DIOPHANTINE INEQUALITIES IN FUNCTION FIELDS

CRAIG V. SPENCER

ABSTRACT. This paper develops the Bentkus-Götze-Freeman variant of the Davenport-Heilbronn method for function fields in order to count $\mathbb{F}_q[t]$ -solutions to diagonal Diophantine inequalities in $\mathbb{F}_q((1/t))$.

1. INTRODUCTION

Over 60 years ago, the Davenport-Heilbronn method (see [2]) was introduced to study non-trivial integral solutions of Diophantine inequalities. Let k and s be positive integers with $k > 1$, and let τ be some fixed positive real number. Suppose that $\lambda_1, \dots, \lambda_s$ are non-zero real numbers, not all in rational ratio. Let $N_0(P, \boldsymbol{\lambda})$ denote the number of solutions $\mathbf{x} \in [-P, P]^s \cap \mathbb{Z}^s$ that satisfy

$$|\lambda_1 x_1^k + \dots + \lambda_s x_s^k| < \tau.$$

Plainly, in the case that k is an even number, we must impose the restriction that the numbers λ_i do not all share the same sign in order to guarantee the existence of a non-trivial solution of $\lambda_1 z_1^k + \dots + \lambda_s z_s^k = 0$ in $\mathbb{R}^s$. In [2], Davenport and Heilbronn proved that if $s > 2^k$, then $N_0(P_n, \boldsymbol{\lambda}) \gg P_n^{s-k}$ for a sequence $(P_n)_{n=1}^\infty$ which increases to infinity. This sequence is determined from the convergents of the continued fraction expansion for an irrational number of the form λ_i/λ_j , and as a result, the sequence $(P_n)_{n=1}^\infty$ may be arbitrarily sparse. In the last decade, the Bentkus-Götze-Freeman version of the Davenport-Heilbronn method (see [1], [5], [6], and [14]) has been used to establish an asymptotic formula for $N_0(P, \boldsymbol{\lambda})$, valid for all large enough values of P , provided that

$$s \geq k^2(\log k + \log \log k + O(1)),$$

and an asymptotic lower bound for $N_0(P, \boldsymbol{\lambda})$, valid for all large enough values of P , provided that

$$s \geq k(\log k + \log \log k + 2 + o(1)).$$

In this paper, we use the Bentkus-Götze-Freeman version of the Davenport-Heilbronn method to study the analogous problem in function fields.

In order to state our main result, it is first necessary to record some notation. Let $\mathbb{A} = \mathbb{F}_q[t]$ denote the ring of polynomials over $\mathbb{F}_q$, the finite field of q elements. Let

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$\mathbb{K}_\infty = \mathbb{F}_q((1/t))$ be the completion of $\mathbb{K} = \mathbb{F}_q(t)$ at the infinite place. In this paper, we wish to exploit the basic observation that $\mathbb{A}$ behaves like $\mathbb{Z}$, that $\mathbb{K}$ behaves like $\mathbb{Q}$, and that $\mathbb{K}_\infty$ behaves like $\mathbb{R}$.

Let k and s be positive integers with $k > 1$. Let p denote the characteristic of $\mathbb{F}_q$. Each non-zero element α in $\mathbb{K}_\infty$ can be written as $\alpha = \sum_{i \leq n} a_i t^i$, where each a_i is an element in $\mathbb{F}_q$ and $a_n \neq 0$. We define $\text{ord } \alpha$ to be n and $\text{lead}(\alpha)$ to be a_n in this situation. Furthermore, we define $\text{res } \alpha$ to be the coefficient of t^{-1} in such an expansion, and we adopt the convention that $\text{ord } 0 = -\infty$. There exists a natural non-Archimedean valuation $\langle x \rangle = q^{\text{ord } x}$ on $\mathbb{K}_\infty$. For any real number u , we will let $\hat{u}$ denote q^u . For a positive number x , we let $\text{Log } x = \max(1, \log x)$. When k has a base- p expansion $k = a_0 + a_1 p + \cdots + a_n p^n$ with $0 \leq a_i \leq p-1$ ($0 \leq i \leq n$), we define $\gamma(k) = \gamma_q(k)$ by

$$\gamma(k) = a_0 + a_1 + \cdots + a_n.$$

Define the constant $B = B_q(k)$ by

$$B_q(k) = \begin{cases} 1, & \text{when } k \leq 2^{\gamma-2}, \\ (1 - 2^{-\gamma(k)})^{-1}, & \text{when } k > 2^{\gamma-2}. \end{cases}$$

Let

$$s_{q,k} = Bk(\text{Log } k + \text{Log Log } k + 2 + B \text{Log Log } k / \text{Log } k).$$

We are now in a position to state the main result of this paper.

Theorem 1.1. *There exists a positive absolute constant C with the following property. Suppose that k and s are natural numbers with $k > 1$,*

$$s \geq s_{q,k} + Ck\sqrt{\text{Log Log } k} / \text{Log } k,$$

and $\text{char}(\mathbb{F}_q) \nmid k$. Let τ be some fixed integer, and let $\lambda_1, \dots, \lambda_s$ be fixed non-zero elements of $\mathbb{K}_\infty$, not all in $\mathbb{F}_q(t)$ -rational ratio. Suppose also that the equation

$$\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$$

has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. Then, for all sufficiently large positive real numbers P , the number of $\mathbb{F}_q[t]$ -solutions $N(P, \boldsymbol{\lambda})$ of

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k \rangle < \hat{\tau}, \tag{1.1}$$

with $\langle x_i \rangle \leq \hat{P}$ ($1 \leq i \leq s$), satisfies

$$N(P, \boldsymbol{\lambda}) \gg \hat{P}^{s-k}. \tag{1.2}$$

In (1.2), the implicit constant may depend on $s, k, q, \boldsymbol{\lambda}$, and τ .

