Final Practice Problems

Spring1999.6) Find and classify the critical points of $f(x, y) = y^2 - x^3y + 3xy$.

$\boxed{\text{Ans } (\sqrt{3}, 0) \longleftarrow \text{ saddle; } (-\sqrt{3}, 0) \longleftarrow \text{ saddle; } (0,0) \longleftarrow \text{ saddle;}}$

$\boxed{(1,-1) \longleftarrow \text{ local min; } (-1,1) \longleftarrow \text{ local min}}$

Spring1999.7) Use the method of Lagrange multipliers to find the largest value and the smallest value of $f(x, y) = x^2 + y^2$ on the curve $x^4 + y^4 = 16$.

$\boxed{\text{Ans: The cp's are } (0, \pm 2), (\pm 2, 0), (\pm 8^{1/4}, \pm 8^{1/4}).}$

$\boxed{\text{Smallest value of } f \text{ is } 4, \text{ largest value of } f \text{ is } 4\sqrt{2}.}$

Spring1999.8) Find the volume of the 3-D region enclosed by the surfaces $y = x^2$, $y = 4$, $z = 5 + x$, $z = 2x$. $\boxed{\text{Ans } \frac{160}{3}}$

Spring1999.9) A mass distribution occupies the region which is above $z = x^2 + y^2$ and under $z = 4$. If the mass density function is $\delta(x, y, z) = z^2$ units of mass/unit volume, calculate the total mass. $\boxed{\text{Ans } 64\pi}$

Spring1999.10) Show that the force field

$$\vec{F} = \left(2xy + \frac{2x}{x^2 + y^2}\right)\vec{i} + \left(x^2 + \frac{2y}{x^2 + y^2} + 1\right)\vec{j}$$

is conservative in the region $(x, y) \neq (0, 0)$ by finding a potential function for it. Now use this potential function to calculate the work done by $\vec{F}$ as it acts on an object which moves from $(2, 1)$ to $(3, 4)$ along any path which does not pass through $(0, 0)$. $\boxed{\text{Ans: After finding a potential function, the work is } 35 + \ln 5.}$

Spring1999.11) Use Green's theorem to evaluate the line integral

$$\int_C (y^2 + 12xy)dx + (6x^2 + y^2)dy$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ directed counterclockwise. $\boxed{\text{Ans } -\frac{1}{3}}$

Spring1999.12) Evaluate the surface integral $\displaystyle\iint_S y\,dS$ where S is the parametrized surface $x = \frac{1}{2}t^2$, $y = 2t$, $z = s$, $0 \le s \le 2$, $0 \le t \le 1$. $\boxed{\text{Ans } \frac{4}{3}(5\sqrt{5} - 4\sqrt{4})}$

Fall2000.12) Verify Stokes' theorem by directly calculating both sides of the equation where the vector field is $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$ and the surface is $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Solution: curl$\vec{F} = ... = -2\vec{k}$.

For $x^2 + y^2 + z^2 = 4$, $\vec{n} = \frac{1}{2}(x, y, z)$.

$$\iint_S (\text{curl}\vec{F}) \cdot \vec{n}\, dS = \iint_S -z dS = \int_0^{2\pi} \int_0^{\pi/2} (-2\cos\phi) 4 \sin\phi d\phi d\theta = ... = -8\pi$$

Or even easier, take $z = \sqrt{4 - x^2 - y^2}$ then

$$\iint_S (\text{curl}\vec{F}) \cdot \vec{n}\, dS = \iint_R [0(-\frac{\partial z}{\partial x}) + 0(-\frac{\partial z}{\partial y}) + (-2)] dx dy$$

$$= \iint_R -2 dx dy = (-2)\text{area}(R) = (-2)4\pi = -8\pi.$$

For the other side we calculate the work done along the boundary:

The boundary is the curve $x^2 + y^2 = 4$, $z = 0$. Parameterize this to get

$x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \leq t \leq 2\pi$
$dx = -2\sin t dt$, $dy = 2\cos t dt$, $dz = 0$.

$$\int_C y dx - x dy + z dz = \int_0^{2\pi} (-4\sin^2 t - 4\cos^2 t) dt = -8\pi.$$

Fall2000.11) Let S be the closed surface, consisting of two pieces, which bounds the 3-dimensional region which is under $z = 4 - x^2 - y^2$ and above the xy-plane. Suppose $\vec{F} = x\vec{i} + y\vec{j} + e^z\vec{k}$. Verify the divergence theorem in this case by calculating both sides of the equation and seeing that they are equal.

