Exam 3 Paradigms

■ Calculating Mass

A mass distribution occupies the region in the first octant which is enclosed by the surfaces $y = x$, $x = 1$, $y = 0$, $z = 1 + x$, $z = 0$. If the mass density function is $\delta = x^2 + y^2$ units of mass/unit volume, calculate the total mass.

Solution: $M = \int\int\int_T (x^2 + y^2)dV = \int\int_R \int_0^{1+x} (x^2 + y^2)dzdA$

$$= \int\int_R (x^2 z + y^2 z)\Big|_{z=0}^{z=1+x} dA = \int\int_R (x^2 + x^3 + y^2 + xy^2)dA$$

$$= \int_0^1 \int_0^x (x^2 + x^3 + y^2 + xy^2)dydx = \int_0^1 (x^2 y + x^3 y + \frac{y^3}{3} + x\frac{y^3}{3})\Big|_{y=0}^{y=x}$$

$$= \int_0^1 (x^3 + x^4 + \frac{x^3}{3} + \frac{x^4}{3})dx = \frac{4}{3}\frac{x^4}{4} + \frac{4}{3}\frac{x^5}{5}\Big|_0^1 = \frac{1}{3} + \frac{4}{15} = \boxed{\frac{9}{15}}$$

Similar problems

(1) A mass distribution occupies the 3-dimensional region which is above $z = x^2 + y^2$ and below the plane $z = 4$. If the mass density function is $\delta = \sqrt{x^2 + y^2}$ units of mass/unit volume, calculate the total mass. $\boxed{\text{Ans } \frac{128\pi}{15}}$

(2) A mass distribution occupies the 3 dimensional region which is above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 4$. If the mass density function is $\delta = z$ units of mass/unit volume, use a triple integral in **spherical coordinates** to calculate the total mass. $\boxed{\text{Ans } 64\pi}$

(3) A mass distribution occupies the region which is enclosed by the surfaces $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$. The mass density function is $\delta = 2z$ units of mass/unit volume. Calculate the total mass. $\boxed{\text{Ans } 2\pi}$

(4) A mass distribution occupies the region which is above the surface $z = \sqrt{x^2 + y^2}$ and under the plane $z = 2$. The mass density function is $\delta = z\sqrt{x^2 + y^2 + z^2}$ units of mass/unit volume. Use a triple integral in **spherical coordinates** to calculate the total mass. $\boxed{\text{Ans } \frac{64\pi}{15}(2\sqrt{2} - 1)}$

■ Surface Area

Find the surface area of that part of the surface $z = y^2$ which is in the first octant and is above the region of the xy-plane bounded by the lines $y = x$, $y = 2$ and $x = 0$.

Solution: Here, you can use the formula for SA in rectangular coordinates: SA= $\int\int_R \sqrt{1 + f_x^2 + f_y^2} dA$; or you could derive this by finding the associated $\vec{N}$ and integrate $|\vec{N}|$. In either case, we get

SA= $\int_0^2 \int_0^y \sqrt{1 + 4y^2} dx dy = \int_0^2 y\sqrt{1 + 4y^2} dy = \boxed{\frac{1}{12}(17\sqrt{17} - 1)}$

In general, you can always calculate $\vec{N}$, and you *need* to do this for a general parametric surface. But for surfaces in rectangular or cylindrical, you may look up $|\vec{N}|$ in your book. For example, $|\vec{N}|$ in cylindrical is

$$|\vec{N}| = \sqrt{r^2 + r^2 \left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2}$$

Similar problems

(1) Find the area of that part of the surface $z = 1 + x^2$ which lies above the region of the xy-plane which is enclosed by the curves $y = x$, $y = 0$ and $x = 2$. $\boxed{\text{Ans } \frac{1}{12}[17\sqrt{17} - 1]}$

(2) Calculate the surface area of the parametrized surface $x = \frac{1}{2}t^2$, $y = 2t$, $z = s$, $0 \le s \le 2$, $0 \le t \le 1$. Use a formula from the integral tables to evaluate the integral. $\boxed{\text{Ans } \sqrt{5} + 4\ln(1 + \sqrt{5}) - 4\ln 2}$

