STEPANOV'S METHOD APPLIED TO BINOMIAL EXPONENTIAL SUMS

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ABSTRACT. For a prime p and binomial $ax^k + bx^l$ with $1 \leq l < k < \frac{1}{32}(p-1)^{\frac{2}{3}}$, we use Stepanov's method to obtain the bound

$$\left| \sum_{x=1}^{p-1} e_p(ax^k + bx^l) \right| \ll \max \left\{ 1, l\Delta^{-\frac{1}{3}} \right\}^{\frac{1}{4}} k^{\frac{1}{4}} p^{\frac{3}{4}},$$

where $\Delta = \frac{k-l}{(k, l, p-1)}$.

1. INTRODUCTION

For a prime p and binomial $f(x) = ax^k + bx^l$ we consider the binomial exponential sum

$$(1.1) \quad S^* = \sum_{x=1}^{p-1} e_p(ax^k + bx^l), \quad p \nmid ab, \quad 1 \leq l < k < p-1,$$

where $e_p(\cdot)$ is the additive character $e_p(\cdot) = e^{2\pi i \cdot / p}$ on the finite field $\mathbb{Z}_p$. For such sums the classical Weil bound [9]

$$(1.2) \quad |S^*| \leq (k-1)p^{\frac{1}{2}}$$

is non-trivial as long as $k < p^{\frac{1}{2}}$. When l is significantly smaller than k a bound of Mordell [7] as refined in [2] can extend this range:

$$(1.3) \quad |S^*| < (kl)^{\frac{1}{4}} p^{\frac{3}{4}},$$

a bound also obtained in the special case $l = 1$ by Karatsuba [6]. For monomial sums Heath-Brown & Konyagin [5] successfully used Stepanov's method to obtain a bound non-trivial for $k \leq p^{\frac{2}{3}}$

$$(1.4) \quad \left| \sum_{x=1}^{p-1} e_p(ax^k) \right| \ll \min \left\{ k^{\frac{5}{8}} p^{\frac{5}{8}}, k^{\frac{3}{8}} p^{\frac{3}{4}} \right\},$$

improving upon the Weil bound (1.2) once $k > p^{\frac{1}{3}}$. Konyagin informs us that he has made additional refinements, further extending this range. We show here how Stepanov's

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method can be similarly used for binomials to improve the Mordell and Weil bounds when $k < \frac{1}{32}(p-1)^{\frac{2}{3}}$. We are able to deal with the slightly more general sums

$$(1.5) \quad S_+ = \sum_{x=1}^{p-1} \chi(x) e_p(ax^k + bx^l), \quad p \nmid ab, \quad 1 \leq l < k < p-1,$$

and

$$(1.6) \quad S_- = \sum_{x=1}^{p-1} \chi(x) e_p(ax^k + bx^{-l}), \quad p \nmid ab, \quad 1 \leq l \leq k, \quad (k+l) < p-1,$$

where $\chi(x)$ is a multiplicative character mod p . For these generalized sums the Weil bound (see [8]) still gives

$$(1.7) \quad |S_+| \leq kp^{\frac{1}{2}}, \quad |S_-| \leq (k+l)p^{\frac{1}{2}},$$

and the Mordell type bound (see [2]),

$$(1.8) \quad |S_+| < (kl)^{\frac{1}{4}} p^{\frac{3}{4}}, \quad |S_-| < (3kl)^{\frac{1}{4}} p^{\frac{3}{4}}.$$

We write

$$(1.9) \quad d = (l, k, p-1),$$

and

$$(1.10) \quad \Delta_+ = \frac{(k-l)}{d}, \quad \Delta_- = \frac{(k+l)}{d}.$$

Theorem 1.1. For $k < \frac{1}{32}(p-1)^{\frac{2}{3}} d^{\frac{1}{3}}$,

$$(1.11) \quad |S_{\pm}| \ll \max \left\{ 1, l\Delta_{\pm}^{-\frac{1}{3}} \right\}^{\frac{1}{4}} k^{\frac{1}{4}} p^{\frac{3}{4}}.$$

Notice that when $k \ll p^{\frac{2}{3}} d^{\frac{1}{3}}$ and Theorem 1.1 applies, our bound (1.11) is plainly never worse in order than the Mordell type bound (1.3), improving on it once $l \gg 1$ and $k-l \gg d$. One particular feature of our result is that we do not need the restrictions $k|(p-1)$ and $(k, l) = 1$ that appear in the binomial bounds (1.14) and (1.15) of Akulinicev [1] and Yu [10]. As a special case, when $l < (k/d)^{\frac{1}{3}}$ we observe that Theorem 1.1 simplifies to $|S_{\pm}| \ll k^{\frac{1}{4}} p^{\frac{3}{4}}$, always non-trivial and surpassing the Weil bound (1.2) when $p^{\frac{1}{3}} \ll k \ll p^{\frac{2}{3}}$, the same range as for the Heath-Brown & Konyagin monomial result (1.4).