Solution: $\text{div}\vec{F} = 2 + e^z$

$$\iiint_T (\text{div}\vec{F})\, dV = \iiint_T (2 + e^z) dV = \iint_R \int_0^{4-r^2} (2 + e^z) dz dA$$

$$= ... = \int_0^{2\pi} \int_0^2 (8 - 2r^2 + e^{4-r^2} - 1) r dr d\theta = ... = 11\pi + \pi e^4.$$

Let S_1 be the top parabaloid and S_2 be the bottom plane. Then,

$$\iint_{S_1} \vec{F} \cdot \vec{n}\, dS = \iint_R (x(2x) + y(2y) + e^z) dx dy = \int_0^{2\pi} \int_0^2 (2r^2 + e^{4-r^2}) r dr d\theta$$

$$= ... = 15\pi + \pi e^4.$$

$$\iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_{S_2} (-e^z) dS = \iint_{S_2} (-1) dS = -\text{area}(S_2) = -4\pi.$$

Thus the total outward flux is $15\pi + \pi e^4 + (-4\pi) = 11\pi + \pi e^4$.

Note that on the above two surface integrals you can plug in what z is on the surface. For example, on S_1, $z = 4 - x^2 - y^2$; and on S_2, $z = 0$.

Fall2000.10) The force field $\vec{F} = y^2\vec{i} + xy\vec{j}$ acts on an object as it moves in the plane. Suppose the object is moved from $(0,0)$ to $(3,6)$ by first going along $y = x^2$ from $(0,0)$ to $(2,4)$ and then along $y = 2x$ from $(2,4)$ to $(3,6)$. Calculate the work done by $\vec{F}$. Ans $\frac{1048}{15}$

Spring1999.1) An object is moving in 3-space in such a way that its acceleration vector as a function of time is $\vec{a} = (\cos t)\vec{i} - (\sin t)\vec{k}$. Suppose you know that at time $t = 0$, the velocity vector is $\vec{v}(0) = \vec{i} + \vec{j} + \vec{k}$ and the position vector is $\vec{r}(0) = \vec{j}$. Find the velocity vector and the position vector as functions of t.

Ans $\vec{v} = (1 + \sin t)\vec{i} + \vec{j} + \cos t\vec{k}, \quad \vec{r} = (1 + t - \cos t)\vec{i} + (1 + t)\vec{j} + (\sin t)\vec{k}$

Spring1999.2) A particle is moving in the plane along the curve $y = (x+1)^2$ in the direction of increasing x. It is moving at a constant speed of 4 ft/sec.

a) Find a_T and a_N when the particle is at $(x, (x+1)^2)$.

Ans $a_T = 0, \quad a_N = \dfrac{32}{[1 + 4(x+1)^2]^{3/2}}$

b) Find the velocity vector and the acceleration vector when the particle is at $(0,1)$.

Ans $\vec{v} = \frac{4}{\sqrt{5}}\vec{i} + \frac{8}{\sqrt{5}}\vec{j}$ and $\vec{a} = -\frac{64}{25}\vec{i} + \frac{32}{25}\vec{j}$ at $(0,1)$

Spring1999.3) The force field $\vec{F} = -y\vec{i} + 2x\vec{j}$ acts on an object as it moves in the xy-plane. Calculate the work done by $\vec{F}$ as the object moves from $(0,0)$ to $(2,7)$ along the following path: First, along $y = x^2$ from $(0,0)$ to $(2,4)$; second, along the vertical line segment from $(2,4)$ to $(2,7)$. Ans 20

Spring1999.4) Let $f(x,y) = x^3 + xy^2 - xy$.

a) If you are moving along the straight line through the point $(x,y) = (1,2)$ toward the point $(x,y) = (4,1)$ what is the instantaneous rate of change of $f(x,y)$ per unit distance that you will observe at $(1,2)$? Ans $\frac{12}{\sqrt{10}}$

b) If you are at $(x,y) = (1,2)$, in which direction should you go it you want $f(x,y)$ to increase most rapidly? Ans $5\vec{i} + 3\vec{j}$

Spring1999.5) Find the equation of the tangent plane to each of the following surfaces at the specified point:

a) To the graph of $f(x,y) = xe^{-xy}$ at $(1,-2,e^2)$.

Ans $3e^2(x-1) - e^2(y+2) - (z - e^2) = 0$

b) To $xy^2 + yz^2 + zx^2 = 5$ at $(1,2,-1)$.

Ans $2(x-1) + 5(y-2) - 3(z+1) = 0$

c) To the parametrized surface $x = s^2 + t$, $y = t^2 + s$, $z = s^3$ at the point obtained when $s = 2$ and $t = 1$.

Ans $-24(x-5) + 12(y-3) + 7(z-8) = 0$

Fall2000.2) An object is moving in 3-space in such a way that its acceleration vector as a function of time is $\vec{a} = \vec{j} + (\cos t)\vec{k}$. Suppose that at time $t = 0$ you know that its velocity vector is $\vec{v}(0) = \vec{i}$ and its position vector is $\vec{r}(0) = \vec{k}$.
a) Find the velocity vector and the position vector as functions of time t.

Ans $\vec{v} = \vec{i} + t\vec{j} + (\sin t)\vec{k}, \quad \vec{r} = t\vec{i} + \frac{t^2}{2}\vec{j} + (2 - \cos t)\vec{k}$

b) Give the parametric equations of motion.

