

# WEIGHTED MULTILINEAR POINCARÉ INEQUALITIES FOR VECTOR FIELDS OF HÖRMANDER TYPE

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ABSTRACT. As the classical  $(p, q)$ -Poincaré inequality is known to fail for  $0 < p < 1$ , we introduce the notion of weighted multilinear Poincaré inequality as a natural alternative when  $m$ -fold products and  $1/m < p$  are considered. We prove such weighted multilinear Poincaré inequalities in the subelliptic context associated to vector fields of Hörmander type. We do so by establishing multilinear representation formulas and weighted estimates for multilinear potential operators in spaces of homogeneous type.

## 1. INTRODUCTION AND MAIN RESULT

The classical Poincaré inequality

$$(1.1) \quad \left( \int_B |u(x) - u_B|^q dx \right)^{1/q} \leq C \left( \int_B |\nabla u(x)|^p dx \right)^{1/p}, \quad u \in C^1(\overline{B}),$$

where  $B$  is an Euclidean ball in  $\mathbf{R}^n$  and  $u_B = \frac{1}{|B|} \int_B u(x) dx$ , holds when  $1 \leq p < n$  and  $q = \frac{np}{n-p}$ . However, simple examples prove that this inequality is false for every  $0 < p < 1$ , see for instance Buckley-Koskela [3, p.224], where it is shown that even the following weaker version of (1.1) fails for  $0 < p < 1$  and any Euclidean ball  $B \subset \mathbf{R}^n$ ,

$$(1.2) \quad \inf_{a \in \mathbf{R}} \left( \int_B |u(x) - a|^q dx \right)^{1/q} \leq C \left( \int_B |\nabla u(x)|^p dx \right)^{1/p}.$$

We mention in passing that (1.1) does hold for some  $0 < p < 1$  if  $u$  satisfies extra conditions such as being a solution to a suitable elliptic PDE (Hajlasz-Koskela [18], Chapter 13) or having  $|\nabla u|$  bounded by a weight with a weak reverse Hölder inequality (Buckley-Koskela [3]).

We now focus on the case  $u = fg$  with  $f, g \in C^1(\overline{B})$ . By the previous comment, the following Poincaré inequality for the product of two functions

$$(1.3) \quad \inf_{a \in \mathbf{R}} \left( \int_B |(fg)(x) - a|^q dx \right)^{1/q} \leq C \left( \int_B |\nabla(fg)(x)|^p dx \right)^{1/p},$$

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( $C$  independent of  $f$  and  $g$ ), also fails for every  $0 < p < 1$  and every Euclidean ball  $B$ . On the other hand, note that for any numbers  $0 < p, r, s, \tilde{r}, \tilde{s} < \infty$  with

$$(1.4) \quad \frac{1}{p} = \frac{1}{r} + \frac{1}{s} = \frac{1}{\tilde{r}} + \frac{1}{\tilde{s}},$$

the inequality

$$(1.5) \quad \left( \int_B |\nabla(fg)|^p \right)^{1/p} \lesssim \left( \int_B |\nabla f|^r dx \right)^{1/r} \left( \int_B |g|^s dx \right)^{1/s} \\ + \left( \int_B |f|^{\tilde{r}} dx \right)^{1/\tilde{r}} \left( \int_B |\nabla g|^{\tilde{s}} dx \right)^{1/\tilde{s}},$$

with constant depending only on  $p$ , holds as a consequence of Hölder's inequality. Hence, a natural alternative to (1.3) for arbitrary functions  $f$  and  $g$  and  $0 < p < 1$  is given by the inequality

$$(1.6) \quad \inf_{a \in \mathbf{R}} \left( \int_B |(fg)(x) - a|^q dx \right)^{1/q} \\ \leq C \left( \left( \int_B |f|^r dx \right)^{1/r} \left( \int_B |\nabla g|^s dx \right)^{1/s} + \left( \int_B |\nabla f|^{\tilde{r}} dx \right)^{1/\tilde{r}} \left( \int_B |g|^{\tilde{s}} dx \right)^{1/\tilde{s}} \right),$$

or the following stronger inequality, which we will call *bilinear Poincaré inequality*

$$(1.7) \quad \left( \int_B |(fg)(x) - f_B g_B|^q dx \right)^{1/q} \\ \leq C \left( \left( \int_B |f|^r dx \right)^{1/r} \left( \int_B |\nabla g|^s dx \right)^{1/s} + \left( \int_B |\nabla f|^{\tilde{r}} dx \right)^{1/\tilde{r}} \left( \int_B |g|^{\tilde{s}} dx \right)^{1/\tilde{s}} \right),$$

where  $p, r, s, \tilde{s}, \tilde{r}$  are related through (1.5) and  $C$  is independent of  $f, g$ , and  $B$ .

The purpose of this article is to derive weighted inequalities of the type (1.7) where  $p$  is allowed to be bigger than  $1/2$  and, more generally,  $p > 1/m$  when  $m$  factor functions are involved. Moreover, we do so in the subelliptic setting associated to vector fields satisfying Hörmander's condition. Since D. Jerison's fundamental work [19] on subelliptic Poincaré inequalities, the research on Poincaré-type inequalities in stratified groups and more general Carnot-Carathéodory structures has continued to gain substantial momentum, see, for instance, [4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 25, 26, 27, 29, 37, 40] and references there in. In particular, Buckley-Koskela-Lu [4] have established the validity of weighted versions of (1.1) for  $0 < p < 1$  in the Carnot-Carathéodory setting under the assumption that the subelliptic gradient of  $u$  satisfies a weak reverse Hölder condition. Along these lines, our main result is motivated by the exploration of inequalities such as (1.7) and the search for a substitute to (1.1) in the case  $0 < p < 1$ , also in the general Carnot-Carathéodory setting, when

the function  $u$  in question is an  $m$ -fold product of differentiable functions, with no extra assumptions. Namely, we prove

**Theorem 1.** *Let  $m \in \mathbf{N}$ . Suppose that  $1 < p_1, \dots, p_m, p, q < \infty$  satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}, \quad \frac{1}{m} < p \leq q < \infty,$$

and  $\Omega \subset \mathbf{R}^n$  is a connected bounded open set. Let  $Y = \{Y_k\}_{k=1}^M$  be a collection of vector fields on  $\Omega$  verifying Hörmander's condition and denote by  $\rho$  the associated Carnot-Carathéodory metric. Let  $u, v_k, k = 1, \dots, m$ , be weights defined on an open set  $\Omega_0 \subset\subset \Omega$  and satisfying condition (1.8) if  $q > 1$  or condition (1.9) if  $q \leq 1$ , where

$$(1.8) \quad \sup_{B=B_\rho(x,r), x \in \overline{\Omega}_0} \text{diam}_\rho(B) |B|^{1/q-1/p} \left( \frac{1}{|B|} \int_B u^{qt} dx \right)^{1/qt} \prod_{j=1}^m \left( \frac{1}{|B|} \int_B v_j^{-tp'_j} dx \right)^{1/tp'_j} < \infty,$$

for some  $t > 1$ ,

$$(1.9) \quad \sup_{B=B_\rho(x,r), x \in \overline{\Omega}_0} \text{diam}_\rho(B) |B|^{1/q-1/p} \left( \frac{1}{|B|} \int_B u^q dx \right)^{1/q} \prod_{j=1}^m \left( \frac{1}{|B|} \int_B v_j^{-tp'_j} dx \right)^{1/tp'_j} < \infty,$$

for some  $t > 1$ , where  $|B|$  denotes the Lebesgue measure of the  $\rho$ -ball  $B$ . Then, there exist positive constants  $r_0$  and  $C$  such that for all  $\rho$ -ball  $B \subset \overline{\Omega}_0$  with radius less than  $r_0$  and for all  $f_k \in C^1(\overline{B})$ ,  $k = 1, \dots, m$ , the following weighted  $m$ -linear subelliptic Poincaré inequality holds true

$$(1.10) \quad \left( \int_B \left( \left| \prod_{k=1}^m f_k - \prod_{k=1}^m f_{kB} \right| u \right)^q dx \right)^{1/q} \leq C \sum_{k=1}^m \left( \int_B (|Y f_k| v_k)^{p_k} dx \right)^{1/p_k} \prod_{i \neq k} \left( \int_B (|f_i| v_i)^{p_i} dx \right)^{1/p_i},$$

where

$$f_{kB} = \frac{1}{|B|} \int_B f_k(x) dx, \quad k = 1, \dots, m.$$

*Remark 1.* We point out that Theorem 1, as well as notion of weighted multilinear Poincaré inequality (1.10), are new even in the Euclidean setting. When  $m = 2$ , Theorem 1 provides a substitute to (1.3) for  $p > 1/2$ , and, in general, for  $p$  as close to 0 as desired, as long as  $m$  factor functions, with  $m > 1/p$ , are considered.

*Remark 2.* It is clear that, when  $p < 1$ , inequality (1.7) cannot follow from an application of the linear Poincaré inequality (1.1). In addition, (1.7) does not seem to follow (at least in a straightforward way) from an application of the linear Poincaré inequality even in cases when  $p > 1$ . Indeed, to illustrate why the linear approach

breaks down, in the Euclidean setting consider the particular choices  $n = 2$ ,  $r = \tilde{r} = 2$ ,  $s = \tilde{s} = 4$ , and  $p = 4/3$ , which yields  $q = 4$ . If we write

$$(1.11) \quad |f(x)g(x) - f_B g_B|^4 \lesssim |f(x)|^4 |g(x) - g_B|^4 + |g_B|^4 |f(x) - f_B|^4$$

and use, for instance, Hölder's inequality with any auxiliary index  $l \geq 1$  in the first summand to get

$$\left( \int_B |f(x)|^4 |g(x) - g_B|^4 dx \right)^{1/4} \leq \left( \int_B |f(x)|^{4l} dx \right)^{1/4l} \left( \int_B |g(x) - g_B|^{4l} dx \right)^{1/4l}$$

we realize that it is impossible to utilize a linear Poincaré inequality of the type

$$(1.12) \quad \left( \int_B |g(x) - g_B|^{4l} dx \right)^{1/4l} \leq C \left( \int_B |\nabla g|^s dx \right)^{1/s},$$

since

$$\frac{1}{s} - \frac{1}{n} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \neq \frac{1}{4l}$$

for any  $l \geq 1$ . We then notice that any attempt to use a linear Poincaré inequality with these exponents  $s$  and  $r$  will be unsuccessful, since in this example we have  $1/s - 1/n < 0$  and  $1/r - 1/n = 0$ . As opposed to separately considering the fractions  $1/r$  and  $1/s$ , the bilinear approach is based on the sum  $1/r + 1/s$ , which verifies

$$\frac{1}{r} + \frac{1}{s} - \frac{1}{n} = \frac{1}{4} = \frac{1}{q}.$$

Also, if we try a different way and write

$$|f(x)g(x) - f_B g_B|^4 \lesssim |f(x)g(x) - (fg)_B|^4 + |(fg)_B - f_B g_B|^4,$$

then the linear Poincaré inequality allows to control the first summand by

$$\left( \int_B |f(x)g(x) - (fg)_B|^4 dx \right)^{1/4} \leq C \left( \int_B |\nabla(fg)(x)|^{4/3} dx \right)^{3/4},$$

which, in turn, can be bounded as in (1.5). However, given any  $l \geq 1$ , for the constant term  $|(fg)_B - f_B g_B|$  we have,

$$\begin{aligned} |(fg)_B - f_B g_B| &= \frac{1}{|B|} \left| \int_B f(x)(g(x) - g_B) dx \right| \leq \frac{1}{|B|} \int_B |f(x)| |g(x) - g_B| dx \\ &\leq \left( \frac{1}{|B|} \int_B |f(x)|^{l'} dx \right)^{1/l'} \left( \frac{1}{|B|} \int_B |g(x) - g_B|^l dx \right)^{1/l}, \end{aligned}$$

so that

$$\begin{aligned} \int_B |(fg)_B - f_B g_B|^4 dx &\leq |B| \left( \frac{1}{|B|} \int_B |f(x)|^{4l'} dx \right)^{1/l'} \left( \frac{1}{|B|} \int_B |g(x) - g_B|^{4l} dx \right)^{1/l} \\ &= \left( \int_B |f(x)|^{4l'} dx \right)^{1/l'} \left( \int_B |g(x) - g_B|^{4l} dx \right)^{1/l}, \end{aligned}$$

where we used Jensen's inequality to avoid loose powers of  $|B|$ . We now see that if we intend to bound the last term by means of the linear Poincaré inequality, we run into the same problem as in (1.12) since  $1/s - 1/n < 0 \neq 1/4l$ .

In a sense, controlling the oscillation  $|f(x)g(x) - f_B g_B|$  (rather than the oscillation  $|f(x)g(x) - (fg)_B|$ ) requires bilinear methods, even for some  $p$  larger than 1.