In [2, Theorem 1.2] we showed that

$$(1.12) \quad |S_{\pm}| \leq p^{\frac{1}{4}} M_{\pm}^{\frac{1}{4}},$$

where

$$M_+ = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_p^{*4} : x_1^k + x_2^k = x_3^k + x_4^k, \quad x_1^l + x_2^l = x_3^l + x_4^l\},$$

and

$$M_- = \#\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}_p^{*4} : x_1^k + x_2^k = x_3^k + x_4^k, \quad x_1^{-l} + x_2^{-l} = x_3^{-l} + x_4^{-l}\},$$

with (1.8) corresponding to the straightforward bounds

$$(1.13) \quad M_+ < kl(p-1)^2, \quad M_- < 3kl(p-1)^2.$$

Hence Theorem 1.1 amounts to showing that in certain cases we can improve these bounds:

Lemma 1.1. *For $k < \frac{1}{32}(p-1)^{\frac{2}{3}}d^{\frac{1}{3}}$,*

$$M_{\pm} \ll \max \left\{ 1, l\Delta_{\pm}^{-\frac{1}{3}} \right\} kp^2.$$

This result is perhaps initially surprising since there are certainly situations in which (1.13) can not be improved; for example when $k = 2l$, $l|(p-1)$ it is readily verified that $M_+ = 2l^2(p-1)^2 - l^3(p-1)$, and when $k = l$ and $2l|(p-1)$ that $M_- = 3l^2(p-1)^2 - 3l^3(p-1)$, so that $M_{\pm} = O(kp^2)$ is not in general true. The proof of Lemma 1.1 occupies sections 2, 3 and 4. Theorem 1.1 follows immediately from Lemma 1.1 and (1.12).

There are two parameters that must be controlled in order to expect a uniform non-trivial bound on $|S_{\pm}|$ for arbitrary f with given exponents k, l . The first is $d = (k, l, p-1)$. In [3, Corollary 2.1] the authors showed that for given $1 < l < k$ with $d > \frac{p}{\log p}$ there exist a, b such that $|S^*|$ is very close to p in size; to be precise $|S^*| > p \left(1 - \frac{2\pi^2}{\lfloor p^{2d/(p-1)} \rfloor^2} \right) - 1$. The second parameter is the quantity

$$d_+^* = (k-l, p-1), \quad d_-^* = (k+l, p-1).$$

The following theorem shows that there is always a collection of exceptional sums where $|S_{\pm}| \sim d_{\pm}^*$, but that aside from these exceptions there is a uniform nontrivial bound when $d_{\pm}^*$ is large:

Proposition 1.1. *a) If $\chi^{(p-1)/d_{\pm}^*} = \chi_0$ and $-b\bar{a} = \alpha^{(k \mp l)}$ for some α in $\mathbb{Z}_p^*$, then*

$$|S_{\pm} - \chi(\alpha)d_{\pm}^*| \leq (p-1-d_{\pm}^*) \frac{d}{d_{\pm}^*} p^{\frac{1}{2}}.$$

b) Otherwise, we have uniformly

$$|S_{\pm}| \leq \frac{d}{d_{\pm}^*} p^{\frac{3}{2}}.$$

Though the bound as stated is only non-trivial when $d_{\pm}^* > dp^{\frac{1}{2}}$, by using (1.4) instead of the Weil bound in (5.1), one can replace the right-hand side of both inequalities by

$$O \left(\min \left\{ (d/d_{\pm}^*)^{\frac{5}{8}} p^{\frac{5}{4}}, (d/d_{\pm}^*)^{\frac{3}{8}} p^{\frac{9}{8}} \right\} \right),$$

extending this range to $d_{\pm}^* \gg dp^{\frac{1}{3}}$ and improving the bound when $d_{\pm}^* \ll dp^{\frac{2}{3}}$.

We deduce from Theorem 1.1 and Proposition 1.1 the following uniform upper bound on $S_{\pm}$, nontrivial for $d < cp^{1/12}$, c a sufficiently small constant.

Corollary 1.1. *a) If $S_{\pm}$ is not one of the exceptional sums in part (a) of Proposition 1.1 or $d_{\pm}^* \leq d^{3/13} p^{51/52}$ then we have uniformly*

$$|S_{\pm}| \ll d^{3/13} p^{51/52}.$$

b) If $S_{\pm}$ is one of the exceptional sums in part (a) of Proposition 1.1 and $d_{\pm}^ > d^{3/13} p^{51/52}$ then*

$$|S_{\pm} - \chi(\alpha)d_{\pm}^*| \ll d^{10/13} p^{27/52}.$$

In particular, we see that if χ is the trivial character or Legendre symbol and

$$f(x) = a \left(x^{\frac{1}{2}(p-1)} - \alpha^{\frac{1}{2}(p-1)} \right) x^{\pm l}$$

for some α in $\mathbb{Z}_p^*$ and $1 \leq l < \frac{1}{2}(p-1)$, then

$$\left| S_{\pm} - \frac{\chi(\alpha)}{2}(p-1) \right| \leq dp^{\frac{1}{2}}.$$

For all remaining cases, if $d < cp^{1/12}$ with c a sufficiently small constant,

$$|S_{\pm}| \leq \frac{1}{3}(p-1) + 2d\sqrt{p}.$$

In comparison, for binomials with $k|(p-1)$ and $(k, l) = 1$ Akulinicev [1] obtained the bound

$$(1.14) \quad |S^*| \leq \frac{p}{\sqrt{k}} + \sqrt{(l, p-1) - 1} p^{\frac{3}{4}}$$

and deduced when $l = 1$ and $k|(p-1)$ the uniform bound

$$|S^*| \leq p^{\frac{5}{6}}.$$

Yu [10] obtained

$$(1.15) \quad |S^*| \ll k^{-\frac{3}{8}} p^{\frac{9}{8}}$$

and hence the uniform bound

$$|S^*| \ll p^{\frac{21}{22}}$$

under the weaker assumptions $k|(p-1)$ and $(k, l) = 1$.

