu, \mu} 2^{j|\mu|} 2^{k|\nu|} (2^j|\xi| + 2^k|\eta|)^{-|\mu| - |\nu|} |\partial_\xi^{\gamma-\mu} \partial_\eta^{\beta-\nu} (\hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta))| \\ & \leq \sum_{\mu \leq \gamma, \nu \leq \beta} \tilde{c}_{\nu, \mu} 2^{(k-j)|\nu|} |\xi|^{-|\mu| - |\nu|} |\partial_\xi^{\gamma-\mu} \partial_\eta^{\beta-\nu} (\hat{\psi}(\xi) \hat{\psi}(2^{j-k}\xi + \eta))|. \end{aligned}$$

Then  $|F(x, y, z)|$  is bounded since  $|j - k| < 2$  and  $\text{supp}(\psi) \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ .  $\square$

**Corollary 3.2.** Let  $T_\sigma$  be as in Theorem 3.1,  $1 < p, q, r < \infty$ , with  $1/p + 1/q = 1/r$ , and  $\alpha > 0$ , then

$$\|T_\sigma(f, g)\|_{B_r^{\alpha, s}} \leq C \|f\|_{B_p^{\alpha, s}} \|g\|_{L^q}.$$

**Proof.** Since  $\alpha > 0$  implies  $\|f\|_{B_p^{\alpha, s}} = \|f\|_{\dot{B}_p^{\alpha, s}} + \|f\|_{L^p}$ , the result follows from (3.5) and the fact that bilinear Hörmander-Mihlin multipliers obey the inequality

$$\|T_\sigma(f, g)\|_{L^r} \leq C \|f\|_{L^p} \|g\|_{L^q},$$

as proved by R. Coifman and Y. Meyer in [8], K. Yabuta in [35], and later extended to other indices by L. Grafakos and R. Torres in [20].  $\square$

Theorem 3.1 does not require that the symbol  $\sigma$  satisfies condition (3.4). If  $\sigma$  satisfies conditions (3.1)–(3.4), then it easily follows that the symbol of  $T_\sigma^2$ , which is given by  $\sigma(\xi, -(\xi + \eta))$ , satisfies conditions (3.1)–(3.3). By duality we then have the following

**Corollary 3.3.** Consider  $1 \leq p, q, r \leq \infty$ ,  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ , and  $\alpha \in \mathbb{R}^n$ . Let  $\sigma(\xi, \eta)$  be an infinitely differentiable function satisfying (3.1) for all  $|\gamma|, |\beta| \leq n + 1$  and the cancelation conditions (3.2)–(3.4), for all multiindices  $\rho$  satisfying  $|\rho| \leq [|\alpha|] + n + 1$ . Then

$$\|T_\sigma(f, g)\|_{L^r} \leq C \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{\dot{B}_q^{-\alpha, s'}}. \tag{3.8}$$

Estimates similar to (3.8), but for non-homogeneous Besov spaces, have been considered by Á. Bényi in [2] when the bilinear multiplier  $\sigma(\xi, \eta)$  is replaced by a bilinear symbol in the forbidden class  $BS_{1,1}^0$  (however, this class and the Hörmander–Mihlin class are not comparable).

### 4. Molecular paraproducts

For  $v \in \mathbb{Z}$  and  $k \in \mathbb{Z}^n$ , let  $P_{vk}$  be the dyadic cube

$$P_{vk} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : k_i \leq 2^v x_i < k_i + 1, i = 1, \dots, n\}. \tag{4.1}$$

The lower left-corner of  $P = P_{vk}$  is denoted by  $x_P = x_{vk} = 2^{-v}k$ , its size by  $|P| = 2^{-vn}$ , and its characteristic function by  $\chi_{P_{vk}}$ . The collection of all dyadic cubes will be denoted by  $\mathcal{D}$ , i.e.  $\mathcal{D} = \{P_{vk} : v \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ . Following [12, p. 48], a smooth molecule of regularity  $M$  and decay  $N > 0$  associated to  $P$  is a function  $\phi_P = \phi_{P_{vk}} = \phi_{vk} : \mathbb{R}^n \rightarrow \mathbb{C}$  that satisfies

$$|\partial^\gamma \phi_{vk}(x)| \leq \frac{C_{\gamma, N} 2^{vn/2} 2^{|\gamma|v}}{(1 + 2^v|x - 2^{-v}k|)^N}, \quad \text{for all } |\gamma| \leq M \text{ and some } N > n. \tag{4.2}$$

