Fourier series and the δ^2 process

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1 Introduction

Because of the widespread utility of Fourier series, it is of interest, especially in applications, to analyze their speed of convergence. Particularly enticing is the prospect of applying methods to accelerate this convergence. Various methods of acceleration of convergence of sequences have been applied with some success to the partial sums of Fourier series.

Throughout, we will consider a function defined on $[-\pi, \pi]$ which is integrable. For such a function f we define the Fourier coefficients by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

for each integer n, and we define the n^{th} partial sum of the Fourier series as

$$S_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e^{ikx},$$

where n is a positive integer and x is in $[-\pi, \pi]$. A fundamental central question of the theory is: when and in what sense does $S_n f(x) \to f(x)$ as $n \to \infty$? When f is square-integrable, this holds in the mean-square sense, that is, $\int_{-\pi}^{\pi} |S_n f(x) - f(x)|^2 dx \to 0$ as $n \to \infty$. The very deep result of Carleson [2] asserts that for square-integrable f, $S_n f(x) \to f(x)$ at every point except for a set of zero Lebesgue measure. Other classical results, due to Dini-Lipschitz, Lebesgue and Dirichlet-Jordan give conditions for pointwise convergence (see e.g., Zygmund [10] for these). Typical of these results is the Dirichlet-Jordan theorem: If f is of bounded variation over $[-\pi, \pi]$, then $S_n f(x)$ converges to $f(x)$ at each point of continuity of f. However, even for fairly smooth functions, this convergence can be quite slow and it is desirable to attempt to find a way to speed this convergence.

For a numerical sequence $\{s_n\}$ which has limit s, we say a transformation t_n of s_n accelerates convergence if there exists a k such that each t_n depends only on $s_0, \ldots, s_{n+k}$ and t_n converges to s faster than s_n. Many sequence transformations have been developed to speed convergence of numerical sequences which arise in many contexts (see e.g. Delahaye [3] or Wimp [9]). In this paper, we discuss a particular non-linear transformation, known as the δ^2 process, usually attributed to Aitken [1], although the idea had appeared earlier in the works of other authors. Generalizations of this transformation, which transform a sequence $\{s_n\}$ into a family of sequences $\{e_k(s_n)\}$ were studied extensively by Shanks [7] and for this reason these transformations are sometimes called Shanks transformations. In this notation, $e_1(s_n)$ is identical to the δ^2 process. This process takes the sequence s_n and transforms it to

$$e_1(s_n) = t_n := \frac{s_{n+1}s_{n-1} - s_n^2}{s_{n+1} + s_{n-1} - 2s_n} = s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})}, \tag{1}$$

where we set $t_n = s_n$ if the denominator of the fraction is zero.

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In the case when the s_n are the partial sums of a geometric series, the transform (1) produces a constant sequence, each term the sum of the series. Shanks [7] generalizes this to show an improved rate of convergence under transform (1) for partial sums of series which he calls "nearly geometric".

Consider the function f on the interval $[-\pi, \pi]$ given by $f(x) = 1$ if $0 \le x \le \pi$, $f(x) = -1$ if $-\pi \le x < 0$. After computation and simplification, its Fourier series is:

$$f \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}. \tag{2}$$

At $x = \frac{\pi}{2}$ we obtain the slowly convergent Leibniz series:

$$1 = \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

Shanks [7] applies the transformation (1) and iterations of this transform to the sequence of partial sums and the results are dramatic: t_8 is accurate to three significant figures, and iterating the transform four times gives a sequence whose fifth term is correct to eight significant figures. By contrast, Shanks notes it would take over 40,000 summands in the original series to obtain this accuracy.

Smith and Ford [8] used numerical tests to compare different methods of convergence acceleration to the partial sums of Fourier series. Using a set of five points they tested a slowly and a rapidly converging Fourier series and found that, on the average, the transformation (1) improves convergence of the slowly converging series but slightly degrades convergence in the rapidly convergent case. Drummond [4] discusses many methods of convergence acceleration and includes discussion of their application to Fourier series.