2. INITIAL DECOMPOSITION OF $M_{\pm}$

We write

$$M_{\pm} = \sum_{0 \leq u_1, u_2 \leq p-1} R_{\pm}(u_1, u_2)^2$$

where

$$R_{\pm}(u_1, u_2) = \#\{(x, y) \in \mathbb{Z}_p^{*2} : x^k - y^k = u_1, x^{\pm l} - y^{\pm l} = u_2\}.$$

Plainly

$$R_{\pm}(0, 0) = \#\{(x, y) \in \mathbb{Z}_p^{*2} : (x\bar{y})^k = 1, (x\bar{y})^{\pm l} = 1\} = d(p-1).$$

When $(u_1, u_2) \neq (0, 0)$ we always have the trivial bound

$$R_{+}(u_1, u_2) \leq (k-d)l,$$

since x must be a root of the non-zero polynomial

$$(x^k - u_1)^{l/d} - (x^l - u_2)^{k/d} = 0$$

of degree at most $(k/d-1)l$ and $y^{(l,k)} = (x^k - u_1)^s (x^l - u_2)^t$ where $ks + lt = (l, k)$. For M_{-} we have $(x^k - u_1)^{l/d} (u_2 x^l - 1)^{k/d} - (-1)^{k/d} x^{kl/d} = 0$ and $R_{-}(u_1, u_2) \leq 2kl$. Hence, writing

$$\ell_{+} = l, \quad \ell_{-} = 2l,$$

we have

$$\sum_{u_1=1}^{p-1} R_{\pm}^2(u_1, 0) < k\ell_{\pm} \sum_{u_1=0}^{p-1} R_{\pm}(u_1, 0) = k\ell_{\pm} \#\{(x, y) \in \mathbb{Z}_p^{*2} : x^{\pm l} = y^{\pm l}\} \leq l(p-1)k\ell_{\pm}.$$

Likewise $\sum_{u_2=1}^{p-1} R_{\pm}^2(0, u_2) < k^2\ell_{\pm}(p-1)$. For the $u_1, u_2 \neq 0$ we write $R_{\pm}(u_1, u_2) = dC_{\pm}(u_1, u_2)$ where

$$C_{\pm}(u_1, u_2) = \#\{x \in \mathbb{Z}_p^{*} : x^k - 1 = u_1 y^k, x^{\pm l} - 1 = u_2 y^{\pm l} \text{ for some } y \in \mathbb{Z}_p^{*}\};$$

to see this observe that counting the number of (x, y) satisfying $(x\bar{y})^k - 1 = u_1 \bar{y}^k, (x\bar{y})^{\pm l} - 1 = u_2 \bar{y}^{\pm l}$, is the same as counting the (x, y) satisfying $x^k - 1 = u_1 y^k, x^{\pm l} - 1 = u_2 y^{\pm l}$, where $y^{(k,l)} = (x^k - 1)^s (x^{\pm l} - 1)^t \bar{u}_1^s \bar{u}_2^t$, $ks \pm lt = (k, l)$ shows that there will be exactly d different y for each x with a solution. Clearly if $u_1 = u^k \tilde{u}_1$ then $u_1 y^k = \tilde{u}_1 (uy)^k$, $u_2 y^{\pm l} = u_2 \bar{u}^{\pm l} (uy)^{\pm l}$ and $C_{\pm}(u_1, u_2) = C_{\pm}(\tilde{u}_1, u_2 \bar{u}^{\pm l})$, while if $u_2 = u^{\pm l} \tilde{u}_2$ with $u^k = 1$ then $u_1 y^k = u_1 (uy)^k$, $u_2 y^{\pm l} = \tilde{u}_2 (uy)^{\pm l}$ and $C_{\pm}(u_1, u_2) = C_{\pm}(u_1, \tilde{u}_2)$. Hence, using α to

denote a primitive root mod p , and writing $k_1 = (k, p-1)$,

$$\begin{aligned} \sum_{1 \leq u_1, u_2 \leq p-1} R_{\pm}(u_1, u_2)^2 &= d^2 \sum_{u_2=1}^{p-1} \sum_{u_1=1}^{p-1} C_{\pm}^2(u_1, u_2) \\ &= d^2 \frac{p-1}{k_1} \sum_{u_2=1}^{p-1} \sum_{j=1}^{k_1} C_{\pm}^2(\alpha^j, u_2) = d(p-1) \sum_{j=1}^{k_1} \sum_{i=1}^{\frac{(p-1)d}{k_1}} C_{\pm}^2(\alpha^j, \alpha^i). \end{aligned}$$

We show that for this latter sum no x will be counted more than once: If

$$x^k - 1 = \alpha^{j_1} y_1^k = \alpha^{j_2} y_2^k, \quad x^{\pm l} - 1 = \alpha^{i_1} y_1^{\pm l} = \alpha^{i_2} y_2^{\pm l},$$

for some $y_1, y_2 \in \mathbb{Z}_p^*$ and i_1, i_2, j_1, j_2 with $1 \leq j_1, j_2 \leq k_1$ and $1 \leq i_1, i_2 \leq (p-1)d/k_1$, then $\alpha^{(j_1-j_2)} = (y_2 \bar{y}_1)^k$ and $j_1 = j_2$. Since $(y_2 \bar{y}_1)^k = 1$ we have $y_2 \bar{y}_1 = \alpha^{(p-1)u/k_1}$ for some u , and

$$\alpha^{(i_1-i_2)} = (y_2 \bar{y}_1)^{\pm l} = \alpha^{\pm \frac{(p-1)}{(k_1/d)} u(l/d)},$$

forcing $(p-1)d/k_1 | (i_1 - i_2)$ and $i_1 = i_2$.