A family of smooth molecules  $\{\phi_P\}_{P \in \mathcal{D}} = \{\phi_{vk}\}_{v \in \mathbb{Z}, k \in \mathbb{Z}^n}$  that satisfies the additional conditions

$$\int \phi_{vk}(x) x^\gamma dx = 0, \quad \text{for all } |\gamma| \leq L, v \in \mathbb{Z}, k \in \mathbb{Z}^n, \tag{4.3}$$

where  $L$  will be specified in particular uses, will be called a family of smooth molecules with cancelation. Let  $\{\phi_Q^1\}, \{\phi_Q^2\}, \{\phi_Q^3\}$  be three families of smooth molecules, the molecular paraproduct (or model paraproduct, [29, p. 23]) associated to these families is defined by

$$T(f, g) = \sum_{Q \in \mathcal{D}} |Q|^{-1/2} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^2 \rangle \phi_Q^3, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{4.4}$$

$T$  has a bilinear kernel given by

$$K(x, y, z) = \sum_{Q \in \mathcal{D}} |Q|^{-1/2} \phi_Q^1(y) \phi_Q^2(z) \phi_Q^3(x). \tag{4.5}$$

A molecular paraproduct has the advantage of involving molecules adapted to dyadic cubes, a more flexible construction than the usual dilations and translations of two fixed profiles  $\psi$  and  $\phi$  defining the Bony paraproduct

$$\Pi(f, g) = \sum_{j \in \mathbb{Z}} (\psi_j * f)(\phi_j * g), \tag{4.6}$$

where  $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$  and  $\psi \in \Psi$  and  $\text{supp}(\hat{\phi}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1/4\}$ . As opposed to the functions  $\phi_j$  and  $\psi_j$  in (4.6), which are  $L^1$ -normalized, the smooth molecules in (4.4) are  $L^2$ -normalized. Nevertheless, the concept of the molecular paraproduct (4.4) includes (modulo smoothing operators) the one of Bony paraproduct. Indeed, given  $\psi$  and  $\phi$  as in (4.4), we reason as follows: consider  $\phi^1 \in \mathcal{S}(\mathbb{R}^n)$  with  $\text{supp}(\hat{\phi}^1) \subset \{\xi \in \mathbb{R}^n : 1/16 \leq |\xi| \leq 9/2\}$  and  $\hat{\phi}^1 \equiv 1$  in  $\{\xi \in \mathbb{R}^n : 1/4 \leq |\xi| \leq 9/4\}$  to obtain

$$\widehat{\phi_j^1}(\xi + \eta) \widehat{\psi_j}(\xi) \widehat{\phi_j}(\eta) = \widehat{\psi_j}(\xi) \widehat{\phi_j}(\eta), \quad j \in \mathbb{Z}, \xi, \eta \in \mathbb{R}^n,$$

and, consequently,

$$\phi_j^1 * ((\psi_j * f)(\phi_j * g)) = (\psi_j * f)(\phi_j * g), \quad j \in \mathbb{Z}. \tag{4.7}$$

Setting  $\phi^2 := \psi$  and  $\phi^3 := \phi$ , and using (4.7), we can write

$$\begin{aligned} \Pi(f, g)(x) &= \sum_{v \in \mathbb{Z}} \phi_v^1 * ((\phi_v^2 * f)(\phi_v^3 * g))(x) \\ &= \iint \left( \sum_{v \in \mathbb{Z}} \int \phi_v^1(x - w) \phi_v^2(w - y) \phi_v^3(w - z) dw \right) f(y)g(z) dy dz \\ &=: \iint K_\Pi(x, y, z) f(y)g(z) dy dz, \end{aligned}$$

whose bilinear kernel can be expanded as

$$\begin{aligned} K_{\Pi}(x, y, z) &= \sum_{v \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \int_{Q_{vk}} 2^{\frac{3vn}{2}} 2^{\frac{vn}{2}} \phi^1(2^v(x-w)) 2^{\frac{vn}{2}} \phi^2(2^v(w-y)) 2^{\frac{vn}{2}} \phi^3(2^v(w-z)) dw \\ &= \sum_{v \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |Q_{vk}|^{-\frac{1}{2}} \frac{1}{|Q_{vk}|} \int_{Q_{vk}} 2^{\frac{vn}{2}} \phi^1(2^v(x-w)) 2^{\frac{vn}{2}} \phi^2(2^v(w-y)) 2^{\frac{vn}{2}} \phi^3(2^v(w-z)) dw \\ &= \sum_{v \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |Q_{vk}|^{-\frac{1}{2}} \phi_Q^1(x) \phi_Q^2(y) \phi_Q^3(z) + E(x, y, z). \end{aligned}$$

Here  $\phi_Q^j(x) = 2^{vn/2} \phi^j(2^v x - k)$ , for  $Q = Q_{vk}$  and  $j = 1, 2, 3$ , are smooth molecules and the error term  $E(x, y, z)$ , depending on the differences  $2^{\frac{vn}{2}} \phi^j(2^v(x-w)) - 2^{\frac{vn}{2}} \phi^j(2^v(x-x_{vk}))$ ,  $j = 1, 2, 3$ , is the kernel of a smoothing operator. Due to the size condition (4.2),  $E(x, y, z)$  is usually disregarded, since during the estimates the averages over  $Q_{vk}$  above can be replaced by the values of the integrand at  $x_{vk}$ .