A few comments about the above theorem are in order. If $p \nmid k$, then $\gamma_q(k) \geq 2$, and it follows that $B_q(k)$ satisfies $1 \leq B_q(k) \leq 4/3$. Also, one should note that our function $s_{q,k}$ corresponds to the quantity $\hat{G}_q(k)$ defined in [10] in the context of Waring's problem in function fields, and our work depends on mean value estimates arising from the use of efficient differencing technology for the latter problem. By incorporating any improvements to this machinery into the arguments of this paper, one would be able to get comparable improvements in Theorem 1.1. In the case that $k < \text{char}(\mathbb{F}_q)$, a combination of Proposition 13 of [9], the amplification method discussed in Section 1 of [14], and the

ideas in this paper would give a result similar to Theorem 1.1 with $s \geq 2^k + 1$. This bound would be stronger than that of Theorem 1.1 for small values of k . Also, by Proposition 6.1, the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ has a non-trivial solution $\mathbf{z} \in \mathbb{K}_\infty^s$ whenever $s \geq k^2 + 1$, whenever $q > k^4$ and $s \geq 2k + 1$, or whenever $(k, q - 1) = 1$ and $s \geq k + 1$. Lastly, it is worth noting that the question of finding solutions to (1.1), where each x_i is a monic, irreducible polynomial in $\mathbb{F}_q[t]$, has already been studied by Hsu in [7] through the use of the Davenport-Heilbronn method. Hsu's paper was restricted to the case that $k < p$ and

$$s \geq \begin{cases} 2^k + 1, & \text{when } 2 \leq k < 11, \\ 2[2k^2 \log k + k^2 \log \log k + 2k^2 - 2k] + 1, & \text{when } k \geq 11. \end{cases}$$

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2. THE DAVENPORT-HEILBRONN METHOD FOR FUNCTION FIELDS

In this section, we set up the Davenport-Heilbronn Method for function fields. We will combine Hsu's version of the Davenport-Heilbronn method (see [7]) with the ideas of Bentkus and Götze (see [1]) and those of Freeman (see [5] and [6]) in order to prove Theorem 1.1.

Define a non-trivial additive character $e_q : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ by $e_q(a) = e^{2\pi i \text{tr}(a)/p}$, where $\text{tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$ denotes the trace map. This character induces a map $e : \mathbb{K}_\infty \rightarrow \mathbb{C}^\times$ defined by $e(\alpha) = e_q(\text{res } \alpha)$. Let $\mathbb{T}$ be the compact additive subgroup of $\mathbb{K}_\infty$ given by $\mathbb{T} = \{\alpha \in \mathbb{K}_\infty : \langle \alpha \rangle < 1\}$, and normalize a Haar measure $d\alpha$ on $\mathbb{K}_\infty$ so that $\int_{\mathbb{T}} d\alpha = 1$. Note that this measure is translation-invariant and that for any integer r , we have

$$\int_{\langle \alpha \rangle < \hat{r}} d\alpha = \hat{r}.$$

By Lemma 2.2 of [7], for $\tau \in \mathbb{Z}$, if we define a function $\chi_\tau : \mathbb{K}_\infty \rightarrow \mathbb{R}$ by

$$\chi_\tau(\alpha) = \begin{cases} \hat{r}, & \text{when } \langle \alpha \rangle < \hat{r}^{-1}, \\ 0, & \text{when } \langle \alpha \rangle \geq \hat{r}^{-1}, \end{cases}$$

we obtain a method of detecting when $\langle \beta \rangle < \hat{r}$ for an element $\beta \in \mathbb{K}_\infty$ by noting that

$$\int_{\mathbb{K}_\infty} e(\alpha\beta) \chi_\tau(\alpha) d\alpha = \begin{cases} 1, & \text{when } \langle \beta \rangle < \hat{r}, \\ 0, & \text{when } \langle \beta \rangle \geq \hat{r}. \end{cases} \quad (2.1)$$

When R and P are positive numbers with $R \leq P$, we define the set of R -smooth polynomials $\mathcal{A}(P, R)$ to be the set of all $x \in \mathbb{A}$ satisfying both $\langle x \rangle \leq \hat{P}$ and the property that whenever $\varpi | x$ for an irreducible polynomial ϖ , then $\langle \varpi \rangle \leq \hat{R}$. We now can define our classical Weyl

sum

$$F(\alpha) = F(\alpha, P) = \sum_{\langle x \rangle \leq \widehat{P}} e(\alpha x^k)$$

and our smooth Weyl sum

$$f(\alpha) = f(\alpha, P, R) = \sum_{x \in \mathcal{A}(P, R)} e(\alpha x^k).$$

Let $F_i(\alpha) = F(\lambda_i \alpha)$ for $1 \leq i \leq s$, and let $f_j(\alpha) = f(\lambda_j \alpha)$ for $3 \leq j \leq s$. It now follows from (2.1) that the integral

$$\int_{\mathbb{K}_\infty} F_1(\alpha) F_2(\alpha) f_3(\alpha) \cdots f_s(\alpha) \chi_\tau(\alpha) d\alpha \quad (2.2)$$

counts the number of solutions $\mathbf{x} \in \mathbb{A}^s$ of

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k \rangle < \widehat{\tau}$$

with $\langle x_i \rangle \leq \widehat{P}$ ($i = 1, 2$) and $x_j \in \mathcal{A}(P, R)$ ($3 \leq j \leq s$). Thus, the integral in (2.2) serves as a lower bound for $N(P, \boldsymbol{\lambda})$. For the remainder of this paper, whenever R appears in a statement, implicitly or explicitly, we are asserting that there exists a positive number η_0 such that the statement holds whenever $R = \eta P$, where $0 < \eta \leq \eta_0$.