*Remark 3.* In the linear case ( $m = 1$ ), representation formulas and Poincaré inequalities imply embedding theorems on Campanato-Morrey spaces, see, for instance, Lu [23, 24] for such embeddings in the Carnot-Carathéodory context. In order to illustrate the multilinear analogs of these embeddings associated to Theorem 1, let us focus on the Euclidean setting and the bilinear case  $m = 2$ . Let  $w \geq 0$  be a weight and for  $p, \lambda > 0$  and  $f \in L^1_{loc}(\mathbf{R}^n, w^p)$ ,  $f$  is said to belong to the weighted Morrey space  $L^{p,\lambda}(w)$  if

$$\|f\|_{L^{p,\lambda}(w)} = \sup_B \left( \frac{1}{|B|^{\lambda/n}} \int_B |f(x)w(x)|^p dx \right)^{1/p} < \infty,$$

and  $f$  is said to belong to the weighted Campanato space  $\mathcal{L}^{p,\lambda}(w)$  if

$$\|f\|_{\mathcal{L}^{p,\lambda}(w)} = \sup_B \inf_{a \in \mathbf{C}} \left( \frac{1}{|B|^{\lambda/n}} \int_B (|f(x) - a|w(x))^p dx \right)^{1/p} < \infty.$$

Then, Theorem 1 (with  $m = 2$  and in the Euclidean setting) implies a variety of weighted inequalities of the form

$$(1.13) \quad \|fg\|_{\mathcal{L}^{p,\lambda}(w)} \lesssim \|\nabla f\|_{L^{p_1,\lambda_1}(u)} \|g\|_{L^{p_2,\lambda_2}(v)} + \|f\|_{L^{p_1,\lambda_1}(u)} \|\nabla g\|_{L^{p_2,\lambda_2}(v)},$$

for a larger class of weights  $u, v, w$  (and, therefore, a larger range of indices  $p, \lambda, p_1, \lambda_1, p_2,$  and  $\lambda_2$ ) than one could possibly obtain by iteration of the linear weighted estimates and Hölder's inequality. See remark 5.

Inequalities of the form (1.13) are related to the so-called Kato-Ponce inequality, where the  $L^p$ -norm of the derivative of the product is being replaced by another measure of the oscillation (i.e., the Campanato norm) of the product, and the Morrey spaces play the role of the Lebesgue spaces.

Regarding the organization of the article, we prove Theorem 1 in § 5 after conveniently adapting the usual approach to the classical Poincaré inequality (1.1). That is, by proving a multilinear analog to the representation formula

$$|f(x) - f_B| \lesssim I_{B,1}(|\nabla f|)(x), \quad x \in B,$$

where  $I_{B,1}(h)(x) = \int_B h(y) |x - y|^{1-n} dy$  (see Corollary 4 in § 5). Then, in § 2 we use the framework of spaces of homogeneous type to introduce a class of multilinear potential operators that includes the multilinear counterpart to  $I_{B,1}$  and we establish their weighted Lebesgue estimates in § 3. These weighted estimates are further conveyed into the context of Orlicz spaces in § 4, producing natural multilinear alternatives to their linear counterparts and allowing for a strictly wider range of indices, see Theorem 3 and Remark 7.

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## 2. MULTILINEAR POTENTIAL OPERATORS IN SPACES OF HOMOGENEOUS TYPE

We introduce in this section the theory of multilinear potential operators in the ample context of spaces of homogeneous type and state their weighted boundedness properties.

Recall that a space of homogeneous type (in the sense of Coifman-Weiss [9]) is a triple  $(X, \rho, \mu)$ , where  $X$  is a nonempty set,  $\rho$  is a quasi-metric defined on  $X$ , that satisfies

$$(2.1) \quad \rho(x, y) \leq \kappa(\rho(x, z) + \rho(z, y)) \quad x, y, z \in X$$

for some  $\kappa \geq 1$  and  $\mu$  is a Borel measure on  $X$  (with respect to the topology defined by  $\rho$ ) such that there exists a constant  $L \geq 0$  verifying

$$(2.2) \quad 0 < \mu(B_\rho(x, 2r)) \leq L \mu(B_\rho(x, r)) < \infty$$

for all  $x \in X$  and  $0 < r < \infty$ , and where  $B_\rho(x, r) = \{x \in X : \rho(x, y) < r\}$  is the  $\rho$ -ball of center  $x$  and radius  $r$ . It can be assumed without loss of generality that the  $\rho$ -balls are open subsets of  $X$ , see [28]. Given a ball  $B = B_\rho(x, r)$  and  $\theta > 0$  we will usually write  $r(B)$  to denote the radius  $r$  and  $\theta B$  to denote  $B_\rho(x, \theta r)$ .

Condition (2.2) is known as the *doubling property* of  $\mu$ . We will also assume that  $\mu$  satisfies the *reverse doubling property*, that is, for all  $\eta > 1$  there are constants  $c(\eta) > 0$  and  $\delta > 0$  such that

$$(2.3) \quad \frac{\mu(B_\rho(x_1, r_1))}{\mu(B_\rho(x_2, r_2))} \geq c(\eta) \left(\frac{r_1}{r_2}\right)^\delta,$$

whenever  $B_\rho(x_2, r_2) \subset B_\rho(x_1, r_1)$ ,  $x_1, x_2 \in X$  and  $0 < r_1, r_2 \leq \eta \operatorname{diam}_\rho(X)$ , where  $\operatorname{diam}_\rho(X) = \sup\{\rho(x, y) : x, y \in X\}$ . Note that  $\eta$  is not needed when  $\operatorname{diam}_\rho(X) = \infty$  and that when  $\operatorname{diam}_\rho(X) < \infty$  the inequality (2.3) for, say,  $\eta = 2$  implies (2.3) for any  $\eta > 1$  with the same value of  $\delta$ .

For  $x, y_1, \dots, y_m \in X$  and  $\mu$ -measurable functions  $f_1, \dots, f_m$  defined on  $X$ , we will write  $\vec{y} = (y_1, \dots, y_m) \in X^m$ ,  $d\mu(\vec{y}) = d\mu(y_1) \dots d\mu(y_m)$ ,  $\vec{f} = (f_1, \dots, f_m)$ ,  $\vec{f}(\vec{y}) = f_1(y_1) \dots f_m(y_m)$ , and  $\rho(x, \vec{y}) = \rho(x, y_1) + \dots + \rho(x, y_m)$ . With some abuse of notation we will write  $\rho(\vec{x}, \vec{y}) = \rho(x_1, y_1) + \dots + \rho(x_m, y_m)$  for  $\vec{x}, \vec{y} \in X^m$ . Given a measurable function  $g$  on  $X$ , we denote the average of  $g$  over a measurable subset  $E \subset X$  by

$$\int_E g \, d\mu = \frac{1}{\mu(E)} \int_E g \, d\mu.$$

For  $\alpha > 0$  we define the multilinear fractional integral operator of order  $\alpha$  as

$$(2.4) \quad \mathcal{I}_{X, \alpha}(\vec{f})(x) = \int_{X^m} \vec{f}(\vec{y}) \frac{(\rho(x, \vec{y}))^\alpha}{(\mu(B_\rho(x, \rho(x, \vec{y}))))^m} d\mu(\vec{y}).$$

More generally, we define multilinear potential operators associated to a nonnegative kernel  $K(x, \vec{y})$  as

$$(2.5) \quad \mathcal{T}(\vec{f})(x) = \int_{X^m} \vec{f}(\vec{y}) K(x, \vec{y}) \, d\mu(\vec{y}).$$

We will always assume that the kernel  $K$  is the restriction of a nonnegative continuous kernel  $\tilde{K}(\vec{x}, \vec{y})$  (i.e.  $K(x, \vec{y}) = \tilde{K}((x, \dots, x), \vec{y})$  for  $(x, \vec{y}) \in X^{m+1}$ ) that satisfies the following growth conditions: for every  $c > 1$  there exists  $C > 1$  such that

$$(2.6) \quad \tilde{K}(\vec{x}, \vec{y}) \leq C\tilde{K}(\vec{z}, \vec{y}) \quad \text{if } \rho(\vec{z}, \vec{y}) \leq c\rho(\vec{x}, \vec{y}), \text{ and}$$

$$\tilde{K}(\vec{x}, \vec{y}) \leq C\tilde{K}(\vec{y}, \vec{z}) \quad \text{if } \rho(\vec{y}, \vec{z}) \leq c\rho(\vec{x}, \vec{y}).$$

The reverse doubling property implies that if the growth condition (2.6) is true for some  $c > 1$ , then it also holds for all  $c > 1$  with a possibly different value of  $C$ .

We notice that the kernel

$$(2.7) \quad K_\alpha(x, \vec{y}) = \frac{\rho(x, \vec{y})^\alpha}{\mu((B_\rho(x, \rho(x, \vec{y})))^m)}$$

associated to the operator (2.4) is the restriction of

$$(2.8) \quad \tilde{K}_\alpha(\vec{x}, \vec{y}) = \frac{\rho(\vec{x}, \vec{y})^\alpha}{\mu(B_\rho(x_1, \rho(\vec{x}, \vec{y}))) \cdots \mu(B_\rho(x_m, \rho(\vec{x}, \vec{y})))}.$$

Following [40] we define the functional  $\varphi$  associated to  $K$  which acts on balls by

$$\varphi(B) = \sup\{K(x, \vec{y}) : (x, \vec{y}) \in B^{m+1}, \rho(x, \vec{y}) \geq cr(B)\}$$

for a sufficiently small positive constant  $c$  and for  $B$  such that  $r(B) \leq \eta \text{diam}_\rho(X)$ , for some fixed  $\eta > 1$ . We note that the reverse doubling property (2.3) ensures that the set  $\{K(x, \vec{y}) : (x, \vec{y}) \in B^{m+1}, \rho(x, \vec{y}) \geq cr(B)\}$  is non-empty if  $c$  is sufficiently small (any  $c$  satisfying  $0 < c^\delta < c(\eta)$  will work).

Under the assumptions (2.6) on  $K$ , we have the following properties of  $\varphi$ .

(P1) If  $\theta \geq 1$  and  $B$  is a  $\rho$ -ball in  $X$  with  $\theta r(B) \leq \eta \text{diam}_\rho(X)$ , and  $(x, \vec{y}) \in (\theta B)^{m+1}$  then  $\varphi(B) \leq C_\theta K(x, y)$  and therefore

$$(2.9) \quad \varphi(B) \leq C\varphi(\theta B).$$

(P2) If  $B' \subset B$  are  $\rho$ -balls in  $X$  with  $r(B'), r(B) \leq \eta \text{diam}_\rho(X)$ , then

$$(2.10) \quad \varphi(B) \leq C\varphi(B').$$

Note that (P1) implies that  $\varphi(B) < \infty$ . Moreover, (2.9) and (2.10) assure that  $\varphi$  is well-defined in the sense that if  $B_\rho(x_1, r_1) = B_\rho(x_2, r_2)$ ,  $0 < r_1, r_2 \leq \eta \text{diam}_\rho(X)$ , then  $\varphi(B_\rho(x_1, r_1)) \approx \varphi(B_\rho(x_2, r_2))$ . We provide a short proof of property (P1) above as the proof of (P2) is similar. Suppose  $(x, \vec{y}) \in (\theta B)^{m+1}$  and  $(s, \vec{t}) \in B^{m+1}$  with  $\rho(s, \vec{t}) \geq cr(B)$ . If  $\rho(s, \vec{y}) \geq \rho(\vec{t}, \vec{y})$ , then

$$\rho(x, \vec{y}) \leq 2m\kappa\theta r(B) \leq 4m\kappa^2\theta c^{-1}\rho(s, \vec{y})$$

so that  $K(s, \vec{y}) \leq CK(x, \vec{y})$ . Further,

$$\rho(s, \vec{y}) \leq 2m\kappa\theta r(B) \leq 2m\kappa\theta c^{-1}\rho(s, \vec{t})$$

which implies  $K(s, \vec{t}) \leq CK(s, \vec{y})$ , and hence

$$K(s, \vec{t}) \leq CK(x, \vec{y}).$$

In the case when  $\rho(s, \vec{y}) \leq \rho(\vec{t}, \vec{y})$ , we have  $\rho(x, \vec{y}) \leq c\rho(\vec{t}, \vec{y})$ . Hence,

$$\tilde{K}(\vec{t}, \vec{y}) \leq C\tilde{K}((x, \dots, x), \vec{y}) = CK(x, \vec{y})$$

and  $\rho(\vec{t}, \vec{y}) \leq 2m\kappa\theta r(B) \leq c\rho(s, \vec{t})$  showing

$$K(s, \vec{t}) \leq C\tilde{K}(\vec{t}, \vec{y}) \leq CK(x, \vec{y}).$$

Taking the supremum over the proper  $(s, \vec{t})$  we have

$$\varphi(B) \leq CK(x, \vec{y}).$$

When  $K(x, \vec{y}) = (|x - y_1| + \dots + |x - y_m|)^{\alpha - nm}$ , we have

$$\varphi(B) \approx r(B)^{\alpha - nm}$$

and when  $K$  is given by (2.7) we have

$$\varphi(B) \approx \frac{r(B)^\alpha}{\mu(B)^m}, \quad r(B) \leq \eta \operatorname{diam}_\rho(X),$$

with constants that depend only on  $\kappa$ ,  $L$ , and  $c$  as in the definition of  $\varphi$  (and therefore on  $c(\eta)$  and  $\delta$ ).