A detailed study of the mapping properties of the form  $T : X \times Y \rightarrow Z$  for molecular paraproducts  $T$ , where  $X, Y$ , and  $Z$  are related functional spaces including Besov, Triebel–Lizorkin, Hardy, Sobolev, and Lebesgue spaces (but not estimate (4.13) in Theorem 4.4 below), can be found in [5]. For the case of Dini continuous molecules, see [27]. End-point results of the form  $T : \dot{F}_p^{\alpha,q} \times Y \rightarrow \dot{F}_p^{\alpha,q}$ , for certain Triebel–Lizorkin spaces  $Y$  are proven in [33] and [34].

Our Besov–Lebesgue estimates for molecular paraproducts will be based on three known almost-orthogonality estimates, which we included here for the reader’s convenience. Namely,

**Proposition 4.1.** (See Frazier–Jawerth [11, Appendix B].) Suppose that  $\varphi_\nu$  and  $\varphi_\mu$  are functions defined on  $\mathbb{R}^n$  such that for some  $x_\nu, x_\mu$  in  $\mathbb{R}^n$ , some  $N_1 > n + L + 1$  with  $L$  a non-negative integer, and some  $N_2 > n$  the following conditions hold:

$$|\varphi_\nu(x)| \leq \frac{2^{vn/2}}{(1 + 2^v|x - x_\nu|)^{\max(N_1, N_2)}}, \tag{4.8}$$

$$\int_{\mathbb{R}^n} \varphi_\nu(x) x^\gamma dx = 0, \quad \text{for all } |\gamma| \leq L, \tag{4.9}$$

and

$$|\partial_x^\gamma \varphi_\mu(x)| \leq \frac{2^{\mu|\gamma|} 2^{\mu n/2}}{(1 + 2^\mu|x - x_\mu|)^{N_2}}, \quad \text{for all } |\gamma| \leq L + 1. \tag{4.10}$$

Then, for  $\nu \geq \mu$  there exists a constant  $C = C(N_1, N_2, L) > 0$  such that the following estimate is valid

$$\left| \int_{\mathbb{R}^n} \varphi_\nu(x) \varphi_\mu(x) dx \right| \leq C \frac{2^{-(\nu-\mu)(L+1+n/2)}}{(1 + 2^\mu|x_\nu - x_\mu|)^{N_2}}. \tag{4.11}$$

**Lemma 4.2.** (See, for instance, [15, p. A-36].) Let  $a, b \in \mathbb{R}^n$ ,  $\mu, \nu \in \mathbb{R}$ , and  $P, Q > n$ . Then

$$\left| \int_{\mathbb{R}^n} \frac{2^{\mu n}}{(1 + 2^\mu|x - a|)^P} \frac{2^{\nu n}}{(1 + 2^\nu|x - b|)^Q} dx \right| \leq C_{P, Q, n} \frac{2^{\min(\mu, \nu)n}}{(1 + 2^{\min(\mu, \nu)}|a - b|)^{\min(P, Q)}}. \tag{4.12}$$

Finally, for three real numbers,  $a_1, a_2, a_3$ , we denote by  $\text{med}(a_1, a_2, a_3)$  one of the  $a_j$ ’s that satisfies  $\min(a_1, a_2, a_3) \leq a_j \leq \max(a_1, a_2, a_3)$ .

**Proposition 4.3.** (See [5, Proposition 3.6].) For every  $N > n + 1$  there is a constant  $C$ , depending only on  $N$  and  $n$ , such that for any  $w = (\gamma, \nu, \mu, \lambda) \in \mathbb{Z}^4$  and any  $x, y, z \in \mathbb{R}^n$

$$\begin{aligned} &\sum_{l \in \mathbb{Z}^n} \frac{2^{-\gamma n} 2^{\nu n/2} 2^{\mu n/2} 2^{\lambda n/2}}{[(1 + 2^\nu|x - 2^{-\gamma}l|)(1 + 2^\mu|y - 2^{-\gamma}l|)(1 + 2^\lambda|z - 2^{-\gamma}l|)]^{5N}} \\ &\leq \frac{C 2^{-\max(\mu, \nu, \lambda)n/2} 2^{\text{med}(\mu, \nu, \lambda)n/2} 2^{\min(\mu, \nu, \lambda)n/2}}{((1 + 2^{\min(\nu, \mu)}|x - y|)(1 + 2^{\min(\mu, \lambda)}|y - z|)(1 + 2^{\min(\nu, \lambda)}|x - z|))^N}. \end{aligned}$$

We are now in position to state our Besov–Lebesgue estimates for molecular paraproducts.