It is worth noting that in the setting of diagonal Diophantine inequalities with solutions in $\mathbb{Z}^s$ (see [14]), one may use s smooth Weyl sums instead of $s - 2$ smooth Weyl sums and two classical Weyl sums. Were one to use only smooth Weyl sums with the approach in this paper, the main challenge arises when proving a result similar to Lemma 3.1 for smooth Weyl sums. The necessary minor arc bound for $f(\alpha)$ is available (see [10, Corollary 13.3]), and one can implement the ideas of [12, Section 7] to obtain a major arc estimate for $f(\alpha)$ that is useful off of a slim set of the major arcs. However, the lack of function field analogues of various sieve bounds due to Fouvry and Tenenbaum in [4] is an impediment to proving an estimate for smooth Weyl sums on a very narrow set of the major arcs along the lines of [12, Lemma 8.5]. Were one to overcome this difficulty and prove an analogue of Lemma 3.1 for smooth Weyl sums, it would lead to a savings of one or two variables in Theorem 1.1.

Let $\mathbf{n}$ denote the set of elements α of $\mathbb{T}$ satisfying the property that whenever a and g are elements of $\mathbb{A}$ such that $\langle g\alpha - a \rangle < \widehat{P}^{1-k}$ and $g \neq 0$, then $\langle g \rangle > \widehat{P}$. We say that a positive number $u > 2k - 2$ is *accessible to the exponent k* when there exists a positive number δ for which

$$\int_{\mathbf{n}} |F(\alpha, P)^2 f(\alpha, P, R)^u| d\alpha \ll \widehat{P}^{u+2-k-\delta}.$$

By Theorem 9.4, Corollary 13.3, and Lemma 14.1 of [10], it follows that there exists a positive absolute constant C such that if

$$u + 5 \geq s_{q,k} + Ck\sqrt{\text{Log Log } k} / \text{Log } k,$$

then u is accessible to the exponent k . Hence, Theorem 1.1 is a consequence of the following result.

Theorem 2.1. *Suppose that k and s are natural numbers with $k > 1$ and $\text{char}(\mathbb{F}_q) \nmid k$. Furthermore, assume that $u > 2k - 2$ is accessible to the exponent k and that $s \geq u + 5$. Let τ be some fixed integer, and let $\lambda_1, \dots, \lambda_s$ be fixed non-zero elements of $\mathbb{K}_\infty$, not all in $\mathbb{F}_q(t)$ -rational ratio. Suppose also that the equation*

$$\lambda_1 z_1^k + \dots + \lambda_s z_s^k = 0 \quad (2.3)$$

has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. Then, for all sufficiently large positive real numbers P , the number of $\mathbb{F}_q[t]$ -solutions $N(P, \boldsymbol{\lambda})$ of

$$\langle \lambda_1 x_1^k + \dots + \lambda_s x_s^k \rangle < \widehat{\tau}, \quad (2.4)$$

with $\langle x_i \rangle \leq \widehat{P}$ ($1 \leq i \leq s$), satisfies

$$N(P, \boldsymbol{\lambda}) \gg \widehat{P}^{s-k}. \quad (2.5)$$

In (2.5), the implicit constant may depend on $s, k, q, \boldsymbol{\lambda}$, and τ .

With the exception of Section 6, which will be used to investigate the solvability of (2.3) in $\mathbb{K}_\infty$, the remainder of this paper will be devoted to proving Theorem 2.1. In order to analyze the integral in (2.2), we split up the subset of $\mathbb{K}_\infty$ for which the integrand is non-zero into two parts. Let

$$S_1(P) = (\text{Log } \widehat{P})^{1/8}.$$

Define the *major arc* by

$$\mathfrak{M} = \{\alpha \in \mathbb{K}_\infty : \langle \alpha \rangle < S_1(P) \widehat{P}^{-k}\}$$

and the *minor arc* by

$$\mathfrak{m} = \{\alpha \in \mathbb{K}_\infty : S_1(P) \widehat{P}^{-k} \leq \langle \alpha \rangle < \widehat{\tau}^{-1}\}.$$

Theorem 2.1 is proved by demonstrating that for large enough values of P , one has

$$\int_{\mathfrak{m}} F_1(\alpha) F_2(\alpha) f_3(\alpha) \dots f_s(\alpha) \chi_\tau(\alpha) d\alpha = o(\widehat{P}^{s-k}) \quad (2.6)$$

and

$$\int_{\mathfrak{M}} F_1(\alpha) F_2(\alpha) f_3(\alpha) \dots f_s(\alpha) \chi_\tau(\alpha) d\alpha \gg \widehat{P}^{s-k}. \quad (2.7)$$

We prove (2.6) in Section 4 (see Lemma 4.2) by combining mean value estimates with a Weyl-type estimate. The mean value estimates depend on the efficient differencing arguments in [10], and the Weyl-type estimate proved in Section 3 stems from the ideas of Bentkus, Götze, and Freeman in [1], [5], and [6]. We prove (2.7) in Section 5 (see Lemma 5.3) by using a line of attack similar to that of [9] and [10].

By multiplying each side of (2.4) by $\langle t^{-j} \rangle$ for some sufficiently large integer j , we may assume that $\widehat{\tau} < 1$ and $0 < \langle \lambda_i \rangle < 1$ for $1 \leq i \leq s$. Since $\lambda_1, \dots, \lambda_s$ are not all in $\mathbb{F}_q(t)$ -rational ratio, there is no loss of generality in supposing that $\lambda_2/\lambda_1 \notin \mathbb{K}$. Also, implicit constants in this paper may depend on $q, k, \boldsymbol{\lambda}, s, u, v$, and τ .

3. A WEYL-TYPE ESTIMATE

In this section, we mimic the work in Section 2 of [14] to show that when $\alpha \in \mathfrak{m}$, one has $|F_1(\alpha)F_2(\alpha)| = o(\widehat{P}^2)$. Recall that $\mathfrak{n}$ denotes the set of elements α of $\mathbb{T}$ satisfying the property that whenever a and g are elements of $\mathbb{A}$ such that $\langle g\alpha - a \rangle < \widehat{P}^{1-k}$ and $g \neq 0$, then $\langle g \rangle > \widehat{P}$. It follows from [11] that there exists a small positive constant $\nu = \nu(q, k)$ such that

$$\sup_{\alpha \in \mathfrak{n}} |F(\alpha)| \ll \widehat{P}^{1-\nu}.$$

Since one cannot use Weyl differencing to obtain such a bound without placing restrictions on the characteristic of $\mathbb{F}_q[t]$, this minor arc estimate is obtained via a combination of Vinogradov's mean value theorem with large sieve technology. It is worth noting that an asymptotic formula for $N(P, \boldsymbol{\lambda})$ would follow from a combination of the results of [11] with the ideas of this paper provided that s is sufficiently large in terms of k . Since solutions of (2.4) are perfectly detected via (2.1), some of the technical difficulties that arise when studying the analogous problem in $\mathbb{Z}$ can be avoided.