(3) Calculate the area of the parametrized surface $x = t$, $y = t^2$, $z = ts^3$, $0 \le t \le 3$, $0 \le s \le 2$. $\boxed{\text{Ans } \frac{2}{3}(37\sqrt{37} - 1)}$

(4) Calculate the area of that part of the paraboloid $z = x^2 + y^2$ which is between the planes $z = 1$ and $z = 9$. $\boxed{\text{Ans } \frac{\pi}{6}(37\sqrt{37} - 5\sqrt{5})}$

■ Potential Functions and Work

Show that the force field

$$\vec{F} = \left(y - \frac{1}{x^2}\right)\vec{i} + \left(x - \frac{1}{y^2}\right)\vec{j}$$

is conservative on the region $x > 0$ and $y > 0$ by finding a potential function for $\vec{F}$. Now use the potential function to find the work done by $\vec{F}$ as it acts on an object which moves along a curve from $(1,1)$ to $(3,2)$ in the region $x > 0$ and $y > 0$.

Solution: We want to find f such that $\nabla f = \vec{F}$. So,

$$\frac{\partial f}{\partial x} = y - \frac{1}{x^2}, \qquad \frac{\partial f}{\partial y} = x - \frac{1}{y^2}$$

Looking at this system, we partial integrate to get

$$f(x,y) = yx + \frac{1}{x} + C(y), \qquad f(x,y) = xy + \frac{1}{y} + \widetilde{C}(x)$$

We compare the two to get

$$f(x,y) = xy + \frac{1}{x} + \frac{1}{y}$$

This is our potential function, and so

$$W = f(3,2) - f(1,1) = \boxed{\frac{23}{6}}$$

Similar problems

(1) Show that the Force field

$$\vec{F} = \left(2 + \frac{2x}{1+y}\right)\vec{i} + \left(2y - \frac{x^2}{(1+y)^2}\right)\vec{j}$$

is conservative in the region $y > -1$ by finding a potential function for it. Now calculate the work done by $\vec{F}$ as it acts on an object which moves from $(1,0)$ to $(9,3)$ along any piecewise smooth curve in the region $y > -1$. $\boxed{\text{Ans } \frac{177}{4}}$

(2) Show that the Force field

$$\vec{F} = -\frac{4+y^2}{(1+x)^2}\vec{i} + \left(2y + \frac{2y}{1+x}\right)\vec{j}$$

is conservative in the region $x > -1$ by finding a potential function for it. Calculate the work $\vec{F}$ does to an object if it moves from $(0,0)$ to $(2,4)$. $\boxed{\text{Ans } \frac{56}{3}}$

■ Work Integrals and Line Integrals

In general, you may not be able to find a potential function, so you need to calculate the work integral directly...

The force field $\vec{F} = (x-y)\vec{i} + (x+y)\vec{j}$ acts on an object as it moves in the plane. Calculate the work done by $\vec{F}$ as the object moves from $(1,0)$ to $(3,1)$ along the following path:
first, along $x^2+y^2=1$ in the first quadrant from $(1,0)$ to $(0,1)$.
second, along the line segment from $(0,1)$ to $(3,1)$.

Solution: $W_C = \int_C (x-y)dx + (x+y)dy$

$C_1 : x = \cos(t),\ y = \sin(t),\ 0 \le t \le \frac{\pi}{2}$
$dx = -\sin(t)dt,\ dy = \cos(t)dt$

$W_{C_1} = \int_0^{\pi/2} [(\cos t - \sin t)(-\sin t) + (\cos t + \sin t)\cos t]dt$
$= \int_0^{\pi/2} (\sin^2 t + \cos^2 t)dt = \frac{\pi}{2}$

