# Regularity for solutions of the two-phase Stefan problem

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**Abstract**

We consider the two-phase Stefan problem $u_t = \Delta\alpha(u)$ where $\alpha(u) = u + 1$ for $u < -1$, $\alpha(u) = 0$ for $-1 \leq u \leq 1$, and $\alpha(u) = u - 1$ for $u \geq 1$. We show that if $u$ is an $L^2_{loc}$ distributional solution then $\alpha(u)$ has $L^2_{loc}$ derivatives in time and space. We also show $|\alpha(u)|$ is subcaloric and conclude that $\alpha(u)$ is continuous.

## 1 Introduction

This paper is devoted to the study of optimal regularity properties of solutions to the degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \Delta\alpha(u) \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^n \times (0, T)$, for some $T > 0$. Here $\alpha(u) = 0$ if $-1 \leq u \leq 1$, $\alpha(u) = u - 1$ for $u > 1$, and $\alpha(u) = u + 1$ for $u < -1$.

For non-negative solutions of (1.1), a collection of techniques is available. The minimal assumption $u \in L^1_{\mathrm{loc}}$ leads to higher integrability, then local boundedness, and then to continuity of $\alpha(u)$. In this case, it is also possible to identify the growth at infinity, and to show existence and uniqueness of solutions to the Cauchy problem in the optimal class (of growth at infinity) for measure data. See [Ko], [AnKo].

Here we prove intrinsic energy estimates for signed local solutions to (1.1). These estimates do not involve initial or boundary data. We will show that if $u \in L^2_{loc}(\Omega)$ is a solution in the sense of distributions of (1.1) (defined precisely below) then $\alpha(u)$ has locally square integrable derivatives in time and space.

In the case of non-negative solutions, the free boundaries $\partial\{u > 0\}$ that arise at discontinuities of $u$ and $\nabla\alpha(u)$ carry a non-negative Radon measure that can be expressed as a divergence of a $L^2_{\mathrm{loc}}$ vector field. That this measure is non-negative makes solutions to the one-phase Stefan problem subcaloric [Ko], [DaKo]. This divergence will give rise to a signed Radon measure if solutions are signed. Nevertheless, we prove related results for the two-phase Stefan problem: namely, that $\alpha(u)^+$, $\alpha(u)^-$, and $|\alpha(u)|$ are subsolutions to the heat equation. As a corollary, locally square integrable solutions are locally bounded. Under the assumption that $u$ is bounded and $\nabla\alpha(u) \in L^2$, Caffarelli and Evans [CE] have shown that $\alpha(u)$ is continuous. Similar results for more general singular parabolic equations were shown by Sacks [Sa], Ziemer [Z] and by DiBenedetto [DiB]. Consequently, a locally $L^2$ weak solution of (1.1) satisfies the hypotheses of these results (of any of these authors) and we conclude the continuity of $\alpha(u)$.

Related to (1.1) is the porous medium equation $u_t = \Delta u^m$, $m > 1$. This has been studied extensively by many authors, but we mention in particular the regularity result of Dahlberg and Kenig [DKe] who showed that a nonnegative $L^m_{loc}$ solution to the porous medium equation is a.e. equal to a continuous function. The methods in this present paper are descendants (via the work of Korten and Andreucci and Korten) of the methods of Dahlberg and Kenig found in [DKe]. However, the fact that we are working with signed solutions complicates matters. To achieve our results we will perform numerous integrations by parts and cannot determine the sign of the resulting boundary terms as in the one-phase case. Consequently we devise a different strategy and introduce new ideas and techniques.

Equation (1.1) is a formulation of the two-phase Stefan problem, describing the flow of heat within a substance which can be in a liquid phase or a solid phase, and for which there is a latent heat to initiate phase change. This allows for the presence of a "mushy zone", that is, a region which is between the liquid and solid phases. In this model $u$ represents the enthalpy and $\alpha(u)$ the temperature. We have assumed that the thermal conductivity in both the solid and liquid phases is the same. These conductivities are determined by the slope of the function $\alpha(u)$ in the regions $u \geq 1$, and $u \leq -1$. The results below all continue to hold (with minor modifications) if the slope of $\alpha(u)$ differs in these regions.