Our first lemma demonstrates that good Diophantine approximations are produced by large Weyl sums.

Lemma 3.1. *There is a positive constant c , depending at most on k and q , with the following property. Suppose that P is a real number, sufficiently large in terms of k and q , and suppose that δ is a positive number with $\widehat{P}^{-\nu/2} \leq \delta \leq 1$. Then, whenever $|F(\alpha)| \geq \delta \widehat{P}$, there exist a and g in $\mathbb{A}$ such that $(a, g) = 1$, $1 \leq \langle g \rangle \leq c\delta^{-k}$, and $\langle g\alpha - a \rangle \leq c\delta^{-k}\widehat{P}^{-k}$.*

Proof. Suppose that α is an element of $\mathbb{K}_\infty$ such that $|F(\alpha)| \geq \delta \widehat{P}$, where δ satisfies the hypothesis of the lemma. By Lemma 3 of [9], there exists a unique choice of a and g in $\mathbb{A}$ such that g is monic, $(a, g) = 1$, $1 \leq \langle g \rangle \leq \widehat{P}^{k-1}$, and $\langle g\alpha - a \rangle < \widehat{P}^{1-k}$.

Suppose that $\langle g \rangle > \widehat{P}$. It follows that $\alpha \in \mathfrak{n}$, implying that $|F(\alpha)| \ll \widehat{P}^{1-\nu}$. When P is sufficiently large in terms of k and q , one has

$$|F(\alpha)| < \frac{1}{2}\widehat{P}^{1-\nu/2} \leq \frac{1}{2}\delta\widehat{P},$$

which would contradict our hypothesis on δ . Hence, we may assume that $\langle g \rangle \leq \widehat{P}$. In the latter circumstance, by Lemma 4.1 of [10], we have

$$F(\alpha) \ll \widehat{P}(\langle g \rangle + \widehat{P}^k \langle g\alpha - a \rangle)^{-1/k}.$$

Thus, there exists a positive constant c such that

$$|F(\alpha)| \leq c^{1/k} \widehat{P}(\langle g \rangle + \widehat{P}^k \langle g\alpha - a \rangle)^{-1/k}.$$

By recalling that $\delta \widehat{P} \leq |F(\alpha)|$, we conclude that

$$\langle g \rangle + \widehat{P}^k \langle g\alpha - a \rangle \leq c\delta^{-k},$$

and the lemma follows. $\square$

We now use the hypothesis that $\lambda_2/\lambda_1 \notin \mathbb{K}$ to begin our study of the product $F_1(\alpha)F_2(\alpha)$ of Weyl sums.

Lemma 3.2. *Suppose that S is a fixed real number with $0 < S < \hat{\tau}^{-1}$. Then, one has*

$$\lim_{P \rightarrow \infty} \sup_{S \leq \langle \alpha \rangle < \hat{\tau}^{-1}} \hat{P}^{-2} |F_1(\alpha) F_2(\alpha)| = 0.$$

Proof. Suppose that the lemma fails. We can then find a real number δ in $(0, q^{-1})$, a sequence $(P_n)_{n=1}^\infty$ of real numbers that increases monotonically to infinity, and a sequence $(\alpha_n)_{n=1}^\infty$ of elements in $\mathbb{K}_\infty$ such that for all n , we have that $S \leq \langle \alpha_n \rangle < \hat{\tau}^{-1}$ and $|F_1(\alpha_n, P_n) F_2(\alpha_n, P_n)| \geq q \delta \hat{P}_n^2$. Since $|F_i(\alpha_n, P_n)| \leq q \hat{P}_n$ for $i \in \{1, 2\}$, one has $|F_i(\alpha_n, P_n)| \geq \delta \hat{P}_n$ for $i \in \{1, 2\}$. When n is large enough, say $n \geq r$, we have that $\hat{P}_n^{-\nu/2} \leq \delta$ and that $P = P_n$ is sufficiently large in the context of Lemma 3.1. By Lemma 3.1, for $i \in \{1, 2\}$ and $n \geq r$, there exist elements a_{in} and g_{in} in $\mathbb{A}$ such that $(a_{in}, g_{in}) = 1$, $1 \leq \langle g_{in} \rangle \leq c \delta^{-k}$, and

$$\langle g_{in} \lambda_i \alpha_n - a_{in} \rangle \leq c \delta^{-k} \hat{P}_n^{-k}.$$

It follows from the inequality for $\langle g_{in} \rangle$ that there are only finitely many possibilities for such g_{in} . For $i \in \{1, 2\}$ and $n \geq r$, by noting that

$$\langle a_{in} \rangle \leq \langle g_{in} \lambda_i \alpha_n \rangle + c \delta^{-k} \hat{P}_n^{-k} \ll 1,$$

we conclude that there are only finitely many choices for a_{in} . Thus, there are only finitely many possibilities for the 4-tuples $(a_{1n}, g_{1n}, a_{2n}, g_{2n})$, and some 4-tuple, say (a_1, g_1, a_2, g_2) , occurs infinitely often.