For a pair u_1, u_2 in $\mathbb{Z}_p^*$ and corresponding vector $\mathbf{u} = (u_1, u_2)$ we shall sometimes write $C_{\pm}(\mathbf{u})$ for $C_{\pm}(u_1, u_2)$. We suppose that there are N non-zero $C_{\pm}(\alpha^j, \alpha^i)$, say $C_{\pm}(\mathbf{u}_1), \dots, C_{\pm}(\mathbf{u}_N)$, as i, j run through $1 \leq j \leq k_1$, $1 \leq i \leq (p-1)d/k_1$ and order them so that

$$C_{\pm}(\mathbf{u}_1) \geq C_{\pm}(\mathbf{u}_2) \geq \dots \geq C_{\pm}(\mathbf{u}_N).$$

Hence

$$(2.1) \quad M_{\pm} \leq d^2(p-1)^2 + 2k^2 \ell_{\pm}(p-1) + d(p-1) \sum_{i=1}^N C_{\pm}^2(\mathbf{u}_i).$$

Also, since the sets defining the $C_{\pm}(\mathbf{u}_i)$ are disjoint,

$$(2.2) \quad \sum_{i=1}^N C_{\pm}(\mathbf{u}_i) \leq (p-1).$$

3. APPLYING THE STEPANOV METHOD

We write

$$d_1 = (l, k), \quad \delta_+ = \frac{k-l}{d_1}, \quad \delta_- = \frac{k+l}{d_1}, \quad \ell_+ = l, \quad \ell_- = 2l,$$

and set

$$(3.1) \quad T = \begin{cases} \left\lfloor 2^{-\frac{9}{2}}(p-1) \left(\frac{\delta_{\pm}^2}{k\ell_{\pm}/d_1} \right)^{-\frac{3}{2}} \right\rfloor, & \text{if } (k\ell_{\pm}/d_1) < \frac{1}{2}\delta_{\pm}^2, \\ \left\lfloor 2^{-7}(p-1) \left(\frac{\delta_{\pm}^2}{k\ell_{\pm}/d_1} \right) \right\rfloor, & \text{if } (k\ell_{\pm}/d_1) \geq \frac{1}{2}\delta_{\pm}^2. \end{cases}$$

We use Stepanov's method to bound the size of the $C_{\pm}(\mathbf{u}_t)$ for $t \leq T$. If $T > N$ we set $C_{\pm}(\mathbf{u}_t) = 0$ for the $N < t \leq T$.

Lemma 3.1. *Suppose that $k < 2^{-5}(p-1)^{\frac{2}{3}}d_1^{\frac{1}{6}}\ell_{\pm}^{\frac{1}{6}}$. Then*

$$(3.2) \quad C_{\pm}(\mathbf{u}_t) \leq 2^{\frac{26}{5}}(p-1)^{\frac{2}{5}}(k\ell_{\pm}/d_1)^{\frac{3}{5}}t^{-\frac{2}{5}}\delta_{\pm}^{-\frac{1}{5}}, \quad 1 \leq t \leq T.$$

In particular $C_{\pm}(\mathbf{u}_T) \leq 2^{\frac{37}{5}}\delta_{\pm}$ when $(k\ell_{\pm}/d_1) < \frac{1}{2}\delta_{\pm}^2$.

Proof. If x is counted in $C_+(u_1, u_2)$ so that $x^k - 1 = u_1 y^k$, $x^l - 1 = u_2 y^l$, for some $y \in \mathbb{Z}_p^*$, then

$$(x^k - 1)^{l/d_1} u_2^{k/d_1} - (x^l - 1)^{k/d_1} u_1^{l/d_1} = 0,$$

while for $C_-(u_1, u_2)$ we have instead $x^{-l} - 1 = u_2 y^{-l}$ and

$$(x^k - 1)^{l/d_1} (x^l - 1)^{k/d_1} - (-1)^{k/d_1} u_1^{l/d_1} u_2^{k/d_1} x^{kl/d_1} = 0.$$

Hence we have the trivial bounds

$$C_{\pm}(\mathbf{u}) \leq \min\{(p-1), k\ell_{\pm}/d_1\},$$

and may assume that

$$(3.3) \quad t^{2/5} > 2^{\frac{26}{5}} \frac{(k\ell_{\pm}/d_1)^{\frac{3}{5}}}{\delta_{\pm}^{\frac{1}{5}}(p-1)^{\frac{3}{5}}}, \quad t^{2/5} > 2^{\frac{26}{5}} \frac{(p-1)^{\frac{2}{5}}}{(k\ell_{\pm}/d_1)^{\frac{2}{5}}\delta_{\pm}^{\frac{1}{5}}}.$$

Consider the polynomials

$$F(x) = \sum_{i < A} \Phi_i(x) f_i(x),$$

with

$$\Phi_i(x) = (x^k - 1)^{(A-i)l/d_1} (x^l - 1)^{ik/d_1}$$

for M_+ and

$$\Phi_i(x) = (x^k - 1)^{il/d_1} (x^l - 1)^{ik/d_1} x^{(A-i)kl/d_1}$$

for M_- , and

$$f_i(x) = \sum_{j < B} \sum_{s, t < C} a_{ijst} x^{j(p-1)+sk+tl},$$

where A, B, C are integer parameters which will be chosen to satisfy

$$(3.4) \quad A, B, C \geq 1,$$

$$(3.5) \quad Ck \leq \frac{1}{2}(p-1),$$

$$(3.6) \quad BC^2 \leq \delta_{\pm},$$

$$(3.7) \quad A\delta_{\pm} \leq (p-1),$$

$$(3.8) \quad B \leq \frac{k\ell_{\pm}}{d_1} \frac{A}{(p-1)},$$

$$(3.9) \quad D(C+D)^2 t < ABC^2.$$

Our goal is to show the existence of a set of coefficients $a_{ijst} \in \mathbb{Z}_p$ such that F is a non-zero polynomial mod p with a high order zero at each x counted in $C_{\pm}(\mathbf{u}_1), \dots, C_{\pm}(\mathbf{u}_t)$.