**Theorem 4.4.** Let  $\{\phi_Q^1\}, \{\phi_Q^2\}, \{\phi_Q^3\}$  be three families of molecules and let  $T$  be its associated molecular paraproduct (4.4). Given  $\alpha \in \mathbb{R}$ , suppose that  $\{\phi_Q^1\}$  and  $\{\phi_Q^3\}$  satisfy (4.3) with some  $L > 2[|\alpha|] - 1$  and (4.2) with  $M = L + 1$  and  $N > 5n + 5$ , and  $\{\phi_Q^2\}$  satisfies (4.2) with  $M = 0$  and  $N > 5n + 5$ . Then, for any  $1 \leq p, q, r, s \leq \infty$  with  $1/p + 1/q = 1/r$ , there exists a constant  $C = C(\alpha, n, p, q, r, s)$  such that

$$\|T(f, g)\|_{\dot{B}_r^{\alpha, s}} \leq C \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{4.13}$$

**Proof.** We first notice that the second transpose of  $T$  is given by

$$T^{*2}(f, g)(x) = \sum_{Q \in \mathcal{D}} |Q|^{-\frac{1}{2}} \langle f, \phi_Q^1 \rangle \langle g, \phi_Q^3 \rangle \phi_Q^2(x)$$

and that, for  $\psi \in \Psi$  verifying (2.3), the functions  $2^{-\frac{nj}{2}} \psi_j(\cdot - x_2)$  and  $2^{-\frac{nk}{2}} \psi_k(x_1 - \cdot)$  satisfy (4.8)–(4.10) for all  $N_1, N_2$ , and  $L$ . Without loss of generality, we can consider only the  $k \leq j$ . Since the case  $k > j$  will follow similarly, as identical conditions are required to the families  $\{\phi_Q^1\}$  and  $\{\phi_Q^3\}$ . Proposition 4.1 yields

$$\begin{aligned} & |T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)| \\ & \leq \sum_{Q \in \mathcal{D}} |Q|^{-\frac{1}{2}} |\langle \psi_j(\cdot - x_2), \phi_Q^1 \rangle| |\langle \psi_k(x_1 - \cdot), \phi_Q^3 \rangle| |\phi_Q^2(x_3)| \\ & \leq \sum_{\substack{Q=Q_{vl} \\ v \in \mathbb{Z}, l \in \mathbb{Z}^n}} \frac{2^{\frac{n}{2}(v+j+k)} 2^{-|j-v|(L+1+\frac{n}{2})} 2^{-|k-v|(L+1+\frac{n}{2})} 2^{\frac{n}{2}v}}{[(1 + 2^{\min(j,v)} |x_2 - 2^{-v}l|)(1 + 2^{\min(k,v)} |x_1 - 2^{-v}l|)(1 + 2^v |x_3 - 2^{-v}l|)]^{N_2}}. \end{aligned}$$

By using  $w = (v, \min(k, v), \min(j, v), v)$  in Proposition 4.3, inequality (4.12), and the fact that  $\min(k, v) \leq \min(j, v) \leq v$ , it follows that

$$\begin{aligned} & |T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)| \\ & \leq C \sum_{v \in \mathbb{Z}} \frac{2^{\frac{n}{2}(j+k+2v)} 2^{-|j-v|(L+1+\frac{n}{2})} 2^{-|k-v|(L+1+\frac{n}{2})}}{[(1 + 2^{\min(k,v)} |x_2 - x_1|)(1 + 2^{\min(k,v)} |x_1 - x_3|)(1 + 2^{\min(j,v)} |x_3 - x_2|)]^{N_2/5}} \\ & \leq C \sum_{v \in \mathbb{Z}} \frac{2^{\frac{n}{2}(j+k+2v)} 2^{-|j-v|(L+1+\frac{n}{2})} 2^{-|k-v|(L+1+\frac{n}{2})}}{[(1 + 2^{\min(k,v)} |x_2 - x_1|)(1 + 2^{\min(j,v)} |x_3 - x_2|)]^{N_2/5}}. \end{aligned}$$

Finally, let  $I(j, k)$  denote the integral of  $|T^{*2}(\psi_j(\cdot - x_2), \psi_k(x_1 - \cdot))(x_3)|$  with respect to any two of the variables  $x_1, x_2, x_3$  to obtain

$$\begin{aligned} I(j, k) & \leq C \sum_{v \in \mathbb{Z}} 2^{\frac{n}{2}(j+k+2v)} 2^{-|j-v|(L+1+\frac{n}{2})} 2^{-|k-v|(L+1+\frac{n}{2})} 2^{-n \min(k, v)} 2^{-n \min(j, v)} \\ & = C \sum_{v \in \mathbb{Z}} 2^{-|j-v|(L+1)} 2^{-|k-v|(L+1)} 2^{\tau(k, j, v)}, \end{aligned}$$

where the power  $\tau(k, j, v)$  is given by

$$\tau(k, j, v) = \frac{n}{2}(k + j + 2v) - \frac{n}{2}|j - v| - \frac{n}{2}|k - v| - n \min(k, v) - n \min(j, v),$$

and, in fact, a brief computation shows that  $\tau(k, j, v) \equiv 0$  for all  $v, j, k \in \mathbb{Z}$ . Consequently,