When $i \in \{1, 2\}$ and $n \geq r$, we have

$$\left\langle \alpha_n - \frac{a_{in}}{g_{in} \lambda_i} \right\rangle \leq \langle g_{in} \lambda_i \rangle^{-1} c \delta^{-k} \hat{P}_n^{-k} \ll \hat{P}_n^{-k}, \quad (3.1)$$

and this implies that

$$\left\langle \frac{a_{1n}}{g_{1n} \lambda_1} - \frac{a_{2n}}{g_{2n} \lambda_2} \right\rangle \ll \hat{P}_n^{-k}.$$

Since $(a_{1n}, g_{1n}, a_{2n}, g_{2n}) = (a_1, g_1, a_2, g_2)$ for infinitely many values of n , we conclude that

$$\frac{a_1}{g_1 \lambda_1} = \frac{a_2}{g_2 \lambda_2}.$$

If $a_1 \neq 0$, then

$$\frac{\lambda_2}{\lambda_1} = \frac{a_2 g_1}{a_1 g_2} \in \mathbb{K},$$

which provides a contradiction. If $a_1 = 0$, by (3.1), there exists some large integer m with $\langle \alpha_m \rangle < S$, contradicting the fact that $S \leq \langle \alpha_n \rangle < \hat{\tau}^{-1}$ for all $n \in \mathbb{N}$. This completes the proof of the lemma. $\square$

We now are in a position to prove our Weyl-type estimate.

Lemma 3.3. *Suppose that $S(P)$ is a function on $(0, \infty)$ that increases monotonically to infinity and satisfies $1 \leq S(P) \leq \hat{P}$. Then, there exists a function $T(P)$ on $(0, \infty)$, depending only on $\lambda_1, \lambda_2, k, q, \tau$, and $S(P)$, that increases monotonically to infinity, satisfies $1 \leq T(P) \leq S(P)$, and satisfies the property that*

$$\sup_{S(P) \hat{P}^{-k} \leq \langle \alpha \rangle < \hat{\tau}^{-1}} |F_1(\alpha) F_2(\alpha)| \ll \hat{P}^2 T(P)^{-\nu/(2k)}.$$

Proof. By Lemma 3.2, for each natural number n , we can find a positive number P_n such that if $P \geq P_n$ and $1/n \leq \langle \alpha \rangle < \widehat{\tau}^{-1}$, then

$$\widehat{P}^{-2} |F_1(\alpha) F_2(\alpha)| \leq \frac{1}{n}.$$

Furthermore, we can choose $(P_n)_{n=1}^\infty$ to be an increasing sequence with $S(P_n) \geq n$ for all n . Define $T(P)$ by setting

$$T(P) = \begin{cases} n, & \text{when } P_n \leq P < P_{n+1}, \\ 1, & \text{when } P < P_1. \end{cases}$$

If $P \geq P_n$ and $T(P)^{-1} \leq \langle \alpha \rangle < \widehat{\tau}^{-1}$, then

$$\widehat{P}^{-2} |F_1(\alpha) F_2(\alpha)| \leq \frac{1}{n}.$$

This implies that

$$\sup_{T(P)^{-1} \leq \langle \alpha \rangle < \widehat{\tau}^{-1}} |F_1(\alpha) F_2(\alpha)| \leq \widehat{P}^2 T(P)^{-1}. \quad (3.2)$$

Suppose now that P is sufficiently large in the context of Lemma 3.1. Note that $S(P) \widehat{P}^{-k} \leq T(P)^{-1}$, and assume that

$$S(P) \widehat{P}^{-k} \leq \langle \alpha \rangle < T(P)^{-1}$$

and

$$|F_1(\alpha)| \geq T(P)^{-\nu/(2k)} \widehat{P}.$$

Since

$$\widehat{P}^{-\nu/2} \leq T(P)^{-\nu/(2k)} \leq 1,$$

by applying Lemma 3.1 with $\delta = T(P)^{-\nu/(2k)}$, there exist elements a and g of $\mathbb{A}$ such that $(a, g) = 1$, $1 \leq \langle g \rangle \leq cT(P)^{\nu/2}$, and

$$\langle g \lambda_1 \alpha - a \rangle \leq cT(P)^{\nu/2} \widehat{P}^{-k}.$$

Hence, by the triangle inequality,

$$\langle a \rangle \ll \langle g \lambda_1 \alpha \rangle + T(P)^{\nu/2} \widehat{P}^{-k} \ll T(P)^{\nu/2-1} + T(P)^{\nu/2} \widehat{P}^{-k}.$$

Since

$$\lim_{P \rightarrow \infty} \left(T(P)^{\nu/2-1} + T(P)^{\nu/2} \widehat{P}^{-k} \right) = 0,$$

it follows that $a = 0$ for large enough values of P . This implies that

$$\langle \alpha \rangle \ll \langle g \lambda_1 \rangle^{-1} cT(P)^{\nu/2} \widehat{P}^{-k} \ll T(P)^{\nu/2} \widehat{P}^{-k},$$

and thus, for large enough values of P , we see that

$$\langle \alpha \rangle < T(P) \widehat{P}^{-k} \leq S(P) \widehat{P}^{-k}.$$

This contradicts the fact that $\langle \alpha \rangle \geq S(P) \widehat{P}^{-k}$. For P sufficiently large in terms of λ_1 , λ_2 , k , q , and $S(P)$, we have therefore shown that whenever

$$S(P) \widehat{P}^{-k} \leq \langle \alpha \rangle < T(P)^{-1},$$

then

$$|F_1(\alpha)| < \widehat{P} T(P)^{-\nu/(2k)}.$$

Hence,

$$\sup_{S(P)\widehat{P}^{-k} \leq \langle \alpha \rangle < T(P)^{-1}} |F_1(\alpha)F_2(\alpha)| \ll \widehat{P}^2 T(P)^{-\nu/(2k)}. \quad (3.3)$$

The lemma now follows by combining (3.2) with (3.3). $\square$

4. THE MINOR ARC

In order to complete our work on the minor arc, we first need to establish a mean value estimate for the smooth Weyl sum $f(\alpha)$.