The monomials in $f_i(x)$ will be distinct since if $j(p-1) + sk + tl = j'(p-1) + s'k + t'l$ then from (3.5)

$$|j - j'|(p-1) = |(s - s')k + (t - t')l| < Ck + Cl < (p-1)$$

and $j = j'$, while from (3.6) we have $C < (k/d_1)$ and

$$k(s - s') = l(t' - t) \Rightarrow (k/d_1)|(t' - t) \Rightarrow t = t', s = s'.$$

In [4] we showed that if $f(x) = x^{k_1}g_1(x)^p + \dots + x^{k_r}g_r(x)^p$ in $\mathbb{Z}_p[x]$ with the k_i distinct mod p , and $f(x)$ has a zero $a \neq 0$ of multiplicity ν , then $\nu = wp + u$ with $u < r$ and w the highest power of $(x-a)$ dividing all the g_i . In particular if $(x-1)^{\mu_i} || f_i$ then $\mu_i = u_i + pv_i$ for some u_i, v_i with $u_i < BC^2$. Hence the power of $(x-1)$ dividing the different $\Phi_i f_i$ will be distinct, since if

$$i\delta_{\pm} + u_i + pv_i = i'\delta_{\pm} + u_{i'} + pv_{i'}$$

then, by (3.6) and (3.7)

$$p|v_i - v_{i'}| = |(u_i - u_{i'}) + (i - i')\delta_{\pm}| < BC^2 + (A-1)\delta_{\pm} \leq A\delta_{\pm} < p,$$

and $v_i = v_{i'}$, while by (3.6)

$$|i - i'|\delta_{\pm} = |u_i - u_{i'}| < BC^2 \leq \delta_{\pm}$$

gives $i = i'$. Hence F is not the zero polynomial in $\mathbb{Z}_p[x]$ unless all the $a_{ijst} = 0$.

Suppose that a is counted in $C_+(u_1, u_2)$. Since $u_1, u_2 \neq 0$ we have $a^k - 1, a^l - 1, a \neq 0$, and for a positive integer $D < p$, writing

$$\nabla = x(x^k - 1)(x^l - 1) \frac{d}{dx},$$

the polynomial $F(x)$ has a zero of order at least D at $x = a$ iff

$$(x^k - 1)^{-lA/d_1} \nabla^r F(x)|_{x=a} = 0, \quad r = 0, \dots, D-1.$$

For a counted in $C_-(u_1, u_2)$ we use instead

$$x^{-klA/d_1} \nabla^r F(x)|_{x=a} = 0, \quad r = 0, \dots, D-1.$$

Of course

$$\begin{aligned}\nabla x^j &= x^j j(x^k - 1)(x^l - 1), \\ \nabla(x^l - 1)^j &= (x^l - 1)^j j l x^l (x^k - 1), \\ \nabla(x^k - 1)^j &= (x^k - 1)^j j k x^k (x^l - 1).\end{aligned}$$

Hence

$$\nabla^r F(x) = \sum_{i < A} \sum_{j < B} \Phi_i(x) x^{j(p-1)} P_{ijr}(x^k, x^l)$$

where the $P_{ijr}(x, y)$ are polynomials of degree at most $(C - 1 + r)$ in x and y whose coefficients will be linear expressions in the original a_{ijst} . Since for a counted in $C_+(u_1, u_2)$ the values

$$(a^k - 1)^{-lA/d_1} \Phi_i(a) a^{j(p-1)} = (u_2^{k/d_1} / u_1^{l/d_1})^i,$$

and for a counted in $C_-(u_1, u_2)$ the values

$$a^{-klA/d_1} \Phi_i(a) a^{j(p-1)} = \left((-1)^{k/d_1} u_1^{l/d_1} u_2^{k/d_1} \right)^i,$$

depend on $\mathbf{u}$ rather than a , we can make $(x^k - 1)^{-lA/d_1} \nabla^r F(x)$ or $x^{-klA/d_1} \nabla^r F(x)$ zero for each a counted in $C_+(\mathbf{u})$ or $C_-(\mathbf{u})$ by making an appropriate polynomial $P_{r,\mathbf{u}}(x^k, x^l)$ of degree at most $(C - 1 + r)$ in x^k and x^l , identically zero mod p ; a requirement imposing $(C + r)^2$ homogeneous linear conditions on the a_{ijst} . Thus for $F(x)$ to have a zero of order D for each element counted in $C_\pm(\mathbf{u})$ we need at most $D(C + D)^2$ linear equations, and for an integer $t \geq 1$, with $tD(C + D)^2$ equations we can ensure that F has a root of order at least D for each element counted in the $C_\pm(\mathbf{u}_1), \dots, C_\pm(\mathbf{u}_t)$. Since the number of variables a_{ijst} is ABC^2 , and elements counted in different $C_\pm(\mathbf{u}_i)$ are distinct, condition (3.9) ensures the existence of a non-zero polynomial $F(x)$ in $\mathbb{Z}_p[x]$ with $\sum_{i=1}^t C_\pm(\mathbf{u}_i)$ zeros, each of order at least D . Hence, from (3.8) and (3.5)

$$D \sum_{i=1}^t C_\pm(\mathbf{u}_i) \leq \deg(F) \leq A(k\ell_\pm/d_1) + (p-1)(B-1) + Ck + Cl \leq 2A(k\ell_\pm/d_1),$$

and

$$(3.10) \quad C_\pm(\mathbf{u}_t) \leq t^{-1} \sum_{i=1}^t C_\pm(\mathbf{u}_i) \leq \frac{2A(k\ell_\pm/d_1)}{Dt}.$$