Finally, we will assume that the functional  $\varphi$  associated to our kernel  $K$  satisfies the following property: there exists  $\epsilon > 0$  such that for all  $C_1 > 1$  there exists  $C_2 > 0$  such that

$$(2.11) \quad \varphi(B')\mu(B')^m \leq C_2 \left( \frac{r(B')}{r(B)} \right)^\epsilon \varphi(B)\mu(B)^m$$

for all balls  $B' \subset B$ , with  $r(B')$ ,  $r(B) < C_1 \operatorname{diam}_\rho(X)$ . Note that the last condition is superfluous when  $\operatorname{diam}_\rho(X) = \infty$ , and that if  $\operatorname{diam}_\rho(X) < \infty$ , it is enough to check (2.11) for only, say,  $C_1 = 2$ , and that  $\epsilon$  can be taken to be independent of  $C_1$ .

*Remark 4.* Notice that

$$K(x, \vec{y}) = (|x - y_1| + \dots + |x - y_m|)^{\alpha - nm}$$

and

$$K_\alpha(x, \vec{y}) = \frac{\rho(x, \vec{y})^\alpha}{\mu(B_\rho(x, \rho(x, \vec{y})))^m}$$

both satisfy (2.11) with  $\epsilon = \alpha$ . In the general case  $K_\alpha$ , if the constant  $C_1$  depends only on  $\kappa$ ,  $L$ , and the constants  $c(\eta)$  and  $\delta$  in (2.3) with  $\eta = C_1$ , so does the corresponding constant  $C_2$ .

We now state our main results concerning weighted boundedness properties for  $\mathcal{T}$ .

**Theorem 2.** *Suppose that  $1 < p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  and  $\frac{1}{m} < p \leq q < \infty$ . Let  $(X, \rho, \mu)$  be a space of homogeneous type that satisfies the reverse doubling property (2.3) and let  $K$  be a kernel such that (2.6) holds with  $\varphi$  satisfying (2.11).*

Furthermore, let  $u, v_k, k = 1, \dots, m$  be weights defined on  $X$  that satisfy condition (2.12) if  $q > 1$  or condition (2.13) if  $q \leq 1$ , where

$$(2.12) \quad \sup_{B \text{ } \rho\text{-ball}} \varphi(B) \mu(B)^{\frac{1}{q} + \frac{1}{p_1'} + \dots + \frac{1}{p_m'}} \left( \frac{1}{\mu(B)} \int_B u^{qt} d\mu \right)^{1/qt} \prod_{j=1}^m \left( \frac{1}{\mu(B)} \int_B v_j^{-tp_j'} d\mu \right)^{1/tp_j'} < \infty,$$

for some  $t > 1$ ,

$$(2.13) \quad \sup_{B \text{ } \rho\text{-ball}} \varphi(B) \mu(B)^{\frac{1}{q} + \frac{1}{p_1'} + \dots + \frac{1}{p_m'}} \left( \frac{1}{\mu(B)} \int_B u^q d\mu \right)^{1/q} \prod_{j=1}^m \left( \frac{1}{\mu(B)} \int_B v_j^{-tp_j'} d\mu \right)^{1/tp_j'} < \infty,$$

for some  $t > 1$ . Then there exists a constant  $C$  such that

$$\left( \int_X \left( |\mathcal{T}\vec{f}| u \right)^q d\mu \right)^{1/q} \leq C \prod_{k=1}^m \left( \int_X (|f_k| v_k)^{p_k} d\mu \right)^{1/p_k}$$

for all  $\vec{f} \in L^{p_1}(X, v_1^{p_1} d\mu) \times \dots \times L^{p_m}(X, v_m^{p_m} d\mu)$ . The constant  $C$  depends only on the constants appearing in (2.1), (2.2), (2.3), (2.6), (2.11), (2.12) and (2.13).

When  $K$  is given by (2.7) as noted before we have  $\varphi(B) \approx \text{diam}_\rho(B)^\alpha / \mu(B)^m$ , hence we have the following result for  $\mathcal{I}_{X,\alpha}$

**Corollary 1.** *Suppose that  $1 < p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  and  $\frac{1}{m} < p \leq q < \infty$ . Let  $(X, \rho, \mu)$  be a space of homogeneous type that satisfies the reverse doubling property (2.3) and assume that the kernel  $\tilde{K}_\alpha$  in (2.8) satisfies the growth conditions (2.6). Let  $u, v_k, k = 1, \dots, m$  be weights defined on  $X$  that satisfy condition (2.14) if  $q > 1$  or condition (2.15) if  $q \leq 1$ , where*

$$(2.14) \quad \sup_{B \text{ } \rho\text{-ball}} \text{diam}_\rho(B)^\alpha \mu(B)^{1/q-1/p} \left( \frac{1}{\mu(B)} \int_B u^{qt} d\mu \right)^{1/qt} \prod_{j=1}^m \left( \frac{1}{\mu(B)} \int_B v_j^{-tp_j'} d\mu \right)^{1/tp_j'} < \infty,$$

for some  $t > 1$ ,

$$(2.15) \quad \sup_{B \text{ } \rho\text{-ball}} \text{diam}_\rho(B)^\alpha \mu(B)^{1/q-1/p} \left( \frac{1}{\mu(B)} \int_B u^q d\mu \right)^{1/q} \prod_{j=1}^m \left( \frac{1}{\mu(B)} \int_B v_j^{-tp_j'} d\mu \right)^{1/tp_j'} < \infty,$$

for some  $t > 1$ . Then there exists a constant  $C$  such that

$$\left( \int_X \left( |\mathcal{I}_{X,\alpha}\vec{f}| u \right)^q d\mu \right)^{1/q} \leq C \prod_{k=1}^m \left( \int_X (|f_k| v_k)^{p_k} d\mu \right)^{1/p_k}$$

for all  $\vec{f} \in L^{p_1}(X, v_1^{p_1} d\mu) \times \dots \times L^{p_m}(X, v_m^{p_m} d\mu)$ . The constant  $C$  depends only on the constants appearing in (2.1), (2.2), (2.3), (2.6), (2.11), (2.14) and (2.15)

*Remark 5.* Moen [30, 31] proved Corollary 1 in the context of  $X = \mathbf{R}^n$  with the Euclidean metric and Lebesgue measure. The multilinear fractional integral operator  $\mathcal{I}_{X,\alpha}$  reduces to the Riesz potential of order  $\alpha$  in  $\mathbf{R}^n$  when  $m = 1$ ,  $X = \mathbf{R}^n$  and  $\mu$  is Lebesgue measure. Namely,

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

Muckenhoupt and Wheeden [32] characterized the one-weight strong type inequality

$$\left( \int_{\mathbf{R}^n} (I_\alpha f w)^q dx \right)^{1/q} \leq C \left( \int_{\mathbf{R}^n} (f w)^p dx \right)^{1/p},$$

for  $f \geq 0$ ,  $1 < p < \frac{n}{\alpha}$  and  $q$  such that  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ . They proved that the above inequality holds if and only if  $w$  belongs to the class  $A_{p,q}$ , this is

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w^q dx \right)^{1/q} \left( \frac{1}{|Q|} \int_Q w^{-p'} dx \right)^{1/p'} < \infty,$$

where the supremum is taken over all cubes  $Q$  in  $\mathbf{R}^n$  with sides parallel to the coordinate axes. The two-weight strong type inequality for  $I_\alpha$

$$\left( \int_{\mathbf{R}^n} (I_\alpha f w)^q dx \right)^{1/q} \leq C \left( \int_{\mathbf{R}^n} (f v)^p dx \right)^{1/p}, \quad f \geq 0,$$

was also extensively studied. For example, Sawyer [39] gave a characterization for  $w$  and  $v$  that basically come to testing the above inequality with  $f = \chi_Q v^{(1-p')p}$  and its dual inequality with  $\chi_Q w^q$ . Sawyer-Wheeden [40] and Pérez-Wheeden [37] studied two-weight conditions for weighted inequalities of fractional integral operators on spaces of homogeneous type. In particular, (2.14) ( $q > 1$ ) reduces to the conditions imposed in Pérez-Wheeden [37, Theorem 2.2] to prove weighted boundedness properties for  $\mathcal{I}_{X,1}$  when  $m = 1$ .

We stress that, even in the Euclidean setting and with the choice  $u = \prod_{k=1}^m v_i$ , using iterations of the linear results mentioned above to prove multilinear ones would lead to considering weights  $(v_1, \dots, v_m)$  in the class

$$W(p, q) := \bigcup_{q_1, \dots, q_m} \prod_{i=1}^m A_{p_i, q_i},$$

where the union is over all  $q_i \geq p_i$  that satisfy  $1/q = 1/q_1 + \dots + 1/q_m$ ,  $1/p_i - 1/q_i = 1/n$ ,  $i = 1, \dots, m$ . However, the class of weights  $u, v_1, \dots, v_m$  (with  $u = \prod_{k=1}^m v_i$ ) satisfying (2.14) is strictly larger than  $W(p, q)$ . See Remark 7.5 in [31] and Section 7 in [20].

### 3. PROOF OF THEOREM 2

We recall the following construction due to M. Christ [8] of dyadic cubes in a general space of homogeneous type  $(X, \rho, \mu)$  with constant  $\kappa \geq 1$  in the quasi-triangle inequality for  $\rho$ . There exists a collection of open subsets  $\mathcal{D} = \{Q_\alpha^k : k \in \mathbf{Z}, \alpha \in I_k\}$

(here, for each  $k \in \mathbf{Z}$ ,  $I_k$  is a countable set of indices), and constants  $A > 2\kappa$ ,  $a_0, a_1 > 0$ , depending only on  $\kappa$ , such that

- (i)  $\mu(X \setminus \bigcup_{\alpha} Q_{\alpha}^k) = 0$  for every  $k \in \mathbf{Z}$ ,
- (ii) given  $Q_{\beta}^l$  and  $Q_{\alpha}^k$  with  $l \leq k$ , then either  $Q_{\beta}^l \subset Q_{\alpha}^k$  or  $Q_{\beta}^l \cap Q_{\alpha}^k = \emptyset$ ,
- (iii) for each  $(k, \alpha)$  and each  $l > k$  there is a unique  $\beta$  such that  $Q_{\alpha}^k \subset Q_{\beta}^l$ ,
- (iv)  $\text{diam}_{\rho}(Q_{\alpha}^k) \leq a_1 A^k$ ,
- (v) each  $Q_{\alpha}^k$  contains some ball  $B_{\rho}(x_{\alpha}^k, a_0 A^k)$ ,

We set  $\mathcal{D}^k = \{Q_{\alpha}^l \in \mathcal{D} : l = k\}$  and note that by property (ii) the family  $\mathcal{D}^k$  may be assumed to be disjoint. If  $Q = Q_{\alpha}^k$  we call  $x_Q = x_{\alpha}^k$  as given in property (v) the center of  $Q$  and define  $B(Q) = B_{\rho}(x_Q, 2\kappa a_1 A^k)$  where  $a_1$  is as given in property (iv). Note that if  $\text{diam}_{\rho}(X) = \infty$  then  $Q \neq X$  for all  $Q \in \mathcal{D}$ , and if  $\text{diam}_{\rho}(X) < \infty$ , there exists  $k_0 \in \mathbf{Z}$  such that  $\mathcal{D}^k = \{X\}$  for all  $k \geq k_0$  and  $X$  is not in  $\mathcal{D}^k$  for  $k \leq k_0$ , in which case we only consider  $k \leq k_0$ .

Observe that if  $Q \in \mathcal{D}^k$ ,  $Q' \in \mathcal{D}^{k'}$  and  $Q \subset Q'$  then  $B(Q) \subset B(Q')$ . To see this, note that by property (ii), we have  $k \leq k'$ . Then if  $y \in B(Q)$

$$\rho(y, x_{Q'}) \leq \kappa(\rho(y, x_Q) + \rho(x_Q, x_{Q'})) \leq \kappa(2\kappa a_1 A^k + a_1 A^{k'}) \leq 2\kappa a_1 A^{k'},$$

where we have used property (iv) for the cube  $Q'$  and that  $A > 2\kappa$ .