Lemma 4.1. *Suppose that $u > 2k - 2$ is accessible to the exponent k and that $s \geq u + 5$. One has*

$$\int_{\mathbb{T}} |f(\alpha)|^{s-2} d\alpha \ll \widehat{P}^{s-2-k}.$$

Proof. Let $v = [s/2] - 2$. By considering the underlying Diophantine equations, we note that

$$\int_{\mathbb{T}} |f(\alpha)|^{2v+2} d\alpha \ll \int_{\mathbb{T}} |F(\alpha)^2 f(\alpha)^{2v}| d\alpha.$$

Since $2v \geq s - 5 \geq u$, we may apply Lemma 6.2 of [10] to establish that

$$\int_{\mathbb{T}} |F(\alpha)^2 f(\alpha)^{2v}| d\alpha \ll \widehat{P}^{2v+2-k},$$

and this implies that

$$\int_{\mathbb{T}} |f(\alpha)|^{2v+2} d\alpha \ll \widehat{P}^{2[s/2]-2-k}.$$

For even values of s , the proof of the lemma is now complete. If s is odd, the lemma follows by noting that

$$\int_{\mathbb{T}} |f(\alpha)|^{s-2} d\alpha \leq \widehat{P} \int_{\mathbb{T}} |f(\alpha)|^{2v+2} d\alpha \ll \widehat{P}^{2[s/2]-1-k}.$$

$\square$

We are now in a position to show that the minor arc contribution is $o(\widehat{P}^{s-k})$, thereby confirming (2.6).

Lemma 4.2. *Suppose that $u > 2k - 2$ is accessible to the exponent k and that $s \geq u + 5$. One has*

$$\int_{\mathfrak{m}} F_1(\alpha)F_2(\alpha)f_3(\alpha) \cdots f_s(\alpha)\chi_{\tau}(\alpha) d\alpha = o(\widehat{P}^{s-k}).$$

Proof. By Hölder's inequality,

$$\int_{\mathfrak{m}} F_1(\alpha)F_2(\alpha)f_3(\alpha) \cdots f_s(\alpha)\chi_{\tau}(\alpha) d\alpha \ll \left(\sup_{\alpha \in \mathfrak{m}} |F_1(\alpha)F_2(\alpha)| \right) \prod_{i=3}^s I_i^{1/(s-2)}, \quad (4.1)$$

where

$$I_i = \int_{\langle \alpha \rangle < \widehat{\tau}^{-1}} |f_i(\alpha)|^{s-2} d\alpha$$

for $3 \leq i \leq s$. Note that $f(\alpha + g) = f(\alpha)$ for all $\alpha \in \mathbb{T}$ and $g \in \mathbb{A}$. By Lemma 4.1, one has

$$\begin{aligned} \int_{\langle \alpha \rangle < \hat{\tau}^{-1}} |f(\alpha)|^{s-2} d\alpha &= \sum_{\substack{x \in \mathbb{A} \\ \langle x \rangle < \hat{\tau}^{-1}}} \int_{\langle \alpha - x \rangle < 1} |f(\alpha)|^{s-2} d\alpha \\ &= \sum_{\substack{x \in \mathbb{A} \\ \langle x \rangle < \hat{\tau}^{-1}}} \int_{\mathbb{T}} |f(\alpha)|^{s-2} d\alpha \ll \hat{P}^{s-2-k}. \end{aligned}$$

For $3 \leq i \leq s$, since $\langle \lambda_i \rangle < 1$, we see that

$$I_i = \langle \lambda_i \rangle^{-1} \int_{\langle \alpha \rangle < \langle \lambda_i \rangle \hat{\tau}^{-1}} |f(\alpha)|^{s-2} d\alpha \ll \int_{\langle \alpha \rangle < \hat{\tau}^{-1}} |f(\alpha)|^{s-2} d\alpha \ll \hat{P}^{s-2-k}. \quad (4.2)$$

By applying Lemma 3.3 with $S(P) = S_1(P)$, we obtain the bound

$$\sup_{\alpha \in \mathfrak{m}} |F_1(\alpha)F_2(\alpha)| = o(P^2). \quad (4.3)$$

The result now follows by combining (4.1), (4.2), and (4.3). $\square$

5. THE MAJOR ARC

We now wish to find an asymptotic for the major arc contribution. Let

$$\mathcal{F}(\alpha) = F_1(\alpha)F_2(\alpha)f_3(\alpha) \cdots f_s(\alpha)$$

and

$$\mathcal{G}(\alpha) = F_1(\alpha) \cdots F_s(\alpha).$$

Since $S_1(P)\hat{P}^{-k} \leq \hat{\tau}^{-1}$, one has

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha) \chi_{\tau}(\alpha) d\alpha = \hat{\tau} \int_{\mathfrak{M}} \mathcal{F}(\alpha) d\alpha.$$

We first wish to compare this integral to the *singular integral*

$$J_{s,k} = \int_{\langle \alpha \rangle < \hat{P}^{1-k}} \mathcal{G}(\alpha) d\alpha.$$

To do this, let $\rho(u)$ denote the Dickman function, which is defined as the unique continuous function on $[0, \infty)$ that satisfies the differential-difference equation $u\rho'(u) = -\rho(u-1)$ ($u > 1$) with the initial condition $\rho(u) = 1$ ($0 \leq u \leq 1$).