It remains to choose parameters satisfying conditions (3.4) through (3.9):

$$\begin{aligned} D = C &= \left\lfloor 2^{-\frac{7}{5}} \frac{(p-1)^{\frac{1}{5}} \delta_{\pm}^{\frac{2}{5}}}{(k\ell_{\pm}/d_1)^{\frac{1}{5}} t^{\frac{1}{5}}} \right\rfloor, \\ B &= \left\lfloor 2^{\frac{4}{5}} \frac{(k\ell_{\pm}/d_1)^{\frac{2}{5}} t^{\frac{2}{5}} \delta_{\pm}^{\frac{1}{5}}}{(p-1)^{\frac{2}{5}}} \right\rfloor, \\ A &= \left\lfloor 2^{\frac{4}{5}} \frac{(p-1)^{\frac{3}{5}} t^{\frac{2}{5}} \delta_{\pm}^{\frac{1}{5}}}{(k\ell_{\pm}/d_1)^{\frac{3}{5}}} \right\rfloor. \end{aligned}$$

These choices ensure restrictions (3.6), (3.8) and (3.9) (the latter simplifies to $4Ct < AB$), and (3.10) gives the stated bound

$$(3.11) \quad C_{\pm}(\mathbf{u}_t) \leq 2^{\frac{26}{5}} (p-1)^{\frac{2}{5}} (k\ell_{\pm}/d_1)^{\frac{3}{5}} t^{-\frac{2}{5}} \delta_{\pm}^{-\frac{1}{5}}.$$

For (3.4) observe that the trivial bounds (3.3) give $A \geq 2^6$, $B \geq 2^6$ respectively, while $C \geq 1$ requires

$$(3.12) \quad t \leq 2^{-7} (p-1) \frac{\delta_{\pm}^2}{(k\ell_{\pm}/d_1)}.$$

The restriction $k < 2^{-5} (p-1)^{\frac{2}{3}} d_1^{\frac{1}{6}} \ell_{\pm}^{\frac{1}{6}}$, and $\delta_{\pm} < 2k/d_1$, ensure that

$$Ck < 2^{-\frac{7}{5}} \frac{(p-1)^{\frac{1}{5}} \delta_{\pm}^{\frac{2}{5}} k}{t^{\frac{1}{5}} (k\ell_{\pm}/d_1)^{\frac{1}{5}}} < \frac{1}{2} \frac{(p-1)^{\frac{1}{5}} k^{\frac{6}{5}}}{t^{\frac{1}{5}} d_1^{\frac{1}{5}} \ell_{\pm}^{\frac{1}{5}}} < \frac{1}{2^7 t^{\frac{1}{5}}} (p-1)$$

so condition (3.5) certainly holds. Finally condition (3.7) is satisfied

$$A\delta_{\pm} < 2^{\frac{9}{5}} \frac{(p-1)^{\frac{3}{5}} t^{\frac{2}{5}} \delta_{\pm}^{\frac{6}{5}}}{(k\ell_{\pm}/d_1)^{\frac{3}{5}}} \leq (p-1)$$

as long as

$$(3.13) \quad t \leq 2^{-\frac{9}{2}} (p-1) \left(\frac{(k\ell_{\pm}/d_1)}{\delta_{\pm}^2} \right)^{\frac{3}{2}}.$$

Restriction (3.13) supercedes (3.12) iff $(k\ell_{\pm}/d_1) < \frac{1}{2} \delta_{\pm}^2$ and the cutoff values T are plain.

Notice that for $(k\ell_{\pm}/d_1) < \frac{1}{2} \delta_{\pm}^2$ and $k < 2^{-5} (p-1)^{\frac{2}{3}} d_1^{\frac{1}{6}} \ell_{\pm}^{\frac{1}{6}}$ we have, using $\delta_{\pm} < 2k/d_1$,

$$T = \left\lfloor 2^{-\frac{9}{2}} (p-1) \left(\frac{(k\ell_{\pm}/d_1)}{\delta_{\pm}^2} \right)^{\frac{3}{2}} \right\rfloor \geq \left\lfloor 2^{-\frac{15}{2}} (p-1) \frac{\ell_{\pm}^{\frac{3}{2}} d_1^{\frac{3}{2}}}{k^{\frac{3}{2}}} \right\rfloor \geq \left\lfloor \ell_{\pm}^{\frac{5}{4}} d_1^{\frac{5}{4}} \right\rfloor \geq 1$$

and the bound (3.11) gives $C(\mathbf{u}_T) \leq 2^{\frac{37}{5}} \delta_{\pm}$.

□

Corollary 3.1. For $k < 2^{-5} (p-1)^{\frac{2}{3}} d_1^{\frac{1}{6}} \ell_{\pm}^{\frac{1}{6}}$,

$$\sum_{i=1}^N C_{\pm}^2(\mathbf{u}_i) \ll \max \left\{ \delta_{\pm}, (k\ell_{\pm}/d_1) \delta_{\pm}^{-\frac{1}{3}} \right\} p.$$

Proof. We assume that $k < 2^{-5}(p-1)^{\frac{2}{3}}d_1^{\frac{1}{6}}l^{\frac{1}{6}}$. When

$$L = \left\lfloor 2^{-7}(p-1) \frac{\delta_{\pm}^{\frac{1}{3}}}{(k\ell_{\pm}/d_1)} \right\rfloor \leq T$$