$$I(j, k) \leq C \sum_{v \in \mathbb{Z}} 2^{-|j-v|(L+1)} 2^{-|k-v|(L+1)} \leq C 2^{-\frac{1}{2}|j-k|(L+1)},$$

and the theorem follows from Lemma 2.1.  $\square$

**Remark 2.** For  $\alpha > 0$  and  $1 < p, q, r < \infty$  with  $1/p + 1/q = 1/r$ , the non-homogeneous version of (4.13) follows as in Corollary 3.2, since molecular paraproducts involving two families of smooth molecules with cancellation then verify

$$\|T(f, g)\|_{L^r} \leq C \|f\|_{L^p} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

see, for instance, [5,13,14,29].

### 5. Bilinear Littlewood–Paley theory

Given a function  $\psi$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  such that  $\int_{\mathbb{R}^n} \psi(x) dx = 0$ , an immediate application of the Fourier transform gives the bound

$$\int_0^\infty \|\Psi_t(f)\|_{L^2}^2 \frac{dt}{t} \leq C \|f\|_{L^2}^2, \tag{5.1}$$

where  $\Psi_t(f)(x) = \int_{\mathbb{R}^n} \psi_t(x-y) f(y) dy$  and  $\psi_t(x) = t^{-n} \psi(x/t)$ . In [28], S. Semmes identified sufficient conditions on a family of functions  $\theta_t(x, y)$ ,  $t > 0$ ,  $x, y \in \mathbb{R}^n$  (more general than  $\psi_t(x-y)$ ) so that the non-convolution operator

$$\Theta_t(f)(x) = \int_{\mathbb{R}^n} \theta_t(x, y) f(y) dy$$

verifies the square function estimate in  $L^2(\mathbb{R}^n)$

$$\int_0^\infty \|\Theta_t(f)\|_{L^2}^2 \frac{dt}{t} \leq C \|f\|_{L^2}^2. \tag{5.2}$$

In the discrete case, when a family  $\theta_k(x, y)$ ,  $k \in \mathbb{Z}$ , is considered, inequality (5.2) then becomes

$$\left( \sum_{k \in \mathbb{Z}} \|\theta_k(f)\|_{L^2}^2 \right)^{1/2} \leq C \|f\|_{L^2}. \tag{5.3}$$

The alluded sufficient conditions have to do with decay, regularity, and cancelation properties of the kernels  $\theta_t(x, y)$  (or  $\theta_k(x, y)$ ). In the following we will assume that  $\{\theta_k\}_{k \in \mathbb{Z}}$  is a family of complex-valued functions defined on  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$  satisfying the following conditions: There are  $L, M, N \in \mathbb{N}$  and constants  $c_\alpha$ , and  $A$ , such that for all  $k \in \mathbb{Z}$ ,

$$|\theta_k(x, y, z)| \leq \frac{A 2^{2nk}}{(1 + 2^k|x-z|)^N (1 + 2^k|x-y|)^N}, \tag{5.4}$$

$$|\partial_y^\alpha \theta_k(x, y, z)| \leq c_\alpha 2^{2nk} 2^{k|\alpha|}, \quad x, y, z \in \mathbb{R}^n, \quad |\alpha| \leq M + 1, \tag{5.5}$$

$$\int_{\mathbb{R}^n} \theta_k(x, y, z) y^\alpha dy = 0, \quad x, z \in \mathbb{R}^n, \quad |\alpha| \leq L. \tag{5.6}$$

Notice that, as opposed to the condition (4.2), condition (5.5) above does not involve any decay in the variables  $x, y$ , or  $z$ .

**Theorem 5.1.** *Let  $\alpha \in \mathbb{R}$  and suppose that the kernels  $\{\theta_k\}$  of the bilinear operators*

$$\Theta_k(f, g)(x) = \iint \theta_k(x, y, z) f(y) g(z) dy dz, \tag{5.7}$$

verify (5.4)–(5.6) with constants  $L, M$ , and  $N$  such that

$$2|\alpha| < \min(M + 1, L + 1), \quad M + n + 1 < N \quad \text{and} \quad L + n + 1 < N, \quad 2n < N.$$

Then, there is a constant  $C$  depending only on  $L, M, A, N, n, s$ , and  $\alpha$ , such that for all  $1 \leq p, q, r, s \leq \infty$  with  $1/p + 1/q = 1/r$ ,

$$\left( \sum_{k \in \mathbb{Z}} 2^{\alpha ks} \|\Theta_k(f, g)\|_{L^r}^s \right)^{1/s} \leq C \|f\|_{\dot{B}_p^{\alpha, s}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{5.8}$$