Lemma 5.1. *Suppose that $s \geq k+1$. One has*

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha) d\alpha - \rho(P/R)^{s-2} J_{s,k} \ll \hat{P}^{s-k} (\text{Log } \hat{P})^{-1/(8k)}.$$

Proof. Let P be large enough so that $P \geq 1$ and

$$2P/\log(2P) < R = \eta P < P - \log(P).$$

For $3 \leq i \leq s$, we deduce from Lemma 4.3 of [10] that

$$\begin{aligned} f_i(\alpha) - \rho(P/R)F_i(\alpha) &\ll \hat{P}(\text{Log } \hat{P})^{-1/2}(1 + \hat{P}^k \langle \lambda_i \alpha \rangle) \\ &\ll \hat{P}(\text{Log } \hat{P})^{-3/8} \end{aligned}$$

for $\alpha \in \mathfrak{M}$. Hence,

$$\mathcal{F}(\alpha) - \rho(P/R)^{s-2} \mathcal{G}(\alpha) \ll \widehat{P}^s (\text{Log } \widehat{P})^{-3/8}$$

for $\alpha \in \mathfrak{M}$. Furthermore, by noting that the measure of $\mathfrak{M}$ is $O(\widehat{P}^{-k} (\text{Log } \widehat{P})^{1/8})$, one has

$$\int_{\mathfrak{M}} \mathcal{F}(\alpha) d\alpha - \rho(P/R)^{s-2} \int_{\mathfrak{M}} \mathcal{G}(\alpha) d\alpha \ll \widehat{P}^{s-k} (\text{Log } \widehat{P})^{-1/4}. \quad (5.1)$$

By Lemma 4.1 of [10], whenever $1 \leq i \leq s$ and $\langle \alpha \rangle \leq \widehat{P}^{1-k}$, we have the bound

$$F_i(\alpha) \ll \widehat{P}(1 + \widehat{P}^k \langle \lambda_i \alpha \rangle)^{-1/k} \ll \widehat{P}(1 + \widehat{P}^k \langle \alpha \rangle)^{-1/k}.$$

Therefore,

$$\int_{\mathfrak{M}} \mathcal{G}(\alpha) d\alpha - J_{s,k} \ll \widehat{P}^s \int_{\mathfrak{T}} (1 + \widehat{P}^k \langle \alpha \rangle)^{-s/k} d\alpha,$$

where $\mathfrak{T} = \{\alpha \in \mathbb{K}_{\infty} : S_1(P) \widehat{P}^{-k} \leq \langle \alpha \rangle\}$. Let $V = \log_q(S_1(P))$. Since the measure of the set of points α in $\mathbb{T}$ with $\langle \alpha \rangle = q^m$ is less than q^{m+1} , we deduce that

$$\begin{aligned} \int_{\mathfrak{M}} \mathcal{G}(\alpha) d\alpha - J_{s,k} &\ll \widehat{P}^s \sum_{V-kP \leq m} q^{m+1} (1 + q^{kP+m})^{-s/k} \\ &\ll \widehat{P}^{s-k} \widehat{V}^{1-s/k} \ll \widehat{P}^{s-k} (\text{Log } \widehat{P})^{-1/(8k)}. \end{aligned}$$

The lemma now follows from (5.1). $\square$

Since $0 < \eta < 1$ and $R = \eta P$, we have $\rho(P/R) \gg 1$. Thus, we are left to show that $J_{s,k} \gg \widehat{P}^{s-k}$ in order to get an asymptotic lower bound of the desired form for the major arc contribution. To do this, we use ingredients from the proof of Lemma 16 in [9].

Lemma 5.2. *Let $s \geq k+1$, and suppose that the equation*

$$\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$$

has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_{\infty}^s$. For sufficiently large values of P , one has $J_{s,k} \gg \widehat{P}^{s-k}$.

Proof. By Lemma 1 of [9],

$$J_{s,k} = \int_{\langle \alpha \rangle < \widehat{P}^{1-k}} F_1(\alpha) \cdots F_s(\alpha) d\alpha = \widehat{P}^{1-k} W, \quad (5.2)$$

where W denotes the number of s -tuples $(x_1, x_2, \dots, x_s)$ in $\mathbb{A}^s$ with

$$\langle \lambda_1 x_1^k + \cdots + \lambda_s x_s^k \rangle < \widehat{P}^{k-1} \quad (5.3)$$

and $\langle x_i \rangle \leq \widehat{P}$ for $1 \leq i \leq s$.

Choose r such that $\langle \lambda_r z_r^k \rangle$ is maximal. Let $d = \text{ord } \lambda_r$ and $w = \text{lead}(\lambda_r)$. For $1 \leq i \leq s$, we define a_i by

$$a_i = \begin{cases} \text{lead}(z_i), & \text{when } \langle \lambda_i z_i^k \rangle = \langle \lambda_r z_r^k \rangle, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq s$, let

$$m_i = \left\lceil \frac{d - \text{ord } \lambda_i + k \text{ ord } z_r}{k} \right\rceil,$$

and let $n = [P] - \max_{1 \leq i \leq s} m_i$. Suppose that P is large enough so that $n + m_i > 0$ for $1 \leq i \leq s$. For $1 \leq i \leq s$, let x_i be an element of $\mathbb{A}$ with $x_i = a_i t^{n+m_i} + y_i$, where $y_i \in \mathbb{A}$ and $\text{ord } y_i < n + m_i$. Let

$$x_r = a_r t^{n+m_r} + b_{n+m_r-1} t^{n+m_r-1} + \cdots + b_0,$$

where each b_i is an element of $\mathbb{F}_q$, and define the coefficients $c_l \in \mathbb{F}_q$ via the relation

$$\lambda_1 x_1^k + \cdots + \lambda_s x_s^k = \sum_{l=-\infty}^{\infty} c_l t^l.$$

The inequality (5.3) is satisfied when $c_l = 0$ for all $l \geq (k-1)P$. For $1 \leq i \leq s$, observe that

$$\text{ord } \lambda_i + k(n + m_i) \leq d + k(n + m_r)$$

with equality holding whenever $\langle \lambda_i z_i^k \rangle = \langle \lambda_r z_r^k \rangle$. Thus, the coefficient $c_l = 0$ for all $l > d + k(n + m_r)$. Furthermore, our choice of $(a_1, \dots, a_s)$ guarantees that $c_{d+k(n+m_r)} = 0$. When

$$d + (k-1)(n + m_r) \leq l < d + k(n + m_r),$$

one has

$$c_l = k w a_r^{k-1} b_{l-d-(k-1)(n+m_r)} + h_l,$$

where h_l is an element of $\mathbb{F}_q$ depending at most on $\lambda, \mathbf{a}, b_i$ with

$$i > l - d - (k-1)(n + m_r),$$

and y_j with $j \neq r$.