Using the energy estimates obtained in the present paper, the authors have obtained an existence and uniqueness theorem for the Cauchy problem for (1.1) on $\mathbb{R}^{n+1}_+$ as long as the initial data (which may be signed measures) are taken on in the sense of conservation laws [KoM]. In [BKoM], the authors together with I. Blank apply the intrinsic energy estimates in their study of the Hele-Shaw problem via a singular limit procedure for one-phase Stefan problems with increasing diffusivities.

We now state our main result. Suppose $u \in L^2_{loc}(\Omega)$ where $\Omega$ is a domain contained in $\mathbb{R}^n \times (0,T)$. We consider distributional solutions of the equation $u_t = \Delta\alpha(u)$, that is, $u$ which satisfy

$$\iint_\Omega \alpha(u)\Delta\varphi + u\varphi_t dx\, dt = 0$$

for every $\varphi \in C^\infty$ with compact support in $\Omega$.

**Theorem 1.1.** *Suppose $u \in L^2_{loc}(\Omega)$ is a solution of $u_t = \Delta\alpha(u)$. Then $\alpha(u)$ is a.e. equal to a continuous function.*

We do not expect, in general, such a result for $u$. As noted in Korten [Ko], the solution to the Cauchy problem $u_t = \Delta\alpha(u)$ on $\mathbb{R}^{n+1}_+$ with initial data $0 \leq u_I(x) \leq 1$ is just $u(x,t) = u_I(x)$. Thus, we cannot expect $u(x,t)$ to be any smoother than $u_I(x)$.

The paper is structured as follows. In section 2 we prove energy estimates for weak solutions of the two phase problem. These show that $\nabla\alpha(u)$ and $\alpha(u)_t$ exist locally in $L^2$. In section 3, we show that $|\alpha(u)|$ is subcaloric. An immediate consequence is that $\alpha(u)$ is locally bounded. This, combined with the energy estimates and previously mentioned theorem of DiBenedetto [DiB] (or others mentioned above) gives the continuity of $\alpha(u)$.

Throughout, the letter $C$ will denote a constant which may vary from line to line.

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## 2 Energy Estimates

We establish that $\alpha(u)$ has derivatives which are locally in $L^2$.

**Theorem 2.1.** *Suppose* $\Omega \subseteq \mathbb{R}^{n+1}_+$ *and* $u \in L^2_{loc}(\Omega)$ *is a distributional solution of* $u_t = \Delta\alpha(u)$ *on* $\Omega$. *Suppose* $r < R$, $T_0 < t_0 < T_1$, *set* $\omega = (t_0, T_1) \times B(x_0, r)$ *and* $\tilde{\omega} = (T_0, T_1) \times B(x_0, R)$, *and suppose the closure of* $\tilde{\omega}$ *is contained in* $\Omega$. *Then* $\nabla\alpha(u)$, $\alpha(u)_t$ *exist in* $L^2(\omega)$ *and there exists a constant* $C$, *depending only on* $\omega$ *and* $\tilde{\omega}$ *such that*

$$\iint_\omega |\nabla\alpha(u)|^2 dx\, dt \le C \iint_{\tilde{\omega}} u^2 dx\, dt \tag{2.1}$$

*and*

$$\iint_\omega \left|\frac{\partial}{\partial t}\alpha(u)\right|^2 dx\, dt \le C \iint_{\tilde{\omega}} u^2 dx\, dt \tag{2.2}$$

*Proof.* Let $\varphi_m(y,s) = \rho_m(y)\tau_m(s)$, $m = 1, 2, \ldots$ where $\rho_m$, $\tau_m$ are smooth mollifiers, radial, centered at 0, compactly supported, and tending to $\delta_0$. For $(x,t) \in \Omega$ and $m$ sufficiently large (depending on $(x,t)$), $\varphi_m(x-y, t-s)$ is a test function supported in $\Omega$ and thus

$$\iint_\Omega u(y,s)\frac{\partial\varphi_m}{\partial t}(x-y,t-s) + \alpha(u(y,s))\Delta\varphi_m(x-y,t-s)dy\, ds = 0. \tag{2.3}$$

In the course of the proof, we will need to define a domain between $\omega$ and $\tilde{\omega}$. To simplify notation, set $\omega_1 = \omega$, $\omega_3 = \tilde{\omega}$ and we will define $\omega_2$ so that $\omega_1 \subset \omega_2 \subset \omega_3$.

