Homework 5 due the 1^{st} Friday of March @ 6 pm

Gradients and directional derivatives:

1.1) Let $f(x,y) = 6x\sqrt{x+y^2}$. Find
a) ∇f (the gradient of f)
b) the directional derivative of f at $(x,y) = (5,2)$ in the direction of the vector $\vec{a} = -2\vec{i} + \vec{j}$.
c) the value of the largest directional derivative if f at $(x,y) = (5,2)$ and a unit vector which points in the direction which gives this largest directional derivative.
1.2) Let $f(x,y,z) = xy + yz + xz$. Find the instantaneous rate of change of $f(x,y,z)$ per unit distance at the point $(x,y,z) = (1,2,3)$ in the direction of the vector $\vec{i} - 4\vec{j} + 2\vec{k}$.
1.3) Given $f(x,y) = ye^{xy}$
a) Find the gradient vector field of $f(x,y)$.
b) At the point $(x,y) = (-1,2)$ find the instantaneous rate of change of $f(x,y)$ per unit distance in the direction of the vector $\vec{v} = \vec{i} + 6\vec{j}$.
c) Find the unit vector which points in the direction you should follow, if starting at $(x,y) = (-1,2)$, you want to achieve the most rapid increase possible for $f(x,y)$.

Tangent planes given by the gradient:

2.1) Find the equation of the tangent plane to the given surface at the specified point.
a) $xyz^2 + x^3 + y^3 - 2z^3 = 3$ at $(-1,2,1)$.
b) $z = \sqrt{x^2 + 2y^2}$ at $(1,2,3)$.
c) $x^2y + y^2z = z^2x + 5$ at $(1,2,3)$.
d) $x^3y + y^3z + xz^3 = 53$ at $(1,2,3)$.
e) $x^3 + y^3 + xyz^3 = 41$ at $(3,2,1)$.
f) $z = 2x\sqrt{x+y^2}$ at $(5,2,30)$.

Lagrange multipliers:

3.1) Use the method of Lagrange multipliers to find the maximum value and the minimum value of $f(x,y) = xy$ on the ellipse $x^2 + 2y^2 = 36$.
3.2) Use the method of Lagrange multipliers to find the largest value and the smallest value of $f(x,y) = x^2 + 2x - y^2$ on the circle $x^2 + y^2 = 16$.
3.3) Use the method of Lagrange multipliers to find the largest value and smallest value of $f(x,y,z) = x + 2y + z^2$ on the ellipsoid $x^2 + y^2 + 2z^2 = 20$.
3.4) Use the method of Lagrange multipliers to find the maximum and the minimum value for $f(x,y) = \frac{1}{3}x^3 - 3x + y^2$ on the circle $x^2 + y^2 = 16$.

Visualizing Lagrange Multipliers:

4.1) Consider the surface $z = x^2 + y^2$. This surface can be subject to many constraints. By visualization alone, figure out how many (x, y) pairs satisfy the the Lagrange multiplier system of equations under these different constraints (you can have 0 or ∞ as an answer):

a) $x + y = -3$
b) $2001x^2 + 1999y^2 = 1$
c) $y = 2001x^2 - 1999$
d) $x = y^2$
e) $y = 10x(x - 1)(x + 1)$
f) $xy = 1$
g) $y = \tan^{-1} x$
h) $y^2 + (x - 2001)^2 = 1$
i) $y^2 = 1 - x^2$

Word Problem?

5.1) The production P of a factory can be represented by the function

$$P(x, y) = 40x^{1/4}y^{3/4}$$

where x is the number of workers (in hundreds) employed at the plant, and y is the amount of money invested (in millions). Currently, the company employs 1 unit of workers and 16 units of money.

a) What is the gradient of the production for $x = 1$, $y = 16$?

b) The company is thinking of increasing the units of money y invested, while cutting the number of workers x by the same amount. Using directional derivatives, describe the effect of this on production.

c) In which direction should the company move instead to get the largest increase in production?

d) What is the largest rate at which the production can grow, if we start with $x = 1$, $y = 16$?