Although with specific series it is possible to get improved, even dramatic results at some points, in this note we show that with the transformation δ^2 the results are, in a general sense, very bad. We will show that for a fairly general set of functions, for a large set of values $x \in [-\pi, \pi]$, the δ^2 process turns the sequence $\{S_n f(x) : n = 1, 2, \dots\}$ into a set with multiple limit points.

Theorem 1. *Suppose that $f \in C^2([-\pi, \pi])$ and that $f(-\pi) \neq f(\pi)$. Consider the sequence $t_n(x)$ formed by applying the transformation (1) to the sequence $S_n f(x)$. Then $t_n(x)$ diverges at every x of the form $x = 2\pi a$, where $a \in [-.5, .5]$ is irrational.*

In fact, the same difficulty also occurs on a dense set of numbers of the form $2\pi q$, where q is rational. At these points, we can be even more precise.

Theorem 2. *Suppose f is as in Theorem 1. Suppose $x := \frac{2\pi j}{k}$ where $\frac{j}{k}$ is in lowest terms and k is odd. Then $t_n(x)$ has three limit points, $f(x)$ and $f(x) \pm \frac{\alpha^2 \sin^2(x/2)}{\alpha \sin(x/2) + 2\beta \cos(x/2)}$, where $\alpha = [f(\pi) - f(-\pi)]/\pi$ and $\beta = [f'(\pi) - f'(-\pi)]/\pi$.*

Remarks. Consider the 2π periodic extension of f to the whole real line. The hypotheses of the theorems imply that this function is in C^2, except for a single jump discontinuity every period. By periodicity, the Theorems remain valid if this jump occurs anywhere in the interval $[-\pi, \pi]$.

At a jump discontinuity of f the partial sums of the Fourier series exhibit the Gibbs phenomenon in a neighborhood of the discontinuity and many authors report difficulties with acceleration methods near this discontinuity. In fact, the Theorems imply that, at least for the δ^2 process, a jump discontinuity causes difficulties just about everywhere.

It is well known that the δ^2 process can turn a convergent sequence into one with multiple convergent subsequences. (See Wimp [9] for an example due to Lubkin.) Thus, our theorems provide many more examples of that phenomenon.

2 Preliminaries, background and technical lemmas

If $S_n f(x)$ denotes the sequence of partial sums of the Fourier series of f, applying the δ^2 transform (1) results in the sequence of functions:

$$t_n(x) = S_nf(x) - \frac{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right)\left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)}{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right) - \left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)}. \quad (3)$$

Our theorems will be established by demonstrating the bad behavior of the fraction on the right. In this section we prove some lemmas which will be used to analyze this term.

Lemma 1. *Let $f \in C^2([-\pi, \pi])$. Then for every integer n,*

$$\hat{f}(n) = \frac{(-1)^{n+1}}{2in}\alpha + \frac{(-1)^n}{2n^2}\beta - \frac{\widehat{f''}(n)}{n^2},$$

where α and β are as in the Theorem 2.

Proof. Integration by parts.

Remark. Notice that $\frac{\widehat{f''}(n)}{n^2} = o(\frac{1}{n^2})$ as $|n| \to \infty$ by the Riemann-Lebesgue lemma. (See, e.g. Zygmund [10].)

Let a be an irrational number and let $\langle a_0, a_1, a_2, \ldots \rangle$ be its unique expression as a continued fraction. Define sequences $\{h_n\}$ and $\{k_n\}$ as follows:

$$\begin{aligned} &h_{-2} = 0, h_{-1} = 1, h_i = a_ih_{i-1} + h_{i-2} \text{ for } i \geq 0, \\ &k_{-2} = 1, k_{-1} = 0, k_i = a_ik_{i-1} + k_{i-2} \text{ for } i \geq 0. \end{aligned} \quad (4)$$

Note that $1 = k_0 \leq k_1 < k_2 < k_3 < \cdots$. The value $r_n := \frac{h_n}{k_n}$ is called the n^{th} convergent to a, and $r_n = \langle a_0, a_1, \ldots, a_n \rangle$. (See Niven and Zuckerman [6], Chapter 7 for these results.)