Ans $x = t, \quad y = \frac{t^2}{2}, \quad z = 2 - \cos t$

c) Find the curvature at time $t = 2\pi$.

Ans $\dfrac{\sqrt{2 + 4\pi^2}}{(1 + 4\pi^2)^{3/2}}$

Fall2000.3) A quantity Q depends upon x and y according to $Q = \dfrac{e^{xy}}{x^2 + y^2}$. Both x and y are changing with time t and suppose at a certain instant we know that $x = 1$, $y = 2$, $\dfrac{dx}{dt} = -2$ and $\dfrac{dy}{dt} = 6$. Use the chain rule to find $\dfrac{dQ}{dt}$ at this instant.

Ans $-\frac{2e^2}{5}$

Fall2000.4) Use the method of Lagrange multipliers to find the maximum value and the minimum value of $f(x, y) = y^2 - 2y - x^2$ on the circle $x^2 + y^2 = 4$.

Ans $f(0, 2) = 0, \quad f(0, -2) = 8 \longleftarrow \text{max} \quad f(\pm\frac{1}{2}\sqrt{15}, \frac{1}{2}) = -\frac{18}{4} \longleftarrow \text{min}$

Fall2000.5) Given three points $P(1, 0, 1)$, $Q(2, 0, 0)$, $R(-1, 2, 2)$
a) Find the area of the triangle having P, Q and R as vertices. Ans $\frac{3}{2}$
b) Find the angle at the vertex P of the triangle having P, Q and R as vertices.

Ans $\theta = 135°$

c) Find the equation of the plane containing the points P, Q and R.

Ans $(2, 1, 2) \cdot (x - 1, y - 0, z - 1) = 0$

Fall2000.7) Find the volume of the 3-dimensional region enclosed by the surfaces $y = x^2$, $y = 4$, and $z = 10 - y$, $z = 0$.

Ans $\frac{1216}{15}$

Fall2000.8) Calculate the area of that part of the surface $z = y^2$ which is above the region of the xy-plane bounded by $y = x$, $x = 0$, and $y = 1$.

Ans $\frac{1}{12}(5\sqrt{5} - 1)$

Verify the divergence theorem for the special case where the vector field is $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and the surface S is the sphere $x^2 + y^2 + z^2 = 4$ by direct calculation of both the surface integral and the triple integral.

Solution: $\text{div}\vec{F} = 1 + 1 + 1 = 3$, so

$$\iiint_T (\text{div}\vec{F})\, dV = \iiint_T 3dV = 3\text{vol}(T) = 3 \cdot \frac{4}{3}\pi 2^3 = 32\pi.$$

To calculate $\iint_S \vec{F} \cdot \vec{n}\, dS$, we find the outward pointing unit normal to

$x^2 + y^2 + z^2 = 4$ is $\vec{n} = \frac{x}{2}\vec{i} + \frac{y}{2}\vec{j} + \frac{z}{2}\vec{k}$. Also,

$$\vec{F} \cdot \vec{n} = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{4}{2} = 2.$$

Thus $\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S 2dS = 2 \iint dS = 2\text{area}(S) = 2(4\pi(2^2)) = 32\pi$

Verify Stokes' Theorem by directly calculating both sides of the equation in the special case where the vector field is $\vec{F} = zy\vec{i} + z\vec{k}$ and the surface is $z = x^2 + y^2$, $z \leq 4$.

Solution: $\text{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ zy & 0 & z \end{vmatrix} = y\vec{j} - z\vec{k}.$

$$\iint_S (\text{curl}\vec{F}) \cdot \vec{n}\, dS = \iint_R (y(-2y) - z)dxdy = \iint_R (-x^2 - 3y^2)dxdy$$

$$= \int_0^{2\pi}\int_0^2 (-r^2\cos^2\theta - 3r^2\sin^2\theta)rdrd\theta = \int_0^{2\pi}\int_0^2 (-\cos^2\theta - 3\sin^2\theta)r^3drd\theta$$

$$= 4\int_0^{2\pi} [-\tfrac{1}{2}(1+\cos 2\theta) - \tfrac{3}{2}(1-\cos 2\theta)]d\theta = 4(-\tfrac{1}{2} - \tfrac{3}{2})2\pi = -16\pi.$$

On the other hand, the boundary curve C is when $x^2 + y^2 = 4$, and $z = 4$. Now parameterize,

$C : x = 2\cos t,\ y = 2\sin t,\ z = 4,\ 0 \leq t \leq 2\pi$
$dx = -2\sin t dt,\ dy = 2\cos t dt,\ dz = 0.$

$$\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C zydx + zdz = \int_0^{2\pi} 4(2\sin t)(-2\sin t)dt$$

$$= -16\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2t)dt = -16(\tfrac{1}{2})2\pi = -16\pi.$$