Let y_j be arbitrarily selected for each $j \neq r$. Since $k w a_r^{k-1} \neq 0$, we can choose b_{n+m_r-1} so that $c_{d+k(n+m_r)-1} = 0$. Similarly, we can now choose b_{n+m_r-2} so that $c_{d+k(n+m_r)-2} = 0$. Continuing in this manner, it is possible to choose x_r in such a way that $c_l = 0$ for all

$$l \geq d + (k-1)(n + m_r).$$

Since d is negative, one has

$$d + (k-1)(n + m_r) < (k-1)P,$$

and it follows that (5.3) holds for $(x_1, \dots, x_s)$. Since y_j was arbitrarily selected for $j \neq r$, for sufficiently large values of P , it follows that $W \gg \hat{P}^{s-1}$, and from (5.2), we conclude that $J_{s,k} \gg \hat{P}^{s-k}$. $\square$

By combining Lemmas 5.1 and 5.2, we obtain the following result, which confirms (2.7).

Lemma 5.3. *Let $s \geq k + 1$, and suppose that the equation*

$$\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$$

has a non-trivial solution $\mathbf{z}$ in $\mathbb{K}_\infty^s$. For sufficiently large values of P , one has

$$\int_{\mathfrak{M}} F_1(\alpha) F_2(\alpha) f_3(\alpha) \cdots f_s(\alpha) \chi_\tau(\alpha) d\alpha \gg \hat{P}^{s-k}.$$

By combining Lemmas 4.2 and 5.3, we have now established Theorem 2.1.

6. THE SOLVABILITY OF $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ IN $\mathbb{K}_\infty$

Let $\psi(q, k)$ denote the minimum integer such that for all $n > \psi(q, k)$ and any choice of $a_1, \dots, a_n \in \mathbb{F}_q$, the equation $a_1 y_1^k + \cdots + a_n y_n^k = 0$ has a non-zero solution $\mathbf{y} \in \mathbb{F}_q^n$. We now use the function $\psi(q, k)$ to discuss the solvability of $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ in $\mathbb{K}_\infty$, which is a necessary hypothesis in Theorems 1.1 and 2.1.

Lemma 6.1. *Let $\lambda_1, \dots, \lambda_s$ be non-zero elements of $\mathbb{K}_\infty$. Whenever $\text{char}(\mathbb{F}_q) \nmid k$ and $s > k\psi(q, k)$, there exists a non-trivial solution $\mathbf{z} \in \mathbb{K}_\infty^s$ of the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$.*

Proof. Suppose that $s > k\psi(q, k)$. Note that for any $l_1, \dots, l_s \in \mathbb{Z}$, we have

$$\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = (\lambda_1 t^{-kl_1})(z_1 t^{l_1})^k + \cdots + (\lambda_s t^{-kl_s})(z_s t^{l_s})^k. \quad (6.1)$$

Hence, without loss of generality, we may assume that $0 \leq \text{ord } \lambda_i < k$ for each $1 \leq i \leq s$, and we can find an integer w with $0 \leq w < k$ such that $\text{ord } \lambda_i = w$ for at least $\lceil s/k \rceil$ distinct choices of i with $1 \leq i \leq s$. By multiplying the equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ by t^{-w} , using (6.1) if necessary, and rearranging the indices if required, there is no loss of generality in supposing that $\text{ord } \lambda_i = 0$ ($1 \leq i \leq n$) and $\text{ord } \lambda_j < 0$ ($n < j \leq s$), where $n \geq s/k > \psi(q, k)$. Therefore, there exist elements $y_1, \dots, y_n \in \mathbb{F}_q$, not all zero, such that

$$\text{lead}(\lambda_1) y_1^k + \cdots + \text{lead}(\lambda_n) y_n^k = 0.$$

By reordering the indices if necessary, we may assume that $y_1 \neq 0$. Let $z_i = y_i$ ($2 \leq i \leq n$) and $z_j = 0$ ($n < j \leq s$). Consider the function

$$f(z) = \lambda_1 z^k + \lambda_2 z_2^k + \lambda_3 z_3^k + \cdots + \lambda_s z_s^k.$$

Since $\text{ord } f(y_1) < 0$ and $\text{ord } f'(y_1) = \text{ord}(k\lambda_1 y_1^{k-1}) = 0$, an application of Hensel's lemma (see [3, Theorem 7.3]) for $\mathbb{F}_q[[1/t]]$, the ring of formal power series in t^{-1} with coefficients in $\mathbb{F}_q$, implies that there exists an element $z_1 \in \mathbb{F}_q[[1/t]] \subset \mathbb{K}_\infty$ such that $\text{ord}(z_1 - y_1) < 0$ and $f(z_1) = 0$. The lemma now follows. $\square$

By Chevalley's theorem (see Theorem 1 of Section 10.2 in [8]), we see that $\psi(q, k) \leq k$. When $q > k^4$, it follows from the work of Weil (see [13]) that $\psi(q, k) \leq 2$. Furthermore, when $(k, q-1) = 1$, the mapping $x \mapsto x^k$ from $\mathbb{F}_q$ to $\mathbb{F}_q$ is a bijection, implying that $\psi(q, k) = 1$. We summarize the results of this section in the following lemma.

Proposition 6.1. *Suppose that $\text{char}(\mathbb{F}_q) \nmid k$, and let $\lambda_1, \dots, \lambda_s$ be non-zero elements of $\mathbb{K}_\infty$. The equation $\lambda_1 z_1^k + \cdots + \lambda_s z_s^k = 0$ has a non-trivial solution $\mathbf{z} \in \mathbb{K}_\infty^s$ whenever one of the following three conditions are met:*

- (1) $s \geq k^2 + 1$,
- (2) $q > k^4$ and $s \geq 2k + 1$,
- (3) $(k, q-1) = 1$ and $s \geq k + 1$.

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SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, 1 EINSTEIN DRIVE, PRINCETON, NJ 08540

E-mail address: craigvspencer@gmail.com