Lemma 2. *Let a be an irrational number. Then there exist infinitely many rational numbers $\frac{h}{k}$ with k odd such that $\left|a - \frac{h}{k}\right| < \frac{1}{k^2}$.*

Proof. Let $\left\{\frac{h_n}{k_n}\right\}$ be the sequence of convergents to a as defined above. By Theorem 7.11 in Niven and Zuckerman [6], we have

$$\left|a - \frac{h_n}{k_n}\right| < \frac{1}{k_nk_{n+1}} < \frac{1}{k_n^2},$$

where the second inequality follows from the fact that $k_n < k_{n+1}$ as noted above. Thus, it suffices to show that there exist infinitely many values n such that k_n is odd. We offer a proof by contradiction. Suppose m is the greatest value for which k_m is odd. Then k_{m+1} and k_{m+2} must be even. But equation (4) gives us $k_{m+2} = a_{m+2}k_{m+1} + k_m$, which is odd. Hence k_{m+2} is both even and odd, a contradiction.

■

Proposition 1. *Let a be irrational and let $x = 2\pi a$. Then there exists a strictly increasing sequence $\{m_n\}$ of positive integers such that*

$$\liminf_{n\to\infty} \left|\frac{\sin m_nx\, \sin(m_n+1)x}{(m_n+1)\sin m_nx + m_n\sin(m_n+1)x}\right| > 0.$$

Proof. By the previous lemma, we can define sequences $\{m_n\}$ and $\{l_n\}$ of integers such that $\{m_n\}$ is strictly increasing, each m_n is positive, and, for every n,

$$\left|a - \frac{l_n}{2m_n+1}\right| < \frac{1}{(2m_n+1)^2} < \frac{1}{4m_n^2 + 2m_n}.$$

Thus, for each n,

$$\begin{aligned}|\sin(m_n+1/2)x| &= |\sin((m_n+1/2)x-l_n\pi)| \le |(m_n+1/2)x-l_n\pi| \\ &= |(2m_n+1)\pi a-l_n\pi| = |(2m_n+1)\pi|\left|a-\frac{l_n}{2m_n+1}\right| < (2m_n+1)\pi\frac{1}{4m_n^2+2m_n} = \frac{\pi}{2m_n}.\end{aligned}$$

Now observe that

$$\sin m_n x = \sin(m_n+1/2)x\cos\frac{x}{2} - \sin\frac{x}{2}\cos(m_n+1/2)x$$
$$\sin(m_n+1)x = \sin(m_n+1/2)x\cos\frac{x}{2} + \sin\frac{x}{2}\cos(m_n+1/2)x.$$

It follows that, for each $n \in \mathbb{N}$,

$$\frac{\sin m_n x\ \sin(m_n+1)x}{(m_n+1)\sin m_n x + m_n\sin(m_n+1)x} = \frac{\sin^2(m_n+1/2)x\ \cos^2\frac{x}{2} - \sin^2\frac{x}{2}\ \cos^2(m_n+1/2)x}{\sin m_n x + 2m_n\cos\frac{x}{2}\ \sin(m_n+1/2)x}.$$

Notice that the denominator of this expression is bounded:

$$|\sin m_n x + 2m_n\cos\frac{x}{2}\sin(m_n+1/2)x| \le 1 + |2m_n\sin(m_n+1/2)x| < 1+\pi.$$

Moreover, the limit of the numerator is clearly $-\sin^2\frac{x}{2}$, since $\sin^2(m_n+1/2)x \to 0$ as $n\to\infty$ and hence $\cos^2(m_n+1/2)x \to 1$ as $n\to\infty$. It follows that this ratio is bounded away from 0 as $n\to\infty$.