The essential steps in the proof of Theorem 5.1 can be taken to also prove

**Corollary 5.2.** *Let  $\Theta_k$  be defined as in (5.7) such that the kernels  $\theta_k$  satisfy (5.4) for some  $N > 2n$ , (5.6) with  $L = 0$ , and the following Hölder regularity condition in the  $y$ -variable*

$$|\theta_k(x, y, z) - \theta_k(x, y', z)| \leq c_\gamma 2^{2nk} (2^k|y - y'|)^\gamma, \tag{5.9}$$

for some  $\gamma \in (0, 1]$  and all  $x, y, y', z \in \mathbb{R}^n$  and  $k \in \mathbb{Z}$ . Then, there is a constant  $C$  depending only on  $s, \gamma$ , and  $n$ , such that for all  $1 \leq p, q, r, s \leq \infty$  with  $1/p + 1/q = 1/r$ ,

$$\left( \sum_{k \in \mathbb{Z}} \|\Theta_k(f, g)\|_{L^r}^s \right)^{1/s} \leq C \|f\|_{\dot{B}_p^{0,s}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n). \tag{5.10}$$

In particular, the case  $s = p = q = 2$  yields

$$\left( \sum_{k \in \mathbb{Z}} \|\Theta_k(f, g)\|_{L^1}^2 \right)^{1/2} \leq C \|f\|_{L^2} \|g\|_{L^2}, \quad f, g \in \mathcal{S}(\mathbb{R}^n), \tag{5.11}$$

which is the natural bilinear version of (5.3).

**Remark 3.** Notice that cancelation in the  $y$ -variable *only* is assumed in Theorem 5.1 and Corollary 5.2. Corollary 5.2 has been proved in the context of spaces of homogeneous type in [26].

**Lemma 5.3.** Let  $l > 0$  and  $\{\theta_k\}$  satisfy (5.5)–(5.6) with constants  $L, M$ , and  $N$  satisfying

$$l \leq \min(M + 1, L + 1, N - n), \quad M + n + 1 < N \quad \text{and} \quad L + n + 1 < N.$$

Then, for all  $x, u, z \in \mathbb{R}^n, j, k \in \mathbb{Z}$ , and  $\psi \in \Psi$ ,

$$\left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy \right| \leq C 2^{-\frac{1}{2}|j-k|} \frac{2^{kn}}{(1 + 2^k|x - z|)^{\frac{N}{2}}} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)}|x - u|)^{\frac{N}{2}}},$$

where  $C$  is a constant depending on  $L, M, A, N, n$ , and  $\psi$ .

**Proof.** It is enough to prove the following two inequalities. For all  $x, z, u \in \mathbb{R}^n, j, k \in \mathbb{Z}$ , we have

$$\left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy \right| \leq C \frac{2^{kn}}{(1 + 2^k|x - z|)^N} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)}|x - u|)^N} \tag{5.12}$$

and

$$\left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy \right| \leq C 2^{-l|j-k|} 2^{kn} 2^{\min(j,k)n}. \tag{5.13}$$

*Proof of (5.12):* Using condition (5.4), the properties of  $\psi$ , and inequality (4.12), we estimate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy \right| &\leq \frac{AC_\psi 2^{nk}}{(1 + 2^k|x - z|)^N} \int_{\mathbb{R}^n} \frac{2^{nk}}{(1 + 2^k|x - y|)^N} \frac{2^{jn}}{(1 + 2^j|y - u|)^N} dy \\ &\leq AC_\psi C_{N,n} \frac{2^{kn}}{(1 + 2^k|x - z|)^N} \frac{2^{\min(j,k)n}}{(1 + 2^{\min(j,k)}|x - u|)^N}. \end{aligned}$$

*Proof of (5.13):* Case  $j > k$ . Using the fact that

$$\int_{\mathbb{R}^n} \psi_j(y - u) (y - u)^\alpha dy = 0, \quad \alpha \in \mathbb{N}_0,$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy &= \int_{\mathbb{R}^n} \left[ \theta_k(x, y, z) - \sum_{|\alpha| \leq M} \frac{1}{\alpha!} \partial_y^\alpha \theta_k(x, u, z) (y - u)^\alpha \right] \psi_j(y - u) dy \\ &= \int_{\mathbb{R}^n} \sum_{|\alpha| = M+1} \frac{1}{\alpha!} \partial_y^\alpha \theta_k(x, \xi, z) (y - u)^\alpha \psi_j(y - u) dy, \end{aligned}$$

where  $\xi$  is in the segment joining  $y$  and  $u$ . By (5.5) and the properties of  $\psi$  we get

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y - u) dy \right| &\leq C_{M,\psi} 2^{2nk} 2^{k(M+1)} \int_{\mathbb{R}^n} |y - u|^{M+1} \frac{2^{jn}}{(1 + 2^j|y - u|)^N} dy \\ &= \int_{|y-u| > 2^{-k}} + \int_{|y-u| \leq 2^{-k}} = I + II. \end{aligned}$$