$C_2 : x = t,\ y = 1,\ 0 \le t \le 3$
$dx = dt,\ dy = 0$

$$W_{C_2} = \int_0^3 (t-1)dt = \frac{t^2}{2} - t\Big|_0^3 = \frac{3}{2}$$

Thus $W = W_{C_1} + W_{C_2} = \boxed{\frac{\pi}{2} + \frac{3}{2}}$

Similar problems:

(1) Evaluate the line integral $\int_C (1+xy)dx + \left(\frac{1}{1+x^2}\right)dy$
for each of the following curves from $(0,0)$ to $(2,4)$.

a) C is the parabola $y = x^2$ from $(0,0)$ to $(2,4)$. $\boxed{\text{Ans } 6 + \ln 5}$
b) C consists of the line segment from $(0,0)$ to $(2,0)$ followed by the line segment from $(2,0)$ to $(2,4)$. $\boxed{\text{Ans } 14/5}$
c) C consists of the line segment from $(0,0)$ to $(0,4)$ followed by the line segment from $(0,4)$ to $(2,4)$. $\boxed{\text{Ans } 14}$

■ Surface Integrals

A general surface integral will look like $\int\int_R f(x,y,z)dS$ where $dS = |\vec{N}|dA$

Evaluate the surface integral $\int\int_S z^2 dS$ where S is the parametrized surface $x = 4\cos t$, $y = 4\sin t$, $z = s$, $0 \le t \le 2\pi$, $0 \le s \le 3$.

Solution: $\vec{r}(s,t) = (4\cos t, 4\sin t, s)$

$$\vec{N} = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -4\sin t & 4\cos t & 0 \end{vmatrix} = (-4\cos t, -4\sin t, 0)$$

Thus $|\vec{N}| = 4$. Hence,

$$\int\int_S z^2 dS = \int\int_R z^2 |\vec{N}| ds dt = \int_0^{2\pi}\int_0^3 s^2 4\, ds dt = \int_0^{2\pi} \frac{4}{3}s^3\Big|_0^3 dt = \boxed{72\pi}$$

Similar problems:

(1) Evaluate the surface integral $\iint_S (z^2 - y^2)\, dS$ where S is the parametrized surface $x = u^2 + v^2$, $y = u - v$, $z = u + v$, $0 \le u \le 1$, $0 \le v \le 1$. $\boxed{\text{Ans } \frac{2}{15}[25\sqrt{5} - 18\sqrt{3} + 1]}$

(2) Evaluate the surface integral $\int\int_S \sqrt{x^2 + y^2} dS$ where S is the parametrized surface $x = s\cos t$, $y = s\sin t$, $z = t$ for $0 \le s \le 1$, $0 \le t \le 2\pi$. $\boxed{\text{Ans } \frac{2\pi}{3}(2\sqrt{2} - 1)}$

(3) Evaluate the surface integral $\int\int_S y dS$ where S is the parametrized surface $x = \frac{1}{2}t^2$, $y = 2t$, $z = s$, $0 \le s \le 2$, $0 \le t \le 1$. $\boxed{\text{Ans } \frac{4}{3}(5\sqrt{5} - 4\sqrt{4})}$

(4) Evaluate the surface integral $\iint_S 12xy\, dS$ where S is that part of $z = \sqrt{25 - x^2 - y^2}$ which lies above the square $0 \le x \le 1$, $0 \le y \le 1$. $\boxed{\text{Ans } 20[25^{3/2} - 2(24)^{3/2} + 23^{3/2}]}$

■ Green's Theorem • ¡Attention! See the problem on Green's Theorem in *vector form* on Additional Problem 3 at the end of these notes •

Green's theorem can be used to evaluate line integrals over a loop, even though the vector field may be somewhat complicated. If you have a vector field in the plane, say $\vec{F} = P\vec{i} + Q\vec{j}$ and C is some closed loop oriented *counterclockwise*, then Green's theorem says $\int_C P dx + Q dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$, where R is the region enclosed by C. Note that if the loop were oriented clockwise, then your integral is the negative of the one above.