■

Lemma 3. *Suppose $x = \frac{2\pi j}{k}$, where j and k are non-zero integers with no common factors and k is odd. Then, for integers n, $\sin nx = -\sin(n+1)x$ if and only if $n = \frac{k-1}{2} + mk$, where m is some integer.*

Proof. Suppose $\sin nx = -\sin(n+1)x$. Note the following identities:

(a) $\sin nx = \sin(n+1/2)x\cos\frac{x}{2} - \sin\frac{x}{2}\cos(n+1/2)x$ and

(b) $\sin(n+1)x = \sin(n+1/2)x\cos\frac{x}{2} + \sin\frac{x}{2}\cos(n+1/2)x$.

We quickly see that $\sin(n+1/2)x = 0$, which implies that $(n+1/2)x = l\pi$ for some integer l. Multiplying both sides by $k/j\pi$ and substituting for x we obtain $2n+1 = lk/j$. Since lk/j must be an odd integer and j and k have no common factors, l/j must be an odd integer. Hence we write $l/j = 2m+1$ for some integer m and obtain $2n+1 = 2mk+k$, which yields $n = \frac{k-1}{2} + mk$, as desired.

To prove the converse, note that whenever $n = \frac{k-1}{2} + mk$, we have $(n+1/2)x = (2m+1)j\pi$, which implies $\sin(n+1/2)x = 0$ and $\cos(n+1/2)x = (-1)^j$. By identities (a) and (b) we have

$$\sin nx = (-1)^{j+1}\sin\frac{x}{2} = -\sin(n+1)x,$$

as desired.

■

Remark: Note the following identities:

(c) $\cos nx = \cos(n+1/2)x\cos\frac{x}{2} + \sin\frac{x}{2}\sin(n+1/2)x$ and

(d) $\cos(n+1)x = \cos(n+1/2)x\cos\frac{x}{2} - \sin\frac{x}{2}\sin(n+1/2)x$.

Now observe that $\sin nx = -\sin(n+1)x$ implies $\sin(n+1/2)x = 0$, which by (c) and (d) implies $\cos nx = \cos(n+1)x = (-1)^j\cos\frac{x}{2}$. This relation will be necessary in the proof of Theorem 2.

3 Proofs of theorems 1 and 2

We first give a proof of Theorem 1. For x as described in the statement of the theorem, we compute the fraction on the right of (3). From Lemma 1 we get

$$\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx} = \frac{(-1)^{n+1}}{n}\alpha \sin nx + \frac{(-1)^n}{n^2}\beta \cos nx + \varepsilon(n),$$

where we have defined $\varepsilon(n) := -\left(\frac{\widehat{f''}(-n)}{n^2}e^{-inx} + \frac{\widehat{f''}(n)}{n^2}e^{inx}\right)$.

Using this notation, we may write the fraction on the right hand side of (3) as:

$$\frac{\left(\frac{(-1)^n}{n+1}\alpha\sin(n+1)x + \frac{(-1)^{n+1}\beta\cos(n+1)x}{(n+1)^2} + \varepsilon(n+1)\right)\left(\frac{(-1)^{(n+1)}}{n}\alpha\sin nx + \frac{(-1)^n\beta\cos nx}{n^2} + \varepsilon(n)\right)}{\left(\frac{(-1)^n}{n+1}\alpha\sin(n+1)x + \frac{(-1)^{(n+1)}\beta\cos(n+1)x}{(n+1)^2} + \varepsilon(n+1)\right) - \left(\frac{(-1)^{n+1}}{n}\alpha\sin nx + \frac{(-1)^n\beta\cos nx}{n^2} + \varepsilon(n)\right)} =$$

$$\frac{(-1)^n\left(\alpha\sin(n+1)x - \frac{\beta\cos(n+1)x}{n+1} + (-1)^n(n+1)\varepsilon(n+1)\right)\left(-\alpha\sin nx + \frac{\beta\cos nx}{n} + (-1)^n n\varepsilon(n)\right)}{\alpha\left(n\sin(n+1)x + (n+1)\sin nx\right) - \beta\left(\frac{n}{n+1}\cos(n+1)x + \frac{(n+1)}{n}\cos nx\right) + n(n+1)(-1)^n\left(\varepsilon(n+1)-\varepsilon(n)\right)}. \quad (5)$$