Estimate for I. We have  $|y - u| > 2^{-k} > 2^{-j}$ . Then  $\frac{1}{(1+2^j|y-u|)^N} \sim \frac{2^{-jN}}{|y-u|^N}$  and therefore

$$I \leq \tilde{C}_{M,\psi} 2^{2nk} 2^{k(M+1)} 2^{j(n-N)} \int_{|y-u|>2^{-k}} |y-u|^{M+1-N} dy.$$

Recalling that  $M + n + 1 < N$ ,  $k < j$  and  $l \leq N - n$ , we then have

$$I \leq C_{M,N,n,\psi} 2^{2nk} 2^{-|k-j|(N-n)} \leq C_{L,N,n} 2^{2nk} 2^{\min(j,k)n} 2^{-|k-j|l}.$$

Estimate for II. We have

$$\begin{aligned} II &= C_{M,\psi} 2^{2nk} 2^{k(M+1)} \int_{|y-u| \leq 2^{-k}} |y-u|^{M+1} \frac{2^{jn}}{(1+2^j|y-u|)^N} dy \\ &= \int_{|y-u| \leq 2^{-j}} + \int_{2^{-j} < |y-u| \leq 2^{-k}} = II_1 + II_2. \end{aligned}$$

For  $II_1$  we use that  $\frac{1}{1+2^j|y-u|} \leq 1$  and we get

$$II_1 \leq C_{M,\psi,n} 2^{2nk} 2^{-|k-j|(M+1)} \leq C_{M,\psi,n} 2^{kn} 2^{\min(j,k)n} 2^{-|k-j|l},$$

where in the last inequality we have used that  $l \leq M + 1$ .

For  $II_2$  we use that  $\frac{1}{(1+2^j|y-u|)^N} \sim \frac{1}{(2^j|y-u|)^N}$ , and after integrating in polar coordinates and recalling that  $M + n - N + 1 < 0$ ,  $k < j$ , and  $l \leq M + 1$ , we get

$$\begin{aligned} II_2 &\leq C_{M,\psi} 2^{2nk} 2^{k(M+1)} 2^{j(n-N)} \int_{2^{-j} < |y-u| \leq 2^{-k}} |y-u|^{M+1-N} \\ &\leq C_{M,N,n,\psi} 2^{2nk} 2^{-|j-k|(M+1)} \leq C_{M,N,n,\psi} 2^{2nk} 2^{\min(j,k)n} 2^{-|j-k|l}. \end{aligned}$$

Case  $j \leq k$ . Using the cancelation property (5.6) for  $\theta_k$ ,

$$\begin{aligned} \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y-u) dy &= \int_{\mathbb{R}^n} \theta_k(x, y, z) \left[ \psi_j(y-u) - \sum_{|\alpha| \leq L} \frac{1}{\alpha!} \partial^\alpha \psi_j(x-u)(y-x)^\alpha \right] dy \\ &= \int_{\mathbb{R}^n} \theta_k(x, y, z) \sum_{|\alpha|=L+1} \frac{1}{\alpha!} \partial^\alpha \psi_j(\xi-u)(y-x)^\alpha dy, \end{aligned}$$

where  $\xi$  is in the segment joining  $x$  and  $y$ . By condition (5.4) we then get

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \theta_k(x, y, z) \psi_j(y-u) dy \right| &\leq AC_{\psi,L} 2^{2nk} 2^{j(n+L+1)} \int_{\mathbb{R}^n} \frac{|y-x|^{L+1}}{(1+2^k|x-y|)^N} dy \\ &= \int_{|x-y|>2^{-j}} + \int_{|x-y| \leq 2^{-j}}. \end{aligned}$$

We now proceed as before obtaining

$$I \leq AC_{\psi,L,N} 2^{kn} 2^{jn} 2^{-|k-j|(N-n)} \leq AC_{\psi,L,N} 2^{kn} 2^{jn} 2^{-|k-j|l},$$

where we have used that  $L + n + 1 \leq N$  and  $l \leq N - n$ , and

$$II \leq AC_{\psi,L,N} 2^{kn} 2^{jn} 2^{-|k-j|(L+1)} \leq AC_{\psi,L,N} 2^{kn} 2^{jn} 2^{-|k-j|l},$$

where we have used that  $L + n + 1 \leq N$  and  $l \leq L + 1$ .  $\square$

**Proof of Theorem 5.1.** As in the proof of Lemma 2.1, let  $K_{jk}(x_1, x_2, x_3)$  be the bilinear kernel of the operator  $(f, g) \mapsto \Theta_k(\psi_j * f, g)$ . That is,

$$K_{jk}(x_1, x_2, x_3) = \int_{\mathbb{R}^n} \theta_k(x_1, y, x_3) \psi_j(y-x_2) dy. \tag{5.14}$$

By Lemma 5.3, for all  $j, k \in \mathbb{Z}$  and  $h = 1, 2, 3$ , we have, for  $2|\alpha| < l \leq \min(M + 1, L + 1)$ ,

$$\sup_{x_h \in \mathbb{R}^n} \iint |K_{jk}(x_1, x_2, x_3)| \prod_{\substack{m=1 \\ m \neq h}}^3 dx_m \leq C 2^{-l|j-k|/2}. \tag{5.15}$$