Use Green's theorem to evaluate

$$\int_C (xy + y^3 \cos x) dx + (3y^2 \sin x + x^2) dy$$

where C is the closed curve consisting of the sides of the triangle having vertices $(0,0)$, $(2,0)$, $(0,2)$, directed counterclockwise.

Solution: $\int_C (xy + y^3 \cos x) dx + (3y^2 \sin x + x^2) dy$

$$= \int\int_R \left[\frac{\partial}{\partial x}(3y^2 \sin x + x^2) - \frac{\partial}{\partial y}(xy + y^3 \cos x) \right] dA$$

$$= \int\int_R [(3y^2 \cos x + 2x) - (x + 3y^2 \cos x)] dA = \int_0^2 \int_0^{2-x} x dy dx$$

$$= \int_0^2 xy \Big|_{y=0}^{y=2-x} dx = \int_0^2 x(2-x) dx = \int_0^2 (2x - x^2) dx = x^2 - \frac{x^3}{3} \Big|_0^2 = \boxed{\frac{4}{3}}$$

Similar problems:

(1) Use Green's theorem to evaluate $\int_C (1 + xy) dx + (x^2 + y) dy$
where C is the triangle with vertices $(0,0)$, $(2,0)$, $(0,2)$, directed counterclockwise. $\boxed{\text{Ans } \frac{4}{3}}$

(2) Use Green's theorem to evaluate $\int_C (x^2 + y^2) dx + (x^2 + y^2 + 2xy) dy$
where C consists of: the x-axis from $(0,0)$ to $(2,0)$, the circle $x^2 + y^2 = 4$ in the first quadrant from $(2,0)$ to $(0,2)$, and the y-axis from $(0,2)$ to $(0,0)$. $\boxed{\text{Ans } \frac{16}{3}}$

■ Flux Across a Surface

Evaluate the surface integral, $\int\int_S (\vec{F}\cdot\vec{n})dS$ where $\vec{n}$ is the upward pointing unit normal vector to the surface S, $\vec{F} = x\vec{i} + y\vec{j} + 5z\vec{k}$, and S is that part of $z = x^2 + y^2$ which is under the plane $z = 4$.

Solution: In this case, you could just use the formula

$$\int\int_S (\vec{F}\cdot\vec{n})dS = \int\int_D [P\left(-\frac{\partial z}{\partial x}\right) + Q\left(-\frac{\partial z}{\partial y}\right) + R]dxdy,$$

where $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$.

However, it is not much work to calculate everything separately; and I hope this illustrates the general process.

First find dS. So, we write down $\vec{r}(x,y) = (x, y, x^2 + y^2)$.

Then $\vec{N} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = -2x\vec{i} - 2y\vec{j} + \vec{k}$. Thus $|\vec{N}| = \sqrt{4x^2 + 4y^2 + 1}$.

Thus $dS = \sqrt{4x^2 + 4y^2 + 1}dxdy$. Also, a possible normal vector is $(f_x, f_y, -1)$, but this is downward pointing. So take the negative to get an upward pointing normal: $(-f_x, -f_y, 1)$. Now divide by the length to get a *unit* normal vector. So the upward pointing unit normal vector is

$$\vec{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$$

In this case $\vec{n} = \frac{(-2x, -2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$. Putting all this together, we have

$$\int\int_S (\vec{F}\cdot\vec{n})dS = \int\int_R (x, y, 5z) \cdot \frac{(-2x, -2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}} \sqrt{4x^2 + 4y^2 + 1}\, dxdy$$

$$= \int\int_R (-2x^2 - 2y^2 + 5(x^2 + y^2))dxdy = \int\int_R 3(x^2 + y^2)dxdy$$

$$= \int_0^{2\pi}\int_0^2 3r^2 \cdot r dr d\theta = \int_0^{2\pi} \frac{3}{4}r^4 \Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 12 d\theta = \boxed{24\pi}$$