Let m_n be the sequence of increasing integers given by Proposition 1. We substitute this sequence for n in the last expression. Notice that $m_n\varepsilon(m_n)$, $(m_n+1)\varepsilon(m_n+1)$, $\frac{\beta\cos(m_n+1)x}{m_n+1}$, $\frac{\beta\cos m_n x}{m_n}$ all go to 0 as $n\to\infty$. Furthermore, $m_n(m_n+1)(\varepsilon(m_n+1)-\varepsilon(m_n)) \to 0$ as $n\to\infty$ by the Riemann-Lebesgue lemma. Both of the terms $\frac{m_n}{m_n+1}\cos(m_n+1)$ and $\frac{(m_n+1)}{m_n}\cos m_n x$ remain bounded as $n\to\infty$. As in the proof of Proposition 1, the numerator is asymptotically $(-1)^{n+1}\alpha^2\sin^2(\frac{x}{2})$ and the denominator remains bounded. As $n\to\infty$, this shows that (5) stays bounded away from 0, and hence by (3), $t_n(x)$ has a subsequence which fails to converge to $f(x)$. To show that $t_n(x)$ diverges, we show that there are subsequences which converge to $f(x)$. Consider the term $(n+1)\sin nx + n\sin(n+1)x$ in the denominator of (5). Rewrite this as: $\sin nx + 2n\cos(\frac{x}{2})\sin(n+\frac{1}{2})x$. By Weyl's equidistribution theorem (see Körner [5]), there are an infinite number of n for which $(n+\frac{1}{2})x \in [-\frac{3\pi}{4}, -\frac{\pi}{4}]\cup[\frac{\pi}{4}, \frac{3\pi}{4}]$. For these n, $|\sin nx + 2n\cos(\frac{x}{2})\sin(n+\frac{1}{2})x|$ is bounded below by a positive constant times n, which forces (5)$\to 0$.

To see Theorem 2, we examine (5) in this case. Consider an $x := \frac{2\pi j}{k}$, where k is odd and the fraction $\frac{j}{k}$ is in lowest terms. Consider n of the form $n = \frac{k-1}{2} + km$ for positive integers m (as in the statement of Lemma 3) in the expression (5). As before, $n\varepsilon(n)$, $(n+1)\varepsilon(n+1)$, $\frac{\beta\cos(n+1)x}{n+1}$, and $\frac{\beta\cos nx}{n}$ all go to 0 as $n\to\infty$ and $n(n+1)(\varepsilon(n+1)-\varepsilon(n))\to 0$ as $n\to\infty$ by the Riemann-Lebesgue lemma. Furthermore, for this x and such n, $\sin nx = (-1)^{j+1}\sin(x/2)$, $\sin(n+1)x = (-1)^j\sin(x/2)$, and $\cos nx = \cos(n+1)x = (-1)^j\cos(x/2)$; thus expression (5) acts asymptotically as $(-1)^{n+j+1}\dfrac{\alpha^2\sin^2(x/2)}{\alpha\sin(x/2)+2\beta\cos(x/2)}$, which has one sign for odd m and the opposite sign for even m. If $n \neq \frac{k-1}{2} + km$ for every integer m, $\sin nx \neq \sin(n+1)x$ and thus in (5) the term $(n+1)\sin nx + n\sin(n+1)x$ is bounded below by a constant times n, which causes (5) $\to 0$ as $n\to\infty$. (We are also using the fact that as n varies $\{nx \pmod{2\pi} : n = 1,2,\dots\}$ is only a finte set.)

Remark. In conclusion, in the case of Theorem 2, $t_n(x)$ has a subsequence, $\{t_n(x) : n = \frac{k-1}{2} + mk$ and $m = 0,1,2,\dots\}$ which produces two limit points different from $f(x)$ and along the subsequence of all other indices $t_n(x)\to f(x)$.