This yields, as in the proof of Lemma 2.1,

$$\|\Theta_k(\Delta_j f, g)\|_{L^r} \leq C 2^{-l|j-k|/2} \|f\|_{L^p} \|g\|_{L^q},$$

and, since  $l/2 > |\alpha|$ , the duality argument in the proof of Lemma 2.1 completes the proof.  $\square$

Our inequalities (5.8) and (5.10) come as an addition to the related known results on bilinear Littlewood–Paley theory. Namely, the inequality

$$\left\| \left( \sum_{k \in \mathbb{Z}} |S_k(f, g)|^2 \right)^{1/2} \right\|_{L^r} \leq C \|\gamma\|_{\ell^\infty} \|f\|_{L^p} \|g\|_{L^q}, \tag{5.16}$$

obtained by G. Diestel in [10] for  $1 < p, q, r < \infty$ ,  $1/r = 1/p + 1/q$ , where the rough paraproduct operator  $S_k$  is defined by

$$S_k(f, g) = \gamma_k \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi) \chi_{[a^k, a^{k-1}]}(\xi) \hat{g}(\eta) \chi_{[-b^k, b^k]}(\eta) e^{2\pi i(\xi+\eta)x} d\xi d\eta,$$

for  $a, b \in (0, 1)$ . And also with the square-function inequality

$$\left\| \left( \sum_{k \in \mathbb{Z}^n} |S_k(f, g)|^2 \right)^{1/2} \right\|_{L^2} \leq C \|f\|_{L^p} \|g\|_{L^q}, \tag{5.17}$$

for all  $2 \leq p, q \leq \infty$ , with  $1/p + 1/q = 1/2$ , proved in the context of Gabor analysis by M. Lacey in [23], with

$$S_k(f, g)(x) = \int_{\mathbb{R}^n} f(x + y) g(x - y) F_k(y) dy, \quad k \in \mathbb{Z}^n,$$

where the smooth function  $F$  has Fourier transform supported on the unit cube of  $\mathbb{R}^n$  and, for  $k \in \mathbb{Z}^n$ ,  $\widehat{F}_k(\xi) := \widehat{F}(\xi - k)$ . For bilinear operators  $\Theta_k$  of the form (5.7), Theorem 5.1 immediately implies

**Corollary 5.4.** *Given  $\alpha \in \mathbb{R}$ , let  $\Theta_k$  be as in Theorem 5.1. Then, for the constant  $C$  as in Theorem 5.1, we have*

$$\left\| \left( \sum_{k \in \mathbb{Z}} 2^{2\alpha k} |\Theta_k(f, g)|^2 \right)^{1/2} \right\|_{L^2} \leq C \|f\|_{\dot{B}_p^{\alpha, 2}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

for  $2 \leq p, q \leq \infty$  with  $1/p + 1/q = 1/2$  and

$$\left\| \left( \sum_{k \in \mathbb{Z}} 2^{2\alpha k} |\Theta_k(f, g)|^2 \right)^{1/2} \right\|_{L^r} \leq C \|f\|_{\dot{B}_p^{\alpha, 1}} \|g\|_{L^q}, \quad f, g \in \mathcal{S}(\mathbb{R}^n),$$

for  $1 \leq p, q, r \leq \infty$  with  $1/p + 1/q = 1/r$ .

**Remark 4.** We point out that the techniques used in this section provide new results even in the linear case. Indeed, by considering a family  $\theta_k(x, y)$ ,  $k \in \mathbb{Z}$ , that satisfies

$$|\theta_k(x, y)| \leq \frac{A 2^{nk}}{(1 + 2^k|x - y|)^N}, \tag{5.18}$$

$$|\partial_y^\alpha \theta_k(x, y)| \leq c_\alpha 2^{nk} 2^{k|\alpha|}, \quad x, y \in \mathbb{R}^n, \quad |\alpha| \leq M + 1, \tag{5.19}$$

$$\int_{\mathbb{R}^n} \theta_k(x, y) y^\alpha dy = 0, \quad x \in \mathbb{R}^n, \quad |\alpha| \leq L \tag{5.20}$$

(for suitable  $L, M$ , and  $N$ ), a bound of the form

$$\left( \sum_{k \in \mathbb{Z}} 2^{\alpha ks} \|\Theta_k(f)\|_{L^p}^s \right)^{1/s} \leq C \|f\|_{\dot{B}_p^{\alpha, s}}, \quad f \in \mathcal{S}(\mathbb{R}^n), \tag{5.21}$$

follows. Thus extending Semmes’s inequality (5.2) to the scale of homogeneous Besov spaces  $\dot{B}_p^{\alpha, s}$  with  $\alpha \in \mathbb{R}$  and  $1 \leq p, s \leq \infty$ .

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