Lemma 3.1 and (2.2) give

$$\begin{aligned} \sum_{i=1}^N C_{\pm}^2(\mathbf{u}_i) &\ll (p-1)^{\frac{4}{5}}(kl/d_1)^{\frac{6}{5}}\delta_{\pm}^{-\frac{2}{5}} \sum_{t \leq L} t^{-\frac{4}{5}} + (p-1)^{\frac{2}{5}}(kl/d_1)^{\frac{3}{5}}\delta_{\pm}^{-\frac{1}{5}}L^{-\frac{2}{5}} \sum_{t > L} C(\mathbf{u}_t) \\ &\ll (p-1)^{\frac{4}{5}}(kl/d_1)^{\frac{6}{5}}\delta_{\pm}^{-\frac{2}{5}}L^{\frac{1}{5}} + (p-1)^{\frac{2}{5}}(kl/d_1)^{\frac{3}{5}}\delta_{\pm}^{-\frac{1}{5}}L^{-\frac{2}{5}}(p-1) \\ &\ll (p-1)(kl/d_1)\delta_{\pm}^{-\frac{1}{3}}. \end{aligned}$$

Clearly $L \leq T = \left\lfloor 2^{-7}(p-1) \frac{\delta_{\pm}^2}{(k\ell_{\pm}/d_1)} \right\rfloor$ when $(k\ell_{\pm}/d_1) \geq \frac{1}{2}\delta_{\pm}^2$, while when $(k\ell_{\pm}/d_1) \leq \frac{1}{2}\delta_{\pm}^2$ we still have $L \leq T = \left\lfloor 2^{-\frac{9}{2}}(p-1) \left(\frac{k\ell_{\pm}}{d_1}\right)^{\frac{3}{2}}\delta_{\pm}^{-3} \right\rfloor$ if $(k\ell_{\pm}/d_1) > \frac{1}{2}\delta_{\pm}^{\frac{4}{3}}$.

When $(k\ell_{\pm}/d_1) < \frac{1}{2}\delta_{\pm}^{\frac{4}{3}}$ we use the bound $C_{\pm}(\mathbf{u}_T) \leq 2^{\frac{37}{5}}\delta_{\pm}$ to obtain

$$\begin{aligned} \sum_{i=1}^N C_{\pm}^2(\mathbf{u}_i) &\ll (p-1)^{\frac{4}{5}}(kl/d_1)^{\frac{6}{5}}\delta_{\pm}^{-\frac{2}{5}} \sum_{t \leq T} t^{-\frac{4}{5}} + \delta_{\pm} \sum_{t > T} C(\mathbf{u}_t) \\ &\ll (p-1)^{\frac{4}{5}}(kl/d_1)^{\frac{6}{5}}\delta_{\pm}^{-\frac{2}{5}}T^{\frac{1}{5}} + \delta_{\pm}(p-1) \\ &\ll (p-1) \frac{(kl/d_1)^{\frac{3}{2}}}{\delta_{\pm}} + \delta_{\pm}(p-1) \ll \delta_{\pm}(p-1). \end{aligned}$$

□

4. COMPLETION OF THE PROOF OF LEMMA 1.1

Assume that $k < \frac{1}{32}(p-1)^{\frac{2}{3}}d^{\frac{1}{3}}$. Since $d \leq d_1$ plainly $d\delta_{\pm} < 2k$ and $(d/d_1)\delta_{\pm}^{-\frac{1}{3}} \leq \Delta_{\pm}^{-\frac{1}{3}}$.

So, from Corollary 3.1 and (2.1), we have

$$M_{\pm} \ll d^2p^2 + k^2lp + \max \left\{ k, kl\Delta_{\pm}^{-\frac{1}{3}} \right\} p^2.$$

Clearly

$$d^2p^2 = kl\Delta_{\pm}^{-\frac{1}{3}}p^2 \left(\frac{d^2}{kl}\Delta_{\pm}^{\frac{1}{3}} \right) < kl\Delta_{\pm}^{-\frac{1}{3}}p^2 \left(2^{\frac{1}{3}}(d/l)(d/k)^{\frac{2}{3}} \right) < kl\Delta_{\pm}^{-\frac{1}{3}}p^2.$$

and

$$k^2lp = kl\Delta_{\pm}^{-\frac{1}{3}}p^2 \left(\frac{k\Delta_{\pm}^{\frac{1}{3}}}{p} \right) < kl\Delta_{\pm}^{-\frac{1}{3}}p^2 \left(\frac{2^{\frac{1}{3}}k^{\frac{4}{3}}}{d^{\frac{1}{3}}p} \right) < kl\Delta_{\pm}^{-\frac{1}{3}}p^2 \left(\frac{d^{\frac{1}{9}}}{p^{\frac{1}{9}}} \right) < kl\Delta_{\pm}^{-\frac{1}{3}}p^2.$$

Hence as claimed

$$M_{\pm} \ll \max \left\{ k, kl\Delta_{\pm}^{-\frac{1}{3}} \right\} p^2.$$

5. PROOF OF PROPOSITION 1.1

We use the technique of Akulinichev to average over the $d_{\pm}^*$ th roots of unity. For any f and χ we have

$$\begin{aligned} (p-1)S_{\pm} &= \sum_{y=1}^{p-1} \chi(xy^{\frac{(p-1)}{d_{\pm}^*}}) \sum_{x=1}^{p-1} e_p \left(f(xy^{\frac{(p-1)}{d_{\pm}^*}}) \right) \\ &= \sum_{x=1}^{p-1} \chi(x) \sum_{y=1}^{p-1} \chi(y)^{\frac{(p-1)}{d_{\pm}^*}} e_p \left((ax^k + bx^{\pm l})y^{\frac{l(p-1)}{d_{\pm}^*}} \right). \end{aligned}$$