Notice that by property (iii) for every  $Q \in \mathcal{D}^k$  there is a unique cube  $Q^* \in \mathcal{D}^{k+1}$ , called the parent of  $Q$ , such that  $Q \subset Q^*$ . Moreover  $\mu(Q) \sim \mu(B(Q)) \sim \mu(B(Q^*))$  since  $\mu$  is doubling and

$$B_{\rho}(x_Q, a_0 A^k) \subset Q \subset B(Q) \subset B(Q^*) \subset (\kappa + 1/2)A B(Q).$$

It is important to observe that if  $Q \in \mathcal{D}^k$  and  $Q \neq X$  there exists  $l > k$  such that if  $Q^{*l} \in \mathcal{D}^l$  is the cube containing  $Q$  given by property (iii) (the  $l$ th ancestor of  $Q$ ) then  $Q \subsetneq Q^{*l}$ . This is clear when  $\text{diam}_{\rho}(X) < \infty$ . When  $\text{diam}_{\rho}(X) = \infty$ , if  $Q = Q^{*l}$  for all  $l > k$  then, by property (v),  $Q$  contains balls of radius  $a_0 A^l$  for all  $l > k$ . However, the radius and diameter of a ball are comparable (a consequence of the reverse doubling property (2.3)), obtaining  $A^l \leq C \text{diam}_{\rho}(Q) < \infty$  for all  $l > k$ , a contradiction.

Our first step towards the proof of Theorem 2 is a discretization of  $\mathcal{I}$ . Let  $(x, \vec{y}) \in (\bigcap_k \bigcup_{\alpha} Q_{\alpha}^k)^{m+1}$  and  $l \in \mathbf{Z}$  be such that

$$A^{l-1} \leq \rho(x, \vec{y}) \leq A^l.$$

There is a dyadic cube  $Q \in \mathcal{D}^l$  with  $x \in Q$ . Let  $x_Q$  be the center of  $B(Q)$ , and  $y_1, \dots, y_m$  be the coordinates of  $\vec{y}$ . Since  $\text{diam}(Q) \leq a_1 A^l$  (and we can assume that  $a_1$  is larger than 1),

$$\rho(x_Q, y_i) \leq \kappa(\rho(x, x_Q) + \rho(x, y_i)) \leq \kappa(a_1 + 1)A^l \leq 2\kappa a_1 A^l$$

for  $1 \leq i \leq m$  and consequently  $\vec{y} \in B(Q)^m$ . Furthermore, since  $(x, \vec{y}) \in B(Q)^{m+1}$  and  $\rho(x, \vec{y}) \geq A^{l-1} = r(B(Q))/2\kappa a_1 A$  we have

$$K(x, \vec{y}) \leq \varphi(B(Q))$$

by the definition of  $\varphi$  (note that  $r(B(Q)) = 2\kappa a_1 A^l \leq 2\kappa a_1 A \rho(x, \vec{y}) \leq 2\kappa a_1 A m \text{diam}_{\rho}(X)$ , so we can choose a structural constant  $\eta \geq 2\kappa a_1 A m$  in the definition of  $\varphi$ ). Since

$x \in Q$  and  $\vec{y} \in B(Q)^m$  it follows that

$$K(x, \vec{y}) \leq \varphi(B(Q))\chi_Q(x)\chi_{B(Q)^m}(\vec{y}) \leq \sum_{Q \in \mathcal{D}} \varphi(B(Q))\chi_Q(x)\chi_{B(Q)^m}(\vec{y})$$

where the last inequality holds for almost all  $(x, \vec{y}) \in X^{m+1}$ . Multiplying by  $\vec{f}(\vec{y}) \geq 0$  and integrating yields

$$(3.1) \quad \mathcal{T}(\vec{f})(x) \leq \sum_{Q \in \mathcal{D}} \varphi(B(Q)) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y})\chi_Q(x).$$

Multiplying by  $u(x)g(x) \geq 0$  and integrating

$$\int_X \mathcal{T}(\vec{f})(x)g(x)u(x) d\mu(x) \leq \sum_{Q \in \mathcal{D}} \varphi(B(Q)) \int_Q g(x)u(x) d\mu(x) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}).$$

Now we switch the summation to a smaller set of dyadic cubes with better disjointness properties. To define this smaller set of dyadic cubes we look at level sets corresponding to a certain multilinear maximal function. Set

$$\mathcal{M}_{B(\mathcal{D})}(\vec{h})(x) = \sup_{Q \in \mathcal{D}: x \in Q} \frac{1}{\mu(B(Q))^m} \int_{B(Q)^m} |\vec{h}(\vec{y})| d\mu(\vec{y}), \quad x \in \bigcup_{Q \in \mathcal{D}} Q.$$

Let  $a > 1$  be a number to be chosen later, and set

$$\mathcal{S}^k = \{x \in \bigcup_{Q \in \mathcal{D}} Q : \mathcal{M}_{B(\mathcal{D})}(\vec{f})(x) > a^k\}.$$

If  $x \in \mathcal{S}^k$ , then there exists  $Q \in \mathcal{D}$  such that  $x \in Q$  and

$$(3.2) \quad \frac{1}{\mu(B(Q))^m} \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) > a^k.$$

In particular, we have  $Q \subset \mathcal{S}^k$  and the fact that  $\int_{X^m} \vec{f}(\vec{y}) d\mu(\vec{y}) < \infty$  and the nested nature of the dyadic cubes in  $\mathcal{D}$  allow to write

$$\mathcal{S}^k = \bigcup_j Q_{k,j},$$

where the cubes  $Q_{k,j}$  belong to  $\mathcal{D}$ , and they are disjoint and maximal relative to inclusion and generation with respect to the property (3.2) (the existence of these maximal cubes is guaranteed by the reverse doubling property (2.3) when  $\text{diam}_\rho(X) = \infty$ ). Notice that if  $Q_{k,j}^*$  is the parent of  $Q_{k,j}$  and  $Q_{k,j} \neq X$ , by the maximality of  $Q_{k,j}$  we have

$$\begin{aligned} a^k &< \frac{1}{\mu(B(Q_{k,j}))^m} \int_{B(Q_{k,j})^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \\ &\leq \frac{c}{\mu(B(Q_{k,j}^*))^m} \int_{B(Q_{k,j}^*)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \\ &\leq ca^k \leq a^{k+1} \end{aligned}$$

if  $a$  is chosen large enough.

The next step is to estimate  $\mu(Q_{k,j} \cap \mathcal{S}^{k+1})$ . Consider  $x \in Q_{k,j} \cap \mathcal{S}^{k+1}$ , then

$$(3.3) \quad \mathcal{M}_{B(\mathcal{D})}(\vec{f})(x) = \sup_{\substack{P \in \mathcal{D} \\ x \in P}} \frac{1}{\mu(B(P))^m} \int_{B(P)^m} \vec{f}(\vec{y}) \, d\mu(\vec{y}) > a^{k+1}$$

and the nested property of dyadic cubes together with the maximality of  $Q_{k,j}$  with respect to the inequality (3.2) imply that if  $P \in \mathcal{D}$  is such that  $x \in P$  and

$$\frac{1}{\mu(B(P))^m} \int_{B(P)^m} \vec{f}(\vec{y}) \, d\mu(\vec{y}) > a^{k+1},$$

then  $P \subset Q_{k,j}$ . Therefore, we have

$$\begin{aligned} a^{k+1} &< \mathcal{M}_{B(\mathcal{D})}(\vec{f})(x) = \sup_{\substack{P \in \mathcal{D} \\ x \in P \subset Q_{k,j}}} \frac{1}{\mu(B(P))^m} \int_{B(P)^m} \vec{f}(\vec{y}) \, d\mu(\vec{y}) \\ &\leq \sup_{\substack{P \in \mathcal{D} \\ x \in P}} \frac{1}{\mu(B(P))^m} \int_{B(P)^m} (f_1 \chi_{B(Q_{k,j})}, \dots, f_m \chi_{B(Q_{k,j})})(\vec{y}) \, d\mu(\vec{y}), \end{aligned}$$

where we have used that  $B(P) \subset B(Q_{k,j})$  for  $P \subset Q_{k,j}$ . Consequently,

$$\begin{aligned} \mu(Q_{k,j} \cap \mathcal{S}^{k+1}) &= \mu(\{x \in Q_{k,j} : \mathcal{M}_{B(\mathcal{D})}(\vec{f})(x) > a^{k+1}\}) \\ &\leq \mu(\{x \in Q_{k,j} : \mathcal{M}_\mu(f_1 \chi_{B(Q_{k,j})}, \dots, f_m \chi_{B(Q_{k,j})})(x) > a^{k+1}\}) \\ &\leq \left( \frac{\|\mathcal{M}_\mu\|}{a^{k+1}} \int_{B(Q_{k,j})^m} \vec{f}(\vec{y}) \, d\mu(\vec{y}) \right)^{1/m} \\ &= \mu(B(Q_{k,j})) \left( \frac{\|\mathcal{M}_\mu\|}{a^{k+1} \mu(B(Q_{k,j}))^m} \int_{B(Q_{k,j})^m} \vec{f}(\vec{y}) \, d\mu(\vec{y}) \right)^{1/m} \\ &\leq \mu(B(Q_{k,j})) \left( \frac{c \|\mathcal{M}_\mu\|}{a} \right)^{1/m} =: \theta \mu(B(Q_{k,j})) \leq \theta \mu(Q_{k,j}), \end{aligned}$$

where  $\mathcal{M}_\mu$  is the multi-sublinear maximal operator

$$\mathcal{M}_\mu(\vec{h})(x) = \sup_{\substack{x \in B \\ B \text{ } \rho\text{-ball}}} \prod_{i=1}^m \frac{1}{\mu(B)} \int_B |h_i(y_i)| \, d\mu(y_i)$$

and  $\|\mathcal{M}_\mu\|$  is the smallest constant in the weak inequality

$$\mu\{x \in X : \mathcal{M}_\mu(\vec{f})(x) > \lambda\}^m \leq \frac{\|\mathcal{M}_\mu\|}{\lambda} \prod_{i=1}^m \|f_i\|_{L^1(\mu)}.$$

Notice that such a constant  $\|\mathcal{M}_\mu\|$  exists because

$$\mathcal{M}_\mu(\vec{f}) \leq \prod_{i=1}^m M_\mu f_i$$

where  $M_\mu$  is the Hardy-Littlewood maximal operator associated to the space of homogeneous type  $(X, \rho, \mu)$ . The constant  $\theta$  can be made smaller than one by choosing  $a$  sufficiently large. In particular, if we set  $E_{k,j} = Q_{k,j} \setminus \mathcal{S}^{k+1}$ , we get

$$(3.4) \quad \mu(E_{k,j}) \geq \gamma \mu(Q_{k,j}), \quad Q_{k,j} \neq X,$$

for some constant  $\gamma \in (0, 1)$  that depends only on structural constants.

Note that if  $\text{diam}_\rho(X) < \infty$  then there exists  $k_1 \in \mathbf{Z}$  such that  $a^{k_1} < \frac{1}{\mu(X)} \int_X f(y) dy \leq a^{k_1+1}$ . If  $\text{diam}_\rho(X) = \infty$ , set  $k_1 = -\infty$ . Next, for  $k > k_1$ ,  $k \in \mathbf{Z}$ , define

$$\mathcal{C}^k = \left\{ Q \in \mathcal{D} : a^k < \frac{1}{\mu(B(Q))^m} \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \leq a^{k+1} \right\},$$

and

$$\mathcal{C}^{k_1} = \begin{cases} \left\{ Q \in \mathcal{D} : \frac{1}{\mu(B(Q))^m} \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \leq a^{k_1+1} \right\}, & k_1 \neq -\infty; \\ \emptyset, & k_1 = -\infty. \end{cases}$$

If  $k > k_1$ , we have  $Q_{k,j} \in \mathcal{C}^k$  for all  $j$  and if  $Q \in \mathcal{C}^k$ ,  $k > k_1$ , then  $Q$  must be contained in  $Q_{k,j}$  for some  $j$ . Returning to the estimate for  $\int_X (\mathcal{T} \vec{f}) gu d\mu$ , we have

$$\begin{aligned} & \int_X (\mathcal{T} \vec{f}) gu d\mu \\ & \leq \sum_{Q \in \mathcal{D}} \varphi(B(Q)) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \int_Q g(x) u(x) d\mu(x) \\ & = \sum_{k \geq k_1} \sum_{Q \in \mathcal{C}^k} \frac{1}{\mu(B(Q))^m} \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \varphi(B(Q)) \mu(B(Q))^m \int_Q gu d\mu \\ & \leq \sum_{k > k_1} a^{k+1} \sum_j \sum_{\substack{Q \in \mathcal{C}^k \\ Q \subset Q_{k,j}}} \varphi(B(Q)) \mu(B(Q))^m \int_Q gu d\mu \\ & \quad + \sum_{Q \in \mathcal{C}^{k_1}} a^{k_1+1} \varphi(B(Q)) \mu(B(Q))^m \int_Q gu d\mu. \end{aligned}$$

We need the following lemma.

