

Exam 2 Practice Problems

Chain rule problems:

- 1) A quantity Q depends on x and y according to $Q = xe^{x^2y}$. Both x and y are changing with time t . Suppose at a certain instant you know that $x = 2$, $y = -1$, $dx/dt = 3$ and $dy/dt = 1/2$. Use the chain rule to find dQ/dt at this instant.
- 2) Given that $z = f(x, y)$ with x and y functions of s and t by $x = st$, $y = s^3 - t^3$ with $s > 0$ and $t > 0$. Suppose you know that at $(x, y) = (4, 0)$, $\frac{\partial f}{\partial x}(4, 0) = 2$ and $\frac{\partial f}{\partial y}(4, 0) = 1$. Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ when $(x, y) = (4, 0)$.

Gradients and directional derivatives:

- 1) Let $f(x, y) = 6x\sqrt{x + y^2}$. Find
 - a) ∇f (the gradient of f)
 - b) the directional derivative of f at $(x, y) = (5, 2)$ in the direction of the vector $\vec{a} = -2\vec{i} + \vec{j}$.
 - c) the value of the largest directional derivative if f at $(x, y) = (5, 2)$ and a unit vector which points in the direction which gives this largest directional derivative.
- 2) Let $f(x, y, z) = xy + yz + xz$. Find the instantaneous rate of change of $f(x, y, z)$ per unit distance at the point $(x, y, z) = (1, 2, 3)$ in the direction of the vector $\vec{i} - 4\vec{j} + 2\vec{k}$.
- 3) Given $f(x, y) = ye^{xy}$
 - a) Find the gradient vector field of $f(x, y)$.
 - b) At the point $(x, y) = (-1, 2)$ find the instantaneous rate of change of $f(x, y)$ per unit distance in the direction of the vector $\vec{v} = \vec{i} + 6\vec{j}$.
 - c) Find the unit vector which points in the direction you should follow, if starting at $(x, y) = (-1, 2)$, you want to achieve the most rapid increase possible for $f(x, y)$.

Tangent planes given by the gradient:

- 1) Find the equation of the tangent plane to the given surface at the specified point.
 - a) $xyz^2 + x^3 + y^3 - 2z^3 = 3$ at $(-1, 2, 1)$.
 - b) $z = \sqrt{x^2 + 2y^2}$ at $(1, 2, 3)$.
 - c) $x^2y + y^2z = z^2x + 5$ at $(1, 2, 3)$.
 - d) $x^3y + y^3z + xz^3 = 53$ at $(1, 2, 3)$.
 - e) $x^3 + y^3 + xyz^3 = 41$ at $(3, 2, 1)$.
 - f) $z = 2x\sqrt{x + y^2}$ at $(5, 2, 30)$.

Lagrange multipliers:

- 1) Use the method of Lagrange multipliers to find the maximum value and the minimum value of $f(x, y) = xy$ on the ellipse $x^2 + 2y^2 = 36$.
- 2) Use the method of Lagrange multipliers to find the largest value and the smallest value of $f(x, y) = x^2 + 2x - y^2$ on the circle $x^2 + y^2 = 16$.
- 3) Use the method of Lagrange multipliers to find the largest value and smallest value of $f(x, y, z) = x + 2y + z^2$ on the ellipsoid $x^2 + y^2 + 2z^2 = 20$.
- 4) Use the method of Lagrange multipliers to find the maximum and the minimum value for $f(x, y) = \frac{1}{3}x^3 - 3x + y^2$ on the circle $x^2 + y^2 = 16$.

Find and classify critical points:

- 1) Find and classify the critical points for $f(x, y) = x^3 + 3xy^2 - 6xy$.
- 2) Find and classify the critical points of $f(x, y) = 4xy - y^4 - x^2$.
- 3) Find and classify the critical points of $f(x, y) = x^3y^2 + 2xy - 2x$.
- 4) Find and classify the critical points of $f(x, y) = \frac{1}{2}x^2y - \frac{1}{2}x^2 + \frac{1}{3}y^3 - \frac{3}{2}y^2$.
- 5) Let $f(x, y) = 12xy - x^2y - y^3$. Find and classify the critical points of $f(x, y)$.
- 6) Find and classify the critical points of $f(x, y) = 4xy - xy^2 - x^3$.

Volumes and areas using double integrals:

- 1) Use a double integral to calculate the volume of the three dimensional region which is under $z = 1 + xy$ and above the region in the first quadrant of the xy -plane which is bounded by $y = 2x$, $x = 0$ and $y = 4$.
- 2) Calculate the volume of the region in the first octant which is enclosed by the surfaces $y = x^2$, $y = 4$, $x = 0$, $z = 3$, and $y + z = 8$.
- 3) Find the volume trapped between the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Use polar coordinates to evaluate the double integral.
- 4) Find the volume of the region which is under the surface $z = x + xy$ and above the region in the first quadrant of the xy -plane that is bounded by $y = 2x$, $y = 0$, and $x = 2$.
- 5) Evaluate the integral $\int \int_R xy \, dA$ where R is the region in the first quadrant enclosed by the parabolas $4y = x^2$ and $4x = y^2$.
- 6) Find the volume of the 3-dimensional region which is under the plane $z = 1 + 2y$ and above the bounded region of the xy -plane which is enclosed by $y = x^2$ and $y = 4$.
- 7) Use a double integral in polar coordinates to calculate the volume of the 3-dimensional region enclosed by the surfaces $z = x^2 + y^2$ and $z = 6 - \sqrt{x^2 + y^2}$.

Reversing the order of integration:

- 1) Evaluate $\int_0^2 \int_{y^2}^4 \sqrt{1 + x\sqrt{x}} \, dx \, dy$ by first reversing the order of integration.
- 2) Evaluate $\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1 + y^3} \, dy \, dx$ by first reversing the order of integration.
- 3) First sketch the region of integration corresponding to the iterated integral $\int_0^2 \int_{x^2}^4 \frac{2x}{1 + y^2} \, dy \, dx$ and then reverse the order of integration to evaluate the integral.