4 Iterated and higher-order transforms

In his extensive study, Shanks [7] also considers the effect of iterating the transform (1) as well as higher order transforms. We will make only a few comments on these.

With regards to the iterated transform, e_1^2, we can assert that again, on a dense set of x, this transform sends $S_nf(x)$ to a sequence with divergent subsequences. Indeed, we only need consider x of the form $x = \frac{2\pi j}{k}$, k odd, and where j and k have no common factors. The remark at the end of the previous section describes the behavior of the sequence $t_n(x) = e_1(S_nf(x))$ and the fact that $e_1^2(S_nf(x))$ has subsequences which still do not converge to $f(x)$ is an immediate consequence of the following:

Lemma 4. *If a_n is a sequence and m, l, k are positive integers with m variable and l, k fixed, $k \geq 2$, such that $a_n \to a$ if $n \neq l + mk$ and $a_n \to a + b$ for $n = l + mk$ then $e_1(a_n) \to a + \frac{b}{2}$ for $n = k + ml$ as $m \to \infty$.*

Proof. Notice that for $n = l + mk$,

$$e_1(a_n) = a_n - \frac{(a_{n+1} - a_n)(a_n - a_{n-1})}{(a_{n+1} - a_n) - (a_n - a_{n-1})} \to (a + b) - \frac{[a - (a + b)][(a + b) - a]}{[a - (a + b)] - [(a + b) - a]} = a + \frac{b}{2}.$$

Remark. Analyzing $e_2(S_n f(x))$ seems to be much more difficult. (See e.g. Shanks [7] for the definition of e_2.) As in Theorems 1 and 2, let $S_n f(x)$ be the sequence of partial sums of a Fourier series of at an $x \in [-\pi, \pi]$, and $t_n(x)$ be the sequence obtained by applying the transformation (1). We can obtain $e_2(S_n f(x))$ via the ε algorithm (see [3]): $\varepsilon_{-1}(n) := 0$ for every n, $\varepsilon_0(n) := S_n f(x)$ and $\varepsilon_{k+1}(n) = \varepsilon_{k-1}(n+1) + \frac{1}{\varepsilon_k(n+1) - \varepsilon_k(n)}$. Then $e_1(S_n f(x)) = t_n(x) = \varepsilon_2(n)$ and $e_2(S_n f(x)) = \varepsilon_4(n)$. For convenience, write s_n for $S_n f(x)$, and t_n for $t_n(x)$. Then computing $\varepsilon_4(n)$ gives:

$$\varepsilon_4(n) = t_{n+2} + \frac{1}{[\frac{1}{t_{n+2} - s_{n+2}}] + [\frac{1}{t_{n+3} - t_{n+2}}] - [\frac{1}{t_{n+2} - t_{n+1}}]}$$

Consider x of the form $x := \frac{2\pi j}{k}$ where j and k have no common factors and k is odd. By Theorem 2, and the remark after its proof, $t_n(x)$ has a subsequence, $n = \frac{k-1}{2} + mk$, $m = 0, 1, 2, \dots$ along which $t_n(x) \to f(x) + \frac{\alpha^2 \sin^2(x/2)}{\alpha \sin(x/2) + 2\beta \cos(x/2)}$. For convenience write this limit as $f(x) + \gamma$. For $n = \frac{k-1}{2} + mk \pm 1$, as $n \to \infty$, $t_n(x) \to f(x)$. With $n - 2$ in place of n in the above formula, and letting $n \to \infty$ along the subsequence $n = \frac{k-1}{2} + mk$, $m = 0, 1, 2, \dots$, yields:

$$\varepsilon_4(n-2) = t_n + \frac{1}{[\frac{1}{t_n - s_n}] + [\frac{1}{t_{n+1} - t_n}] - [\frac{1}{t_n - t_{n-1}}]} \to f(x) + \gamma + \frac{1}{\frac{1}{\gamma} + \frac{1}{-\gamma} - \frac{1}{\gamma}} = f(x).$$

So curiously, the stray subsequence of $t_n(x)$ which diverged from $f(x)$ has been transformed back to the correct limit. However, the analysis is too delicate to tell whether there are now new subsequences which go astray, and certainly in the case when the limit is the actual value $f(x)$ we cannot determine whether convergence has been accelerated.