**Lemma 1.** *If  $\varphi$  satisfies (2.11) with constants  $\epsilon$ ,  $C_1$  and  $C_2$  then there exists a constant  $C = C(C_1, C_2, \epsilon, A)$  such that for each  $Q_0 \in \mathcal{D}$  with  $r(B(Q_0)) \leq C_1 \text{diam}_\rho(X)$*

$$\sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \varphi(B(Q)) \mu(B(Q))^m \int_Q gu d\mu \leq C \varphi(B(Q_0)) \mu(B(Q_0))^m \int_{Q_0} gu d\mu.$$

*Proof.* Note that if  $Q \in \mathcal{D}$  and  $Q \subset Q_0$ , then  $r(B(Q)) \leq r(B(Q_0)) \leq C_1 \text{diam}_\rho(X)$  and recall that  $B(Q) \subset B(Q_0)$ . We now use the condition (2.11) on  $\varphi$  to get

$$\sum_{\substack{Q \in \mathcal{D} \\ Q \subset Q_0}} \varphi(B(Q)) \mu(B(Q))^m \int_Q gu d\mu$$

$$\begin{aligned}
&= \sum_{l=0}^{\infty} \sum_{\substack{Q \subset Q_0 \\ \ell(Q)=A^{-l}\ell(Q_0)}} \varphi(B(Q)) \mu(B(Q))^m \int_Q gu \, d\mu \\
&\leq C_2 \varphi(B(Q_0)) \mu(B(Q_0))^m \sum_{l=0}^{\infty} A^{-l\epsilon} \sum_{\substack{Q \subset Q_0 \\ \ell(Q)=A^{-l}\ell(Q_0)}} \int_Q gu \, d\mu \\
&\leq C_2 \left( \sum_{l=0}^{\infty} A^{-l\epsilon} \right) \varphi(B(Q_0)) \mu(B(Q_0))^m \int_{Q_0} gu \, d\mu.
\end{aligned}$$

□

Lemma 1 with  $C_1 = \eta$ , where  $\eta = 2\kappa a_1 Am$  is the structural constant chosen above, yields

$$\begin{aligned}
\int_X (\mathcal{T}\vec{f})gu \, d\mu &\leq C \sum_{k,j,k>k_1} \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \int_{B(Q_{k,j})} f_i(y_i) \, d\mu(y_i) \\
&\quad \times \int_{Q_{k,j}} gu \, d\mu \, \mu(Q_{k,j}) + C_{k_1},
\end{aligned}$$

where  $C_{k_1} = 0$  if  $k_1 = -\infty$  and

$$C_{k_1} = \varphi(X) \mu(X)^m \prod_{i=1}^m \int_X f_i(y_i) \, d\mu \int_X gu \, d\mu \, \mu(X)$$

if  $k_1 \neq -\infty$ . We have thus fully discretized  $\mathcal{T}$  and are ready to put everything together to get the estimates for the case  $q \geq 1$ . If  $k_1 \neq -\infty$ ,  $C_{k_1}$  can be handled in the same way as the terms in the sum on  $k$  and  $j$ , so we will assume that  $k_1 = -\infty$  and therefore  $C_{k_1} = 0$ . Let  $[u, \vec{v}]$  represent the finite quantity in the weight condition (2.12). Using Hölder inequality and (2.12) we have

$$\begin{aligned}
&\int_X (\mathcal{T}\vec{f})gu \, d\mu \\
&\leq C \sum_{k,j} \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \left( \int_{B(Q_{k,j})} v_i^{-tp'_i} \, d\mu \right)^{1/(tp'_i)} \left( \int_{Q_{k,j}} u^{tq} \, d\mu \right)^{1/(tq)} \\
&\quad \times \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} \, d\mu(y_i) \right)^{1/(tp'_i)'} \left( \int_{Q_{k,j}} g^{(tq)'} \, d\mu \right)^{1/(tq)'} \mu(Q_{k,j}) \\
&\leq c[u, \vec{v}] \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} \, d\mu(y_i) \right)^{\frac{1}{(tp'_i)'}} \left( \int_{Q_{k,j}} g^{(qt)'} \, d\mu \right)^{\frac{1}{(tq)'}} \mu(Q_{k,j})^{\frac{1}{q'} + \frac{1}{p}} \\
&\leq c[u, \vec{v}] \left( \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} \, d\mu(y_i) \right)^{q/(tp'_i)'} \mu(Q_{k,j})^{q/p} \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& \times \left( \sum_{k,j} \left( \int_{Q_{k,j}} g^{(qt)'} d\mu \right)^{q'/(tq)'} \mu(Q_{k,j}) \right)^{1/q'} \\
& \leq c[u, \vec{v}] \left( \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp_i)'} d\mu(y_i) \right)^{p/(tp_i)'} \mu(E_{k,j}) \right)^{1/p} \\
& \quad \times \left( \sum_{k,j} \left( \int_{Q_{k,j}} g^{(qt)'} d\mu \right)^{q'/(tq)'} \mu(E_{k,j}) \right)^{1/q'} \\
& \leq c[u, \vec{v}] \prod_{i=1}^m \left( \int_X M_{(tp_i)'}(f_i v_i)^{p_i} d\mu \right)^{1/p_i} \left( \int_X M_{(tq)'}(g)^{q'} d\mu \right)^{1/q'},
\end{aligned}$$

where in the last line  $M_s(g) = M_\mu(|g|^s)^{1/s}$  is the  $L^s(\mu)$  average maximal function. Notice that since  $t > 1$  we have

$$\int_X (\mathcal{T}\vec{f})g u d\mu \leq c[u, \vec{v}] \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}(\mu)} \|g\|_{L^{q'}(\mu)}.$$

By duality we finally obtain

$$\|u\mathcal{T}(\vec{f})\|_{L^q(\mu)} \leq C \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}(\mu)}.$$

Next, we address the case when  $q \leq 1$ . Since  $q \leq 1$ , using (3.1) we have

$$\mathcal{T}\vec{f}(x)^q \leq \sum_{Q \in \mathcal{D}} \left( \varphi(B(Q)) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \right)^q \chi_Q(x)$$

and hence

$$\int_X (u\mathcal{T}\vec{f})^q d\mu \leq \sum_{Q \in \mathcal{D}} \left( \varphi(B(Q)) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \right)^q \int_Q u^q d\mu.$$

We may now proceed as in the case  $q > 1$ , with  $\mathcal{C}^k$  and  $Q_{k,j}$  defined exactly as above. We assume again that  $k_1 = -\infty$ ; as before, the extra term that appears when  $k_1 \neq -\infty$  can be handled in the same way as the terms in the sum in  $k$  and  $j$ . Then

$$\begin{aligned}
& \int_X (u\mathcal{T}\vec{f})^q d\mu \leq \sum_{Q \in \mathcal{D}} \left( \varphi(B(Q)) \int_{B(Q)^m} \vec{f}(\vec{y}) d\mu(\vec{y}) \right)^q \int_Q u^q d\mu \\
& = \sum_{Q \in \mathcal{D}} \left( \varphi(B(Q)) \mu(B(Q))^m \prod_{i=1}^m \int_{B(Q)} f_i d\mu \right)^q \int_Q u^q d\mu \\
& \leq \sum_k a^{(k+1)q} \sum_j \sum_{\substack{Q \in \mathcal{C}^k \\ Q \subset Q_{k,j}}} \varphi(B(Q))^q \mu(B(Q))^{mq} \int_Q u^q d\mu
\end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{k,j} \varphi(B(Q_{k,j}))^q \mu(B(Q_{k,j}))^{mq} \left( \prod_{i=1}^m \int_{B(Q_{k,j})} f_i d\mu \right)^q \int_{Q_{k,j}} u^q d\mu \\
&= c \sum_{k,j} \left[ \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \int_{B(Q_{k,j})} f_i d\mu \left( \int_{Q_{k,j}} u^q d\mu \right)^{1/q} \right]^q \mu(Q_{k,j}).
\end{aligned}$$

where the second to last inequality follows from a slight adaption of Lemma 1. We now use Hölder's inequality with  $tp'_i$  and  $(tp'_i)'$  and condition (2.13). Let  $[u, \vec{v}]$  represent the finite quantity from (2.13), we obtain

$$\begin{aligned}
&\int_X (u\mathcal{T}\vec{f})^q d\mu \\
&\leq c \sum_{k,j} \left[ \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \left( \int_{B(Q_{k,j})} v_i^{-tp'_i} d\mu \right)^{1/tp'_i} \left( \int_{Q_{k,j}} u^q d\mu \right)^{1/q} \right]^q \\
&\quad \times \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} d\mu \right)^{q/(tp'_i)'} \mu(Q_{k,j}) \\
&\leq c [u, \vec{v}]^q \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} d\mu \right)^{q/(tp'_i)'} \mu(Q_{k,j})^{q/p} \\
&\leq c [u, \vec{v}]^q \left( \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} d\mu \right)^{p/(tp'_i)'} \mu(Q_{k,j}) \right)^{q/p} \\
&\leq c [u, \vec{v}]^q \left( \sum_{k,j} \prod_{i=1}^m \left( \int_{B(Q_{k,j})} (f_i v_i)^{(tp'_i)'} d\mu \right)^{p/(tp'_i)'} \mu(E_{k,j}) \right)^{q/p} \\
&\leq c [u, \vec{v}]^q \prod_{i=1}^m \left( \int_X M_{(tp'_i)'}(f_i v_i)^{p_i} d\mu \right)^{q/p_i} \\
&\leq c [u, \vec{v}]^q \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}(\mu)}^q.
\end{aligned}$$

Thus concluding the proof of the case  $q \leq 1$ .  $\square$

#### 4. MULTILINEAR POTENTIAL OPERATORS IN ORLICZ SPACES

The aim of this section is twofold. We will show that it is possible to substantially generalize conditions (2.12) and (2.13) by resorting to the theory of Orlicz spaces and we will introduce the natural multilinear counterparts to some linear weighted estimates in the context of Orlicz spaces studied in [36, 37]. These multilinear estimates will allow for a strictly wider range of indices than in the linear case, see Theorem 3 and Remark 7 below.

We briefly recall some basic facts about Orlicz spaces, and refer the reader to [1] and [38] for a detailed account of the spaces. A function  $\psi : [0, \infty) \rightarrow [0, \infty)$  is called a Young function if it is continuous, convex, increasing,  $\psi(0) = 0$  and  $\psi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Moreover, we shall assume  $\psi$  is normalized so that  $\psi(1) = 1$  and  $\psi$  satisfies the doubling condition, namely there exists constants  $C$  and  $N$  such that

$$\psi(2t) \leq C\psi(t), \quad \text{for all } t \geq N.$$

For each such function  $\psi$  there exists a complementary Young function, denoted  $\bar{\psi}$ , such that

$$t \leq \psi^{-1}(t)\bar{\psi}^{-1}(t) \leq 2t, \quad t > 0.$$

The Orlicz space  $L_\psi = L_\psi(X, \mu)$  is the class of all functions such that

$$\int_X \psi\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) < \infty$$

for some  $\lambda > 0$ . The space  $L_\psi$  is a Banach space equipped with the norm,

$$\|f\|_\psi = \inf \left\{ \lambda > 0 : \int_X \psi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\}.$$

The space  $L_{\bar{\psi}}$  is called *the conjugate space of  $L_\psi$* . Orlicz spaces satisfy the generalized Hölder inequality

$$\int_X |fg| d\mu \leq c\|f\|_\psi\|g\|_{\bar{\psi}}.$$

Notice that if  $\psi(t) = t^r$  for  $r \geq 1$  then  $L_\psi = L^r(X, d\mu)$  and the complementary function  $\bar{\psi}(t) = t^{r'}$  with conjugate space  $L_{\bar{\psi}} = L^{r'}(X, d\mu)$ . Other interesting examples include  $\psi(t) = t^r [\log(1+t)]^{-1-\epsilon}$  for which the complementary is  $\bar{\psi}(t) = t^{p'} [\log(1+t)]^{(p'-1)(1+\epsilon)}$ .