If $ax^k + bx^{\pm l} \neq 0$ then Weil's bound gives

$$(5.1) \quad \left| \sum_{y=1}^{p-1} \chi(y)^{\frac{(p-1)}{d_{\pm}^*}} e_p \left((ax^k + bx^{\pm l})y^{\frac{l(p-1)}{d_{\pm}^*}} \right) \right| \leq \left(\frac{l(p-1)}{d_{\pm}^*}, p-1 \right) p^{\frac{1}{2}} = \frac{d(p-1)}{d_{\pm}^*} p^{\frac{1}{2}}.$$

If $-b\bar{a} = \alpha^{(k \mp l)}$ for some α in $\mathbb{Z}_p^*$ then we also have the $d_{\pm}^*$ values of x with $ax^k + bx^{\pm l} = 0$. If $\chi^{(p-1)/d_{\pm}^*} \neq \chi_0$ then for these values of x the inner sum $\sum_{y=1}^{p-1} \chi(y)^{\frac{(p-1)}{d_{\pm}^*}}$ is zero, while if $\chi^{(p-1)/d_{\pm}^*} = \chi_0$ this sum is $(p-1)$ and $\chi(x) = \chi(\alpha)$.

6. PROOF OF COROLLARY 1.1

First we show that if $d_{\pm}^* < \frac{1}{4}p^{\frac{8}{13}}/d^{\frac{5}{13}}$, then uniformly

$$(6.1) \quad |S_{\pm}| \ll d^{\frac{3}{13}} p^{\frac{51}{52}}.$$

The strategy is to make an appropriate change of variables of the type $x \rightarrow x^m$ so that Theorem 1.1 can be applied effectively. From the box principle there exists an integer m such that

$$mk \equiv \alpha \pmod{p-1}, \quad \pm ml \equiv \beta \pmod{p-1},$$

(plus sign for S_+ and minus for S_-) with

$$|\alpha| \leq 2(pd)^{\frac{8}{13}}, \quad |\beta| \leq 2(pd)^{\frac{5}{13}},$$

and $m \not\equiv 0 \pmod{(p-1)/d}$ (so that α, β are not both 0). We can assume that $2(pd)^{\frac{8}{13}} < \frac{1}{32}(p-1)^{\frac{2}{3}}$, else $d^{\frac{3}{13}}p^{\frac{51}{52}} \gg p$ and the bound is trivial.

Suppose that $(m, p-1) = \nu$ and write $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^m = \{w_1, \dots, w_{\nu}\}$ so that

$$S_{\pm} = \sum_{i=1}^{\nu} \frac{\chi(w_i)}{\nu} \sum_{x=1}^{p-1} \chi(x)^m e_p \left(aw_i^k x^{\alpha} + bw_i^{\pm l} x^{\beta} \right).$$

If $\beta = 0$ (or similarly $\alpha = 0$) then applying the Heath-Brown & Konyagin bound (1.4) to the inner sum gives

$$|S_{\pm}| \ll |\alpha|^{\frac{3}{8}} p^{\frac{3}{4}} \ll d^{\frac{3}{13}} p^{\frac{51}{52}},$$

(strictly speaking their result does not contain a χ but it is easily seen that Lemma 1 of [5], and hence the result, still holds with the addition of a multiplicative character).

So we can assume that $\alpha, \beta \neq 0$ and, since $d_{\pm}^* < \frac{1}{4}p^{\frac{8}{13}}/d^{\frac{5}{13}}$ ensures that $|\beta| < (p-1)/d_{\pm}^*$ (and hence $m \not\equiv 0 \pmod{(p-1)/d_{\pm}^*}$), that $\alpha \not\equiv \beta \pmod{p-1}$. If $|\alpha| < 2|\beta|$ then by the Mordell bound (1.8)

$$|S_{\pm}| \ll (|\alpha||\beta|)^{\frac{1}{4}} p^{\frac{3}{4}} \ll |\beta|^{\frac{1}{2}} p^{\frac{3}{4}} \ll d^{\frac{5}{26}} p^{\frac{49}{52}}.$$

So we can assume that $0 < |\beta| < \frac{1}{2}|\alpha|$, with $|\alpha| < \frac{1}{32}(p-1)^{\frac{2}{3}}$. Hence

$$\Delta_{\pm} = \frac{|\alpha| \mp |\beta|}{(\alpha, \beta, p-1)} > \frac{1}{2} \frac{|\alpha|}{|\beta|}$$

and from Theorem 1.1,

$$\begin{aligned} |S_{\pm}| &\ll \max \left\{ 1, |\beta| (|\alpha|/|\beta|)^{-\frac{1}{3}} \right\}^{\frac{1}{4}} |\alpha|^{\frac{1}{4}} p^{\frac{3}{4}} = \max \left\{ |\alpha|^{\frac{1}{4}}, |\alpha|^{\frac{1}{6}} |\beta|^{\frac{1}{3}} \right\} p^{\frac{3}{4}} \\ &\ll \max \left\{ d^{\frac{2}{13}} p^{\frac{2}{13}}, d^{\frac{3}{13}} p^{\frac{3}{13}} \right\} p^{\frac{3}{4}} \ll d^{\frac{3}{13}} p^{\frac{51}{52}}. \end{aligned}$$

Suppose now that $d_{\pm}^* > \frac{1}{4}p^{\frac{8}{13}}/d^{\frac{5}{13}}$. We may also assume $d < p^{\frac{1}{12}}$ for otherwise (6.1) is trivial. Then

$$\frac{d}{d_{\pm}^*} p^{\frac{3}{2}} < 4d^{\frac{18}{13}} p^{\frac{23}{26}} = 4d^{\frac{3}{13}} d^{\frac{15}{13}} p^{\frac{23}{26}} < 4d^{\frac{3}{13}} p^{\frac{51}{52}}.$$

The corollary follows immediately from Proposition 1.1.

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