Some Solutions to Exam 2 Problems

1) Let $F(x,y,z) = x\sqrt{x^2+y^2+z^2}$

a) Find the gradient vector field of F

$$\boxed{\nabla F = \left(\sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}}, \frac{xy}{\sqrt{x^2+y^2+z^2}}, \frac{xz}{\sqrt{x^2+y^2+z^2}}\right)}$$

b) Find the directional derivative of $F(x,y,z)$ at $(x,y,z) = (3,4,12)$ in the direction of the vector $\vec{v} = -\vec{i} + \vec{j} + \vec{k}$.

A unit vector in the direction $\vec{v}$ is $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{(-1,1,1)}{\sqrt{3}}$

Also, $\nabla F(3,4,12) = \left(13 + \dfrac{9}{13}, \dfrac{12}{13}, \dfrac{36}{13}\right)$

Thus $D_{\vec{u}}F(3,4,12) = \nabla F(3,4,12) \cdot \vec{u} = \left(13 + \dfrac{9}{13}, \dfrac{12}{13}, \dfrac{36}{13}\right) \cdot \dfrac{(-1,1,1)}{\sqrt{3}} = \boxed{\frac{-10}{\sqrt{3}}}$

c) Find a unit vector which points in the direction that $F(x,y,z)$ is changing most rapidly at the point $(0,3,4)$. What is the value of the directional derivative of $F(x,y,z)$ in this direction at (0,3,4)?

First calculate $\nabla F(0,3,4) = (5,0,0)$

Then the maximum value of the directional derivative is $|\nabla F(0,3,4)| = 5$,

and it happens in the direction $\dfrac{\nabla F(0,3,4)}{|\nabla F(0,3,4)|} = (1,0,0)$.

In other words, $\dfrac{\partial F}{\partial x}(0,3,4) = 5$ is maximal.

2) Use the method of Lagrange multipliers to find the largest value and the smallest value of $f(x,y) = \frac{1}{3}x^3 - 2x - y^2$ on the ellipse $x^2 + 2y^2 = 9$.

The system of equations is

$$\begin{cases} x^2 - 2 = \lambda 2x \\ -2y = \lambda 4y \\ x^2 + 2y^2 = 9 \end{cases}$$

The second equation factors and gives:

$0 = \lambda 4y + 2y \Longrightarrow 0 = 2\lambda y + y \Longrightarrow 0 = (2\lambda + 1)y$. Thus,

$y = 0 \Longrightarrow x = \pm 3$

Thus we obtain the critical points $(\pm 3, 0)$. Also,

$\lambda = -\dfrac{1}{2} \Longrightarrow x^2 - 2 = -x \Longrightarrow x^2 + x - 2 = 0 \Longrightarrow (x+2)(x-1) = 0 \Longrightarrow x = -2$ or $x = 1$.

In the case $x = -2$, we obtain from the constraint:

$4 + 2y^2 = 9 \Longrightarrow 2y^2 = 5 \Longrightarrow y^2 = \dfrac{5}{2} \Longrightarrow y = \pm\sqrt{\dfrac{5}{2}}$

Thus we find the critical points $(-2, \pm\sqrt{\dfrac{5}{2}})$.

Similarly, the case $x = 1$ yields:

$1 + 2y^2 = 9 \Longrightarrow 2y^2 = 8 \Longrightarrow y^2 = 4 \Longrightarrow y = \pm 2$.

And this gives the critical points $(1, \pm 2)$.

Finally, we test the values of these six critical points:

$f(3,0) = 3 \longleftarrow$ maximum

$f(-3,0) = -3$

$f(-2, \pm\sqrt{\dfrac{5}{2}}) = -\dfrac{7}{6}$

$f(1, \pm 2) = -\dfrac{17}{3} \longleftarrow$ minimum

3) Find and classify the critical points of $f(x, y) = 12xy - x^2y - y^3$.

Setting both partials equal to 0 gives the system:

$$\begin{cases} 12y - 2xy = 0 \\ 12x - x^2 - 3y^2 = 0 \end{cases}$$

The first equation factors into $2y(6 - x) = 0$. Thus,

$y = 0 \Longrightarrow 12x - x^2 = 0 \Longrightarrow x(12 - x) = 0 \Longrightarrow x = 0$ or $x = 12$

Hence we obtain the critical points $(0, 0)$, $(12, 0)$. Also,

$x = 6 \Longrightarrow 72 - 36 - 3y^2 = 0 \Longrightarrow 36 - 3y^2 = 0 \Longrightarrow y^2 = 12 \Longrightarrow y = \pm\sqrt{12}$

This gives us the critical points $(6, \pm\sqrt{12})$

Note that $\Delta = 12y^2 - (12 - 2x)^2$

Check:
$\Delta(0, 0) < 0$
$\Delta(12, 0) < 0$
$\Delta(6, \sqrt{12}) > 0$ and $f_{xx}(6, \sqrt{12}) < 0$
$\Delta(6, -\sqrt{12}) > 0$ and $f_{xx}(6, -\sqrt{12}) > 0$

Thus,

$(0, 0)$ saddle
$(12, 0)$ saddle
$(6, \sqrt{12})$ local max
$(6, -\sqrt{12})$ local min