Given a ball  $B \subset X$  we define the  $L_\psi$  average over  $B$  by

$$\|f\|_{\psi, B} = \inf \left\{ \lambda > 0 : \int_B \psi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\}.$$

Once we have defined an average over a single ball we may define a corresponding maximal function by

$$M_\psi f(x) = \sup_{B: x \in B} \|f\|_{\psi, B}$$

where the supremum is over all balls  $B$  that contain  $x$ . Notice that when  $\psi(t) = t^r$  we have

$$\|f\|_{\psi, B} = \left( \int_B |f|^r d\mu \right)^{1/r},$$

and hence  $M_\psi f(x) = M_\mu(|f|^r)^{1/r}$ . Furthermore, in this case,

$$M_\psi : L^p(X, d\mu) \rightarrow L^p(X, d\mu)$$

if and only if  $p > r$ . For a general  $\psi$ , Pérez and Wheeden [37] established the following characterization:

$$\int_X (M_\psi f)^p d\mu \leq C \int_X |f|^p d\mu$$

for all  $f \in L^p(X, d\mu)$  if and only if there is a constant  $c > 0$  such that

$$(4.1) \quad \int_c^\infty \frac{\psi(t) dt}{t^p} \approx \int_c^\infty \left( \frac{t^{p'}}{\bar{\psi}(t)} \right)^{p-1} \frac{dt}{t} < \infty.$$

In the context of Orlicz spaces we have

**Theorem 3.** *Suppose that  $1 < p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$  and  $\frac{1}{m} < p \leq q < \infty$ . Let  $(X, \rho, \mu)$  be a space of homogeneous type,  $K$  is a kernel such that (2.6) holds with  $\varphi$  satisfying (2.11) and  $\Psi, \Phi_1, \dots, \Phi_m$  be Young functions satisfying*

$$(4.2) \quad \int_c^\infty \left( \frac{t^q}{\Psi(t)} \right)^{q'-1} \frac{dt}{t} < \infty,$$

and

$$(4.3) \quad \int_c^\infty \left( \frac{t^{p'_i}}{\Phi_i(t)} \right)^{p_i-1} \frac{dt}{t} < \infty \quad 1 \leq i \leq m$$

for some  $c > 0$ . Furthermore, let  $u, v_k, k = 1, \dots, m$  be weights defined on  $X$  that satisfy condition (4.4) if  $q > 1$  or condition (4.5) if  $q \leq 1$ , where

$$(4.4) \quad \sup_{B \text{ } \rho\text{-ball}} \varphi(B) \mu(B)^{1/q+1/p'_1+\dots+1/p'_m} \|u\|_{\Psi, B} \prod_{j=1}^m \|v_j^{-1}\|_{\Phi_j, B} < \infty;$$

$$(4.5) \quad \sup_{B \text{ } \rho\text{-ball}} \varphi(B) \mu(B)^{1/q+1/p'_1+\dots+1/p'_m} \left( \int_B u^q d\mu \right)^{1/q} \prod_{j=1}^m \|v_j^{-1}\|_{\Phi_j, B} < \infty.$$

Then there exists a constant  $C$  such that

$$\left( \int_X (|\mathcal{T}\vec{f}| u)^q d\mu \right)^{1/q} \leq C \prod_{k=1}^m \left( \int_X (|f_k| v_k)^{p_k} d\mu \right)^{1/p_k}$$

for all  $\vec{f} \in L^{p_1}(X, v_1^{p_1}) \times \dots \times L^{p_m}(X, v_m^{p_m})$ .

*Remark 6.* Theorem 2 is contained in Theorem 3 since it corresponds to

$$\Psi(t) = t^{r^q}, \Phi_1(t) = t^{rp'_1}, \dots, \Phi_m(t) = t^{rp'_m}$$

whose complementary functions satisfy conditions (4.2) and (4.3).

We provide a brief sketch of the proof of Theorem 3 when  $q > 1$  and  $\text{diam}_\rho(X) = \infty$ . The proof when  $q < 1$  will be similar to that of Theorem 2.

*Proof of Theorem 3.* The same decomposition techniques as in the proof of Theorem 2 yield

$$\begin{aligned} \int_X (\mathcal{T}\vec{f})gu d\mu &\leq C \sum_{k,j} \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \int_{B(Q_{k,j})} f_i(y_i) d\mu(y_i) \\ &\quad \times \int_{Q_{k,j}} gu d\mu \mu(Q_{k,j}). \end{aligned}$$

Using the generalized Hölder inequality for Orlicz spaces we have

$$\begin{aligned}
& \int_X (\mathcal{T}\vec{f})gu \, d\mu \\
& \leq C \sum_{k,j} \varphi(B(Q_{k,j})) \mu(B(Q_{k,j}))^m \prod_{i=1}^m \|f_i v_i\|_{\overline{\Phi}_i, B(Q_{k,j})} \|v_i^{-1}\|_{\Phi_i, B(Q_{k,j})} \\
& \quad \times \|g\|_{\overline{\Psi}, B(Q_{k,j})} \|u\|_{\Psi, B(Q_{k,j})} \mu(Q_{k,j}) \\
& \leq c \sum_{k,j} \prod_{i=1}^m \|f_i v_i\|_{\overline{\Phi}_i, B(Q_{k,j})} \|g\|_{\overline{\Psi}, B(Q_{k,j})} \mu(Q_{k,j})^{1/q'+1/p} \\
& \leq c \left( \sum_{k,j} \left( \prod_{i=1}^m \|f_i v_i\|_{\overline{\Phi}_i, B(Q_{k,j})} \right)^q \mu(Q_{k,j})^{q/p} \right)^{1/q} \left( \sum_{k,j} \|g\|_{\overline{\Psi}, B(Q_{k,j})}^{q'} \mu(Q_{k,j}) \right)^{1/q'} \\
& \leq c \left( \sum_{k,j} \left( \prod_{i=1}^m \|f_i v_i\|_{\overline{\Phi}_i, B(Q_{k,j})} \right)^p \mu(E_{k,j}) \right)^{1/p} \left( \sum_{k,j} \|g\|_{\overline{\Psi}, B(Q_{k,j})}^{q'} \mu(E_{k,j}) \right)^{1/q'} \\
& \leq c \left( \prod_{i=1}^m \left( \int_X M_{\overline{\Phi}_i}(f_i v_i)^{p_i} \, d\mu \right)^{1/p_i} \right) \left( \int_X (M_{\overline{\Psi}}g)^{q'} \, d\mu \right)^{1/q'} \\
& \leq C \left( \prod_{i=1}^m \|f_i v_i\|_{L^{p_i}(\mu)} \right) \|g\|_{L^{q'}(\mu)}
\end{aligned}$$

where the last line follows since  $\overline{\Psi}, \overline{\Phi}_1, \dots, \overline{\Phi}_m$  satisfy (4.2) and (4.3) so

$$\begin{cases} M_{\overline{\Psi}} : L^{q'}(X, d\mu) & \rightarrow & L^{q'}(X, d\mu) \\ M_{\overline{\Phi}_1} : L^{p_1}(X, d\mu) & \rightarrow & L^{p_1}(X, d\mu) \\ & & \vdots \\ M_{\overline{\Phi}_m} : L^{p_m}(X, d\mu) & \rightarrow & L^{p_m}(X, d\mu). \end{cases}$$

□

We now give some applications of Theorem 3. For simplicity let  $q = p$ , and let  $\Psi, \Phi_1, \dots, \Phi_m$  be the Young functions defined by

$$\Psi(t) = t^p (\log(1+t))^{p-1+\epsilon}, \Phi_1(t) = t^{p_1} (\log(1+t))^{p_1-1+\epsilon}, \dots, \Phi_m(t) = t^{p_m} (\log(1+t))^{p_m-1+\epsilon}.$$

Notice that these functions satisfy conditions (4.2) and (4.3). We denote the Orlicz spaces as

$$L_{\Psi} = L^p(\log L)^{p-1+\epsilon}, L_{\Phi_1} = L^{p_1}(\log L)^{p_1-1+\epsilon}, \dots, L_{\Phi_m} = L^{p_m}(\log L)^{p_m-1+\epsilon}.$$

Thus as a corollary we have the following result.

**Corollary 2.** *Suppose that  $1 < p_1, \dots, p_m < \infty$ ,  $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ ,  $(X, \rho, \mu)$  is a space of homogeneous type, and  $K$  is a kernel such that (2.6) holds with  $\varphi$  satisfying*

(2.11). Let  $\epsilon > 0$  and  $u, v_1, \dots, v_m$  be weights that satisfy

$$(4.6) \quad \sup_B \varphi(B) \mu(B)^{1/q+1/p'_1+\dots+1/p'_m} \|u\|_{L^p(\log L)^{p-1+\epsilon, B}} \prod_{i=1}^m \|v_i^{-1}\|_{L^{p'_i}(\log L)^{p'_i-1+\epsilon, B}} < \infty$$

if  $p > 1$ , or

$$(4.7) \quad \sup_B \varphi(B) \mu(B)^{1/q+1/p'_1+\dots+1/p'_m} \left( \int_B u^p d\mu \right)^{1/p} \prod_{i=1}^m \|v_i^{-1}\|_{L^{p'_i}(\log L)^{p'_i-1+\epsilon, B}} < \infty$$

if  $p \leq 1$ . Then

$$\left( \int_X \left( |u \mathcal{T} \vec{f}| \right)^p d\mu \right)^{1/p} \leq C \prod_{i=1}^m \left( \int_X (|f_i| v_i)^{p_i} d\mu \right)^{1/p_i}$$

for all  $\vec{f} \in L^{p_1}(X, v_1^{p_1} d\mu) \times \dots \times L^{p_m}(X, v_m^{p_m} d\mu)$ .

We now use these  $L(\log L)$  results to obtain a different estimate in terms of the fractional maximal function of a weight. Let  $\gamma$  be a functional on the balls of  $X$  and define  $M_\gamma$  as in [37] by

$$M_\gamma f(x) = \sup_{B: x \in B} \gamma(B) \int_B |f| d\mu.$$

When  $X = \mathbf{R}^n$ ,  $\gamma(B) = |B|^{\alpha/n-1}$  corresponds to the fractional maximal operator  $M_\alpha$ . Let  $M^k$  denote the  $k$ -th iterate of the Hardy-Littlewood maximal operator, i.e.,  $M^k = M_\mu \circ \dots \circ M_\mu$  ( $k$  times). Also, for  $1 < p < \infty$ ,  $[p]$  will denote the greatest integer less than or equal to  $p$ . We have the following result.

**Corollary 3.** *Let  $1 < p_1, \dots, p_m < \infty$ ,  $1/p = 1/p_1 + \dots + 1/p_m$ , and  $\mathcal{T}$ ,  $K$ , and  $\varphi$  be as in Theorem 2. Furthermore, suppose  $0 < \alpha_1, \dots, \alpha_m < 1$  and  $\alpha_1 + \dots + \alpha_m = 1$ , and  $\tilde{\varphi}_i$  is the functional:  $B \mapsto (\varphi(B)^{\alpha_i} \mu(B))^{p_i} / \mu(B)$  and  $w$  be any weight. Then if  $p > 1$  we have*

$$(4.8) \quad \left( \int_X |\mathcal{T} \vec{f}|^p w d\mu \right)^{1/p} \leq C \prod_{i=1}^m \left( \int_X |f_i|^{p_i} M_{\tilde{\varphi}_i}(M^{[p]} w) d\mu \right)^{1/p_i}$$

and if  $p \leq 1$

$$(4.9) \quad \left( \int_X |\mathcal{T} \vec{f}|^p w d\mu \right)^{1/p} \leq C \prod_{i=1}^m \left( \int_X |f_i|^{p_i} M_{\tilde{\varphi}_i}(w) d\mu \right)^{1/p_i}.$$

Before we present the proof of Corollary 3 a few remarks are in order.

*Remark 7.* Inequalities (4.9) and (4.8) are new even in the Euclidean setting and they constitute the multilinear counterparts to the linear ones in [37, Theorem 2.5].

Also, in the Euclidean setting and when  $\tilde{\varphi}_i(B) \approx r(B)^{p_i \alpha_i} / |B|$ ,  $i = 1, \dots, m$  for  $\alpha_1 + \dots + \alpha_m = \alpha$  (i.e.,  $\mathcal{T}$  is the multilinear fractional integral operator  $\mathcal{I}_\alpha$ ), inequalities (4.9) and (4.8) read as follows: If  $p > 1$

$$(4.10) \quad \left( \int_{\mathbf{R}^n} |\mathcal{I}_\alpha \vec{f}|^p w dx \right)^{1/p} \leq C \prod_{i=1}^m \left( \int_{\mathbf{R}^n} |f_i|^{p_i} M_{p_i \alpha_i}(M^{[p]} w) dx \right)^{1/p_i}$$

and if  $p \leq 1$

$$(4.11) \quad \left( \int_{\mathbf{R}^n} |\mathcal{I}_\alpha \vec{f}|^p w \, dx \right)^{1/p} \leq C \prod_{i=1}^m \left( \int_{\mathbf{R}^n} |f_i|^{p_i} M_{p_i \alpha_i}(w) \, dx \right)^{1/p_i}.$$

In turn, (4.10) and (4.11) arise as the multilinear versions of the linear inequalities of the form

$$(4.12) \quad \int_{\mathbf{R}^n} |I_\alpha f|^p w \, dx \leq C \int_{\mathbf{R}^n} |f|^p M_{\alpha p}(M^{[p]}w) \, dx,$$

which were addressed in [36] for  $p > 1$ . It must be observed that in the linear case ( $m = 1$ ), inequality (4.11) with  $p = 1$  is false, see [7, Theorem 2.1]. Therefore, if  $m > 1$ , inequality (4.11) (and, more generally, inequality (4.9)) allows for a range of indices forbidden in the linear case.

Finally, notice that inequality (4.10) does not follow (at least directly) from the fact that,

$$\mathcal{I}_\alpha \vec{f} \leq I_{\alpha_1} f_1 \cdots I_{\alpha_m} f_m.$$

Indeed, if one uses this product bound, followed by Hölder's inequality and (4.12), one obtains

$$(4.13) \quad \begin{aligned} \left( \int_{\mathbf{R}^n} |\mathcal{I}_\alpha \vec{f}|^p w \, dx \right)^{1/p} &\leq \left( \int_{\mathbf{R}^n} \prod_{i=1}^m |I_{\alpha_i} f_i| w \, dx \right)^{1/p} \\ &\leq \prod_{i=1}^m \left( \int_{\mathbf{R}^n} |I_{\alpha_i} f_i|^{p_i} w \, dx \right)^{1/p_i} \\ &\leq \prod_{i=1}^m \left( \int_{\mathbf{R}^n} |f_i|^{p_i} M_{p_i \alpha_i}(M^{[p_i]}w) \, dx \right)^{1/p_i}. \end{aligned}$$

However, since  $p < p_i$  inequality (4.10) is sharper than (4.13).