5 A function with two jumps

Recall the example of equation (2). This case is not covered by Theorem 1 or Theorem 2 because it has two jumps when considered as a 2π periodic function. Nevertheless, it exhibits problems under the Shanks transformation analogous to problems we have already seen. In some sense, we "cheat" by transforming the series consisting of only the odd terms in the Fourier series of f. But all the even Fourier coefficients in this example are 0, so if we were to take the series consisting of all the terms in the Fourier series of f, then (1) would give us $t_n = s_n$. This would give us no insight. Hence our "cheat" is justified.

Lemma 5. *Let $x := \dfrac{\pi j}{4k}$, where j, k are integers with no common factors and j is odd. Then for integers n, $\sin(2n-1)x = \sin(2n+1)x$ if and only if $n = mk, m$ odd, in which case this common value is $(-1)^r \cos x$, where $r = (mj - 1)/2$.*

Proof. First, suppose $\sin(2n-1)x = \sin(2n+1)x$. From the identities

(a) $\sin(2n-1)x = \sin 2nx \cos x - \sin x \cos 2nx$ and (b) $\sin(2n+1)x = \sin 2nx \cos x + \sin x \cos 2nx$,

we quickly see that $\cos 2nx = 0$. It follows that $2nx = l\pi + \pi/2$ for some integer l. Multiply by $2k/\pi j$ and substitute for x to get $n = (2l+1)k/j$. Since n is an integer, and j and k have no common factors, we must have that $m := (2l+1)/j$ is an integer. Since mj is odd, it follows that m is odd; this proves the first implication of the lemma.

We prove the converse by noting that whenever $n = mk, m$ odd, we have $2nx = mj\pi/2$, in which case $\cos 2nx = 0$ and $\sin 2nx = (-1)^r$ where $r := (mj-1)/2$. Thus, by identities (a) and (b) we have

$$\sin(2n-1)x = \sin(2n+1)x = \sin 2nx \cos x = (-1)^r \cos x,$$

as desired.

■

Proposition 2. *Let f be defined as in equation (2). Suppose $x \in [-\pi, \pi]$ satisfies the hypotheses of the previous lemma. Then $t_n(x)$ has three limit points: $f(x)$ and $f(x) \pm \frac{2}{\pi}\cos x$.*

Proof. The n^{th} partial sum of the Fourier series of f at x is

$$S_n f(x) = \frac{4}{\pi} \sum_{k=1}^{n} \frac{\sin(2k-1)x}{2k-1}.$$

Substituting this into equation (1), we get

$$t_n(x) = S_n f(x) - \frac{4}{\pi} \frac{\sin(2n-1)x \, \sin(2n+1)x}{(2n-1)\sin(2n+1)x - (2n+1)\sin(2n-1)x}.$$

We know that $S_n f(x) \to f(x)$. Now as $n \to \infty$ on subsequences for which $\sin(2n-1)x - \sin(2n+1)x \neq 0$, the denominator of the second term in the above equation is unbounded, forcing the ratio to go to zero. Therefore, $t_n(x) \to f(x)$ as $n \to \infty$ on those subsequences.

Now suppose n is increasing on a subsequence of odd multiples of k. Then by Lemma 5, $\sin(2n-1)x = \sin(2n+1)x = \pm\cos x$ for each n. Thus the second term in the above equation is simply $\pm\frac{2}{\pi}\cos x$ for each n along this subsequence, and so as $n \to \infty$, $t_n(x) \to f(x) \pm \frac{2}{\pi}\cos x$.

■

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