■ Additional Problems

(1) Evaluate the surface integral, $\int\int_{\Sigma}(\vec{F}\cdot\vec{n})dS$ where $\vec{n}$ is the upward pointing unit normal vector to the surface Σ, $\vec{F}=-2x\vec{i}+2y\vec{j}+[-4x^2-4y^2+(x^2+y^2)^4]\vec{k}$, and Σ is that part of the hyperbolic paraboloid $z=x^2-y^2+1$ which is over the unit disk $x^2+y^2\leq 1$. $\boxed{\text{Ans } \frac{\pi}{5}}$

(2) Evaluate the surface integral, $\int\int_{\Sigma}(\vec{F}\cdot\vec{n})dS$ where $\vec{n}$ is the upward pointing unit normal vector to the surface Σ, with
$\vec{F}=\frac{z}{x^2+y^2+z^2}\vec{i}+\frac{z}{x^2+y^2+z^2}\vec{j}+\frac{z}{x^2+y^2+z^2}\vec{k}$,
and Σ is the upper hemisphere $z=\sqrt{1-x^2-y^2}$. $\boxed{\text{Ans } \frac{2\pi}{3}}$

(3) Use Green's theorem in vector form to calculate the outward flux $\int_C \vec{F}\cdot\vec{n}\,ds$ where $\vec{F}=xy^2\vec{i}+x^2y\vec{j}$ and C is the circle $x^2+y^2=4$. $\boxed{\text{Ans } 8\pi}$ [Here, Green's theorem says $\int_C \vec{F}\cdot\vec{n}\,ds=\int\int_R \text{div}\vec{F}\,dA$ where R is the region enclosed by the loop, and the loop is oriented counterclockwise. This is a special case of the *divergence theorem.*]

(4) Show that the force field $\vec{F}=\left(\frac{1}{y}+2x\right)\vec{i}+\left(3y^2-\frac{x}{y^2}\right)\vec{j}$ is conservative in the region $y\neq 0$ by finding a potential function for it. Now use this potential function to calculate the work done by $\vec{F}$ as it acts on an object which moves from $(x,y)=(2,1)$ to $(x,y)=(1,3)$ along any curve in the upper half plane. $\boxed{\text{Ans } \frac{64}{3}}$

(5) A mass distribution occupies the region in the first octant which is enclosed by the surfaces $y=x^2$, $y=x$, $z=1$, $z=2+x$. If the mass density function is $\delta(x,y,z)=10y$ units of mass/unit volume, calculate the total mass in the region. $\boxed{\text{Ans } \frac{13}{12}}$

(6) Use Green's theorem to evaluate the line integral

$$\int_C (2xy+ye^x+3y)dx+(x^2+e^x)dy$$

where C is the circle $(x-1)^2+(y-3)^2=9$ directed counterclockwise. $\boxed{\text{Ans } -27\pi}$

(7) Evaluate the surface integral $\int\int_S x dS$ where S is the parametrized surface

$$x = s\cos t, \quad y = s\sin t, \quad z = t, \quad 0 \le s \le 4, \quad 0 \le t \le \frac{\pi}{4}.$$

Ans $\frac{1}{3\sqrt{2}}(17\sqrt{17} - 1)$

(8) A mass distribution occupies the region which is between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. If the mass density function is $\delta = z^4$ units of mass/units of volume, find the total mass. Ans $\frac{508\pi}{35}$

(9) The force field $\vec{F} = xy\vec{i} + x^2\vec{j}$ acts on an object as it is moved in the plane. Find the work done by $\vec{F}$ as the object moves from $(0, 0)$ to $(4, 1)$ along the curve which is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the line segment from $(1, 1)$ to $(4, 1)$. Ans $\frac{33}{4}$

(10) A piece of surface Σ is described in cylindrical coordinates as $z = r\sin\theta$ where $0 \le r \le 1$ and $0 \le \theta \le \pi$. Find the area of Σ. [Remember, you can look up $|\vec{N}|$ in cylindrical, but here $dS = |\vec{N}| dr d\theta$, and **not** $|\vec{N}| r dr d\theta$, that's the point of this problem.] Ans $\frac{\pi}{\sqrt{2}}$