*Proof of Corollary 3.* In order to prove Corollary 3 we will show that there exists  $\epsilon > 0$  such that the weights

$$u = w^{1/p}, v_1 = M_{\tilde{\varphi}_1}(w)^{1/p_1}, \dots, v_m = M_{\tilde{\varphi}_m}(w)^{1/p_m}$$

satisfy (4.7) if  $p \leq 1$  or the weights

$$u = w^{1/p}, v_1 = M_{\tilde{\varphi}_1}(M^{[p]}w)^{1/p_1}, \dots, v_m = M_{\tilde{\varphi}_m}(M^{[p]}w)^{1/p_m}$$

satisfy (4.6) if  $p > 1$ . We start with the case  $p \leq 1$ . Notice that for any ball  $B$  and  $x \in B$  we have

$$M_{\tilde{\varphi}_i}(w)(x) \geq \frac{(\varphi(B)^{\alpha_i} \mu(B))^{p_i}}{\mu(B)} \int_B w \, d\mu.$$

Hence,

$$\prod_{i=1}^m \|M_{\tilde{\varphi}_i}(w)^{-1/p_i}\|_{L^{p'_i}(\log L)^{p'_i-1+\epsilon}, B} \leq \frac{1}{\varphi(B) \mu(B)^m} \prod_{i=1}^m \left( \int_B w \, d\mu \right)^{-1/p_i}$$

from which (4.7) follows. Now for the case  $p > 1$ . Let  $\delta > 0$  and  $\hat{B} = (1 + \delta)\kappa B$ . For any  $x \in B$  we have

$$(M_{\hat{\varphi}_i}(M^{[p]}w)(x))^{-1/p_i} \leq \frac{1}{\varphi(\hat{B})^{\alpha_i} \mu(\hat{B})} \left( \int_{\hat{B}} M^{[p]}w \, d\mu \right)^{-1/p_i}.$$

Hence,

$$\begin{aligned} & \varphi(B)\mu(B)^m \|w^{1/p}\|_{L^p(\log L)^{p-1+\epsilon}, B} \prod_{i=1}^m \|M_{\hat{\varphi}_i}(M^{[p]}w)^{-1/p_i}\|_{L^{p'_i}(\log L)^{p'_i-1+\epsilon}, B} \\ & \leq C \|w^{1/p}\|_{L^p(\log L)^{p-1+\epsilon}, B} \prod_{i=1}^m \left( \int_{\hat{B}} M^{[p]}w \, d\mu \right)^{-1/p_i} \\ & = C \|w^{1/p}\|_{L^p(\log L)^{p-1+\epsilon}, B} \left( \int_{\hat{B}} M^{[p]}w \, d\mu \right)^{-1/p}, \end{aligned}$$

where we have used the reverse doubling properties of  $\varphi$  and  $\mu$  (see (2.3) and (2.9)). Choosing  $\epsilon = [p] - p + 1 > 0$  we have

$$\|w^{1/p}\|_{L^p(\log L)^{p-1+\epsilon}, B}^p = \|w\|_{L(\log L)^{p-1+\epsilon}, B} = \|w\|_{L(\log L)^{[p]}, B}.$$

However, Lemma 8.5 in [37] shows that for any  $\delta > 0$ ,

$$\|w\|_{L(\log L)^{[p]}, B} \leq C \int_{\hat{B}} M^{[p]}w \, d\mu.$$

□

## 5. PROOF OF THEOREM 1

Let  $\Omega$  be an open connected subset of  $\mathbf{R}^n$  and  $Y = \{Y_k\}_{k=1}^M$  a family of real-valued, infinitely differentiable vector fields. We identify the  $Y_j$ 's with the first order differential operators acting on Lipschitz functions defined on  $\Omega$  by the formula

$$Y_k f(x) = Y_k(x) \cdot \nabla f(x), \quad k = 1, \dots, M,$$

and we set  $Yf = (Y_1f, Y_2f, \dots, Y_Mf)$  and

$$|Yf(x)| = \left( \sum_{k=1}^M |Y_k f(x)|^2 \right)^{1/2}, \quad x \in \Omega.$$

*Definition 1.* Let  $\Omega$  and  $Y$  be as above.  $Y$  is said to satisfy Hörmander's condition in  $\Omega$  if there exists an integer  $M_0$  such that the family of commutators of vector fields in  $Y$  up to length  $M_0$ , i.e., the family of vector fields  $Y_1, Y_2, \dots, Y_{M_0}, [Y_{k_1}Y_{k_2}], \dots, [Y_{k_1}, [Y_{k_2}, [\dots, Y_{k_{M_0}}]] \dots]$ , span  $\mathbf{R}^n$  at every point of  $\Omega$ .

Suppose that  $Y = \{Y_k\}_{k=1}^M$  satisfies Hörmander's condition in  $\Omega$ . Let  $C_Y$  be the family of absolutely continuous curves  $\zeta : [a, b] \rightarrow \Omega$ ,  $a \leq b$ , such that there exist measurable functions  $c_j(t)$ ,  $a \leq t \leq b$ ,  $j = 1, \dots, M$ , satisfying  $\sum_{j=1}^M c_j(t)^2 \leq 1$  and  $\zeta'(t) = \sum_{j=1}^M c_j(t)Y_j(\zeta(t))$  for almost every  $t \in [a, b]$ . If  $x, y \in \Omega$  define

$$\rho(x, y) = \inf\{T > 0 : \text{there exists } \zeta \in C_Y \text{ with } \zeta(0) = x \text{ and } \zeta(T) = y\}.$$

The function  $\rho$  is in fact a metric in  $\Omega$  called the Carnot-Carathéodory metric associated to  $Y$ . A detailed study of the geometry of Carnot-Carathéodory spaces can be found in Nagel-Stein-Wainger [33].

*Remark 8.* Let  $Y$  satisfy Hörmander's condition in  $\Omega$  with integer  $M_0$  and let  $\rho$  be the associated Carnot-Carathéodory metric. Nagel-Stein-Wainger [33] proved that for every compact set  $K \subset \Omega$  there exist positive constants  $R_0$ ,  $C$ ,  $C_1$  and  $C_2$  depending on  $K$  such that

$$|B_\rho(x, 2r)| \leq C |B_\rho(x, r)|, \quad x \in K, r < R_0$$

and

$$C_1 |x - y| \leq \rho(x, y) \leq C_2 |x - y|^{1/M_0}, \quad x, y \in K.$$

When  $\Omega$  is bounded, as noted by Bramanti-Brandolini [2, p.534], the last inequality implies that one can actually take  $R_0 = \infty$ . Moreover, Bramanti-Brandolini [2, p.533] proved that if  $B$  is a  $\rho$ -ball contained in  $K$  and  $\Omega$  is bounded then there exists a positive constant  $C = C_{\Omega, K, Y}$  such that

$$(5.1) \quad |B_\rho(x, r) \cap B| \geq C |B_\rho(x, r)|, \quad x \in B, 0 < r < \text{diam}_\rho(B).$$

As a consequence, the triple  $(B, \rho, \text{Lebesgue measure})$  becomes a space of homogeneous type for all  $\rho$ -balls  $B$  contained in  $K$  and with uniform doubling constants that depend on  $\Omega$ ,  $K$  and  $Y$ . It is often found in the literature the claim that for any compact set  $K \subset \Omega$  the triple  $(K, \rho, \text{Lebesgue measure})$  constitutes a space of homogeneous type. However, that is not true in general. Indeed, just in the Euclidean setting, simple examples where  $K \subset B(0, 1) \subset \mathbf{R}^n$  has an exponentially pronounced cusp will show so. On the other hand, some regularity properties for  $\partial K$  will ensure, in general, that  $(K, \rho, \text{Lebesgue measure})$  is a space of homogeneous type. One such property, and one that every  $\rho$ -ball  $B \subset \Omega$  does possess, is expressed by inequality (5.1).

*Remark 9.* A reverse doubling property in the context of Carnot-Carathéodory spaces was proved by Franchi-Wheeden [16, pp.82-89]. More precisely, let  $Y$  be a collection of vector fields satisfying Hörmander's condition in  $\Omega$  and let  $\rho$  be the associated Carnot-Carathéodory metric. If  $\Omega_0 \subset\subset \Omega$  is an open bounded set and  $\tau > 1$ , then there are positive constants  $R_0 = R_0(\Omega, \Omega_0, Y)$  and  $C = C(\Omega, \Omega_0, Y)$  such that if  $B$  is a  $\rho$ -ball contained in  $\Omega_0$  of radius smaller than  $R_0$  and  $B_1$  and  $B_2$  are  $\rho$ -balls such  $B_1 \subset B_2 \subset \tau B$  then

$$(5.2) \quad \frac{|B_2|}{|B_1|} \geq C \frac{r(B_2)}{r(B_1)}.$$

Let  $\Omega_1 \subset \mathbf{R}^{n_1}$  and  $\Omega_2 \subset \mathbf{R}^{n_2}$  be open connected sets. Given two families of vector fields  $Y^{(1)}$  on  $\Omega_1$  and  $Y^{(2)}$  on  $\Omega_2$  the union of the two sets is defined as the collection  $Y$  of vector fields defined on  $\Omega_1 \times \Omega_2 \subset \mathbf{R}^{n_1+n_2}$  obtained by adjoining zero coordinates appropriately to the vectors in  $Y^{(1)}$  and  $Y^{(2)}$  to obtain vectors in  $\mathbf{R}^{n_1+n_2}$ . We note that if  $Y^{(1)}$  and  $Y^{(2)}$  satisfy Hörmander's condition in  $\Omega_1$  and  $\Omega_2$ , respectively, then so does  $Y$  in  $\Omega_1 \times \Omega_2$ . The following lemma (Lu-Wheeden [25, Lemma 1]) describes

the relation between the Carnot-Carathéodory metrics associated to  $Y^{(1)}$ ,  $Y^{(2)}$  and  $Y$ .

**Lemma 2.** *Let  $d_1$  and  $d_2$  be Carnot-Carathéodory metrics associated with Hörmander vector fields  $Y^{(1)}$  and  $Y^{(2)}$  in  $\Omega_1$  and  $\Omega_2$ , respectively. Let  $d$  be the metric in  $\Omega_1 \times \Omega_2$  associated with the union  $Y$  of the two collections. Then if  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  are any two points in  $\Omega_1 \times \Omega_2$ ,*

$$d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

*Remark 10.* Let  $\vec{x} = (x_1, x_2)$  with  $x_1 \in \Omega_1$  and  $x_2 \in \Omega_2$  and  $r > 0$ . Lemma 2 implies that  $B_d(\vec{x}, r) = B_{d_1}(x_1, r) \times B_{d_2}(x_2, r)$ .

We now deduce a multilinear representation formula in the setting of Carnot-Carathéodory spaces. Theorem 1 will follow from Corollary 4 below and the weighted boundedness properties of the multilinear fractional operators in Corollary 1.

**Theorem 4** (Representation formula). *Suppose  $Y$  is a collection of vector fields on a connected bounded open set  $\Omega \subset \mathbf{R}^n$  satisfying Hörmander's condition,  $\rho$  is the associated Carnot-Carathéodory metric and  $\Omega_0 \subset\subset \Omega$  is an open set. There exist positive constants  $r_0 = r_0(\Omega, \Omega_0, Y)$  and  $C_{\Omega, \Omega_0, Y}$  such that for all  $\rho$ -ball  $B \subset \overline{\Omega_0}$  with radius less than  $r_0$  and for all  $f \in C^1(\overline{B})$ ,*

$$(5.3) \quad |f(x) - f_B| \leq C_{\Omega, \Omega_0, Y} \int_B |Yf(y)| \frac{\rho(x, y)}{|B_\rho(x, \rho(x, y))|} dy, \quad x \in B.$$

*Proof.* This is essentially a consequence of Theorem 1 in Lu-Wheeden [26], we only need to check that the hypotheses (H1)-(H3) in that theorem hold true for any  $\rho$ -ball  $B \subset \overline{\Omega_0}$  with radius sufficiently small and with constants depending only on  $\Omega$  and  $Y$ .

Hypothesis (H1): In our context (H1) can be stated as the existence of positive constants  $a_1 \geq 1$  and  $C_1 > 0$  such that for all  $\rho$ -balls  $\tilde{B}$  with  $a_1 \tilde{B} \subset B$

$$(5.4) \quad \int_{\tilde{B}} |f - f_{\tilde{B}}| dx \leq C_1 r(\tilde{B}) \int_{a_1 \tilde{B}} |Yf| dx,$$

where, again,  $r(\tilde{B})$  denotes the radius of  $\tilde{B}$ . Inequality (5.4) holds true as a consequence of Jerison's Poincaré estimate in [19] if the radius of  $B$  is sufficiently small. More precisely, for every compact set  $K \subset \Omega$  there are constants  $C_{K, Y}$  and  $r_{K, Y}$  such that for  $u \in \text{Lip}(\tilde{B})$

$$\int_{\tilde{B}} |f - f_{\tilde{B}}| dx \leq C_{K, Y} r(\tilde{B}) \int_{2\tilde{B}} |Yf| dx,$$

whenever  $\tilde{B}$  is a  $\rho$ -ball centered at  $K$  and radius  $r(\tilde{B}) < r_{K, Y}$  (see Hajlasz-Koskela [18, Theorem 11.20]). Then (H1) follows with  $a_1 = 2$  and  $C_1 = C_{\overline{\Omega_0}, Y}$  if we choose  $K = \overline{\Omega_0}$  and  $B \subset \overline{\Omega_0}$  with radius smaller than  $r_{\overline{\Omega_0}, Y}$ . In fact, Jerison proved the inequality with the  $L^2$  norms on both sides, but the same arguments work with the  $L^1$  norm. He also proved that one can take  $a_1 = 1$ .

Hypothesis (H2): (H2) is the reverse doubling condition (5.2), which, as mentioned before, was proved in Franchi-Wheeden [16, pp.82-89].

Hypothesis (H3): (H3) is the ‘segment property’ for every ball  $\tilde{B} \subset B$ , which holds for any metric induced by a collection of Carnot-Carathéodory vector fields (see Franchi-Wheeden [16, p.66 and Example 3 (p.82)] and note that the notion of ‘segment property’ in Franchi-Wheeden[16] easily implies the notion of ‘segment’ property in Lu-Wheeden [26, p.580]).

We have then checked that hypotheses (H1)-(H3) hold with constants depending only on  $\Omega_0$  and  $Y$  and for any  $\rho$ -ball  $B \subset \overline{\Omega_0}$  with radius smaller than  $r_0$  defined as the minimum of the upper bounds obtained for the radii in (H1) and (H2). Therefore, the representation formula (5.3) holds true by Theorem 1 in Lu-Wheeden [26].

It must be noticed that (5.3) holds true for every  $x \in B$  and not only for a.e.  $x \in B$ . The proof of Theorem 1 in Lu-Wheeden [26] depends on the representation formula of Lemma 3 in Lu-Wheeden [26]. Lemma 3, in turn, is based on Theorem 1 in Franchi-Wheeden [16] and a close examination of its proof shows that, in our case, it actually holds for every  $x \in B$  since we are assuming that  $f$  is continuous and therefore every point in  $B$  is a Lebesgue point of  $f$ . Another explanation for the fact that (5.3) holds for every  $x \in B$  would be that from Lu-Wheeden [26], the reasoning above gives that (5.3) holds a.e. in  $B$ . This and the fact that both sides of the inequality are continuous in  $x$  give that (5.3) holds for every  $x \in B$ .  $\square$

**Corollary 4** (Multilinear representation formula). *Suppose  $Y$  is a collection of vector fields on an open bounded connected set  $\Omega \subset \mathbf{R}^n$  satisfying Hörmander’s condition and  $\tilde{Y}$  is the vector field defined on  $\Omega^m$  that is the union of  $m$  copies of  $Y$ . Denote by  $\rho$  the Carnot-Carathéodory metric in  $\Omega$  associated to  $Y$  and by  $\tilde{\rho}$  the Carnot-Carathéodory metric in  $\Omega^m$  associated to  $\tilde{Y}$ . Let  $\Omega_0 \subset\subset \Omega$  be an open set. There exist positive constants  $r_0 = r_0(\Omega, \Omega_0, Y)$  and  $C_{\Omega, \Omega_0, Y}$  such that for all  $\rho$ -ball  $B \subset \Omega_0$  with radius less than  $r_0$  and for all  $f_k \in C^1(\overline{B})$ ,  $k = 1, \dots, m$ ,*

$$(5.5) \quad \left| \prod_{k=1}^m f_k(x) - \prod_{k=1}^m f_{kB} \right| \leq C_{\Omega, \Omega_0, Y} \sum_{k=1}^m \mathcal{I}_{B,1}(f_1 \chi_B, \dots, (Y f_k) \chi_B, \dots, f_m \chi_B)(x),$$

for all  $x \in B$ .

*Proof.* Let  $r_0$  be given by Theorem 4 when applied to  $\Omega^m$ ,  $\Omega_0^m$  and  $\tilde{Y}$ . If  $B$  is a  $\rho$ -ball of radius less than  $r_0$  contained in  $\Omega_0$ , by Lemma 2,  $B^m$  is a  $\tilde{\rho}$ -ball contained in  $\Omega_0^m$  of radius less than  $r_0$ . Theorem 4 with  $f(\vec{y}) = \prod_{k=1}^m f_k(y_k)$  gives

$$\begin{aligned} \left| f(\vec{x}) - \frac{1}{|B^m|} \int_{B^m} f(\vec{y}) d\vec{y} \right| &\leq C_{\Omega, \Omega_0, Y} \int_{B^m} |\tilde{Y} f(\vec{y})| \frac{\tilde{\rho}(\vec{x}, \vec{y})}{|B_{\tilde{\rho}}(\vec{x}, \tilde{\rho}(\vec{x}, \vec{y}))|} d\vec{y} \\ &= C_{\Omega, \Omega_0, Y} \int_{B^m} |\tilde{Y} f(\vec{y})| \frac{\tilde{\rho}(\vec{x}, \vec{y})}{\prod_{k=1}^m |B_{\rho}(x_k, \tilde{\rho}(\vec{x}, \vec{y}))|} d\vec{y}, \end{aligned}$$

for all  $\vec{x} = (x_1, \dots, x_m) \in B^m$ . Taking  $x_1 = \dots = x_m = x \in B$  we obtain

$$\begin{aligned} \left| \prod_{k=1}^m f_k(x) - \prod_{k=1}^m f_{kB} \right| &\leq C_{\Omega, \Omega_0, Y} \int_{B^m} \left| \tilde{Y} f(\vec{y}) \right| \frac{\tilde{\rho}(\vec{x}, \vec{y})}{|B_\rho(x, \tilde{\rho}(\vec{x}, \vec{y}))|^m} d\vec{y} \\ &\lesssim \int_{B^m} \sum_{k=1}^m |f_1(y_1) \chi_B(y_1) \cdots (Y f_k(y_k)) \chi_B(y_k) \cdots f_m(y_m) \chi_B(y_m)| \frac{\tilde{\rho}(\vec{x}, \vec{y})}{|B_\rho(x, \tilde{\rho}(\vec{x}, \vec{y}))|^m} d\vec{y}, \end{aligned}$$

where the constants depend only on  $\Omega$ ,  $\Omega_0$  and  $Y$ . Since  $\tilde{\rho}(\vec{x}, \vec{y}) \sim \rho(x, \vec{y})$ , it follows from Remark 8 that  $|B_\rho(x, \tilde{\rho}(\vec{x}, \vec{y}))| \sim |B_\rho(x, \rho(x, \vec{y}))|$  uniformly for  $x \in \Omega_0$ , and therefore we obtain (5.5).  $\square$

*Proof of Theorem 1.* Let  $r_0$  be the minimum of the radii given by Remark 9 and by Corollary 4 when applied to  $\Omega$ ,  $\Omega_0$  and  $Y$  as in the hypotheses of Theorem 1. Let  $B$  be a  $\rho$ -ball contained in  $\Omega_0$  of radius smaller than  $r_0$ . The Poincaré inequality (1.10) will follow from the multilinear representation formula (5.5) once we have checked that  $(B, \rho, \text{Lebesgue measure})$  satisfies the hypotheses of Corollary 1 with uniform constants depending only on  $\Omega$ ,  $\Omega_0$  and  $Y$ .

As noted in Remark 8,  $(B, \rho, \text{Lebesgue measure})$  is a space of homogeneous type with doubling constant uniform in  $B$  depending on  $\Omega$ ,  $\Omega_0$  and  $Y$ .

The reverse doubling condition (2.3) in this context means that there are positive constants  $c$  and  $\delta$ , depending only on  $\Omega$ ,  $\Omega_0$  and  $Y$ , such that

$$\frac{|B_\rho(x_1, r_1) \cap B|}{|B_\rho(x_2, r_2) \cap B|} \geq c \left( \frac{r_1}{r_2} \right)^\delta,$$

whenever  $B_\rho(x_2, r_2) \subset B_\rho(x_1, r_1)$ ,  $x_1, x_2 \in B$ , and  $0 < r_1, r_2 \leq 2 \text{diam}_\rho(B)$ . By (5.1), this reduces to prove that there are positive constants  $c$  and  $\delta$ , depending only on  $\Omega$ ,  $\Omega_0$  and  $Y$ , such that

$$(5.6) \quad \frac{|B_\rho(x_1, r_1)|}{|B_\rho(x_2, r_2)|} \geq c \left( \frac{r_1}{r_2} \right)^\delta,$$

whenever  $B_\rho(x_2, r_2) \subset B_\rho(x_1, r_1)$ ,  $x_1, x_2 \in B$ , and  $0 < r_1, r_2 \leq 2 \text{diam}_\rho(B)$ . This is the result proved by Franchi-Wheeden [16] with  $\delta = 1$  and  $\tau = 5$  as indicated in Remark 9.

As explained in Remark 4, (2.11) is satisfied with  $\epsilon = \alpha$  and  $C_2$  depends only on structural constants independent of  $B$  if  $C_1 = 2\kappa a_1 A m$  as given in the proof of Theorem 2 at the moment of applying the result of Lemma 1.

The growth condition (2.6) for the kernel (2.8) in this context means that for every positive constant  $C_1$  there exists a positive constant  $C_2 = C_2(\Omega, \Omega_0, Y)$  such that for all  $\vec{x}, \vec{y}, \vec{z} \in B^m$ ,

$$\begin{aligned} \frac{\rho(\vec{x}, \vec{y})}{\prod_{k=1}^m |B_\rho(x_k, \rho(\vec{x}, \vec{y})) \cap B|} &\leq C_2 \frac{\rho(\vec{z}, \vec{y})}{\prod_{k=1}^m |B_\rho(z_k, \rho(\vec{z}, \vec{y})) \cap B|}, \quad \rho(\vec{z}, \vec{y}) \leq C_1 \rho(\vec{x}, \vec{y}) \\ \frac{\rho(\vec{x}, \vec{y})}{\prod_{k=1}^m |B_\rho(x_k, \rho(\vec{x}, \vec{y})) \cap B|} &\leq C_2 \frac{\rho(\vec{y}, \vec{z})}{\prod_{k=1}^m |B_\rho(y_k, \rho(\vec{y}, \vec{z})) \cap B|}, \quad \rho(\vec{y}, \vec{z}) \leq C_1 \rho(\vec{x}, \vec{y}). \end{aligned}$$

Consider the Carnot-Carathéodory space given by  $\Omega^m$  and the Hörmander vector field  $\tilde{Y}$  made of  $m$  copies of  $Y$  and let  $\tilde{\rho}$  be the associated Carnot-Carathéodory metric. Recalling that  $(B, \rho, \text{Lebesgue measure})$  is a space of homogeneous type with uniform constants, using Lemma 2 and (5.1), the growth condition stated above reduces to have that for every positive constant  $C_1$  there exists a positive constant  $C_2 = C_2(\Omega, \Omega_0, Y)$  such that for all  $\vec{x}, \vec{y}, \vec{z} \in B^m$ ,

$$\begin{aligned} \frac{\tilde{\rho}(\vec{x}, \vec{y})}{|B_{\tilde{\rho}}(\vec{x}, \tilde{\rho}(\vec{x}, \vec{y}))|} &\leq C_2 \frac{\tilde{\rho}(\vec{z}, \vec{y})}{|B_{\tilde{\rho}}(\vec{z}, \tilde{\rho}(\vec{z}, \vec{y}))|}, & \text{if } \tilde{\rho}(\vec{z}, \vec{y}) \leq C_1 \tilde{\rho}(\vec{x}, \vec{y}) \\ \frac{\tilde{\rho}(\vec{x}, \vec{y})}{|B_{\tilde{\rho}}(\vec{x}, \tilde{\rho}(\vec{x}, \vec{y}))|} &\leq C_2 \frac{\tilde{\rho}(\vec{y}, \vec{z})}{|B_{\tilde{\rho}}(\vec{y}, \tilde{\rho}(\vec{y}, \vec{z}))|}, & \text{if } \tilde{\rho}(\vec{y}, \vec{z}) \leq C_1 \tilde{\rho}(\vec{x}, \vec{y}). \end{aligned}$$

These inequalities follow from the reverse doubling property in the Carnot-Carathéodory space given by  $\Omega^m$  and  $\tilde{Y}$  (see Remark 9) and the doubling property of Lebesgue measure on  $\rho$ -balls with center in  $\Omega_0$ .

Finally, the weight conditions (2.14) and (2.15) with  $\alpha = 1$  for balls in  $(B, \rho)$  and with constants that do not depend on  $B$ , follow from (1.8) and (1.9), respectively, and (5.1).  $\square$

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