For $m = 1, 2, 3, \ldots$ set

$$u_m(x,t) = \iint_\Omega u(y,s)\chi_{\omega_3}(y,s)\varphi_m(x-y,t-s)dy\, ds.$$

Then for all $(x,t)$ and $m$, $|u_m(x,t)| \le M(u\chi_{\omega_3}(x,t))$ where $M$ denotes the Hardy-Littlewood maximal function (in both variables $(x,t)$).

Choose $a$, $T_0 < a < \frac{T_0+t_0}{2}$ such that

$$\int_{B(x_0,R)} |M(u\chi_{\omega_3})(x,a)|^2 dx \leq C \iint_{\omega_3} |M(u\chi_{\omega_3})(x,t)|^2 dx\, dt. \tag{2.4}$$

Here $C$ is chosen to depend only on $\omega_1$ and $\omega_3$.

In a similar fashion, set $w_m = \alpha(u)\chi_{\omega_3} * \varphi_m$. Then, by (2.3) on any compact subset of $\omega_3$, $\frac{\partial}{\partial t}u_m - \Delta w_m = 0$ for all $m$ sufficiently large. Using cylindrical coordinates we can choose an $r_1$, $\frac{r+R}{2} < r_1 < R$ so that

$$\int_{\partial B(x_0,r_1)\times(T_0,T_1)} |M(\alpha(u)\chi_{\omega_3})|^2 d\sigma \leq C \iint_{\omega_3} |M(\alpha(u)\chi_{\omega_3})(x,t)|^2 dx\, dt, \tag{2.5}$$

where again, $C$ depends only on $\omega_1$ and $\omega_3$.

Then by (2.4), for all sufficiently large $m$,

$$\begin{aligned} \int_{B(x_0,r_1)} u_m(x,a)^2 dx &\leq \int_{B(x_0,R)} |M(u\chi_{\omega_3})(x,a)|^2 dx \\ &\leq C \iint_{\omega_3} |M(u\chi_{\omega_3})(x,t)|^2 dx\, dt \\ &\leq C \iint_{\omega_3} u^2 dx\, dt \end{aligned} \tag{2.6}$$

Likewise, by (2.5)

$$\begin{aligned} \int_{\partial B(x_0,r_1)\times(a,T_1)} w_m^2 d\sigma &\leq \int_{\partial B(x_0,r_1)\times(T_0,T_1)} |M(\alpha(u)\chi_{\omega_3})|^2 d\sigma \\ &\leq C \iint_{\omega_3} |M(\alpha(u)\chi_{\omega_3})|^2 dx\, dt \\ &\leq C \iint_{\omega_3} |\alpha(u)|^2 dx\, dt. \end{aligned} \tag{2.7}$$

Let $\alpha_m(s)$ be a smooth regularization of $\alpha(s)$ such that $\alpha_m(s) = \alpha(s)$ for $|s| \geq 1 + \frac{1}{m}$, $\alpha_m(s)$ is strictly increasing and $\alpha_m(s) \neq 0$ except for $s = 0$. Put $\omega_2 = B(x_0, r_1) \times (a, T_1)$. Let $v_m$ be a classical solution (see [LSU]) to

$$\begin{cases} v_t = \Delta\alpha_m(v) & \text{on } \omega_2 \\ v(x,a) = u_m(x,a) & x \in B(x_0,r_1) \\ \alpha_m(v) = w_m & \text{on } \partial B(x_0,r_1) \times (a,T_1). \end{cases}$$

Set $\phi(x) = \frac{1}{2n}(|x-x_0|^2 - r_1^2)$. Then $\phi = 0$ on $\partial B(x_0,r_1)$, $\Delta\phi = 1$ on $B(x_0,r_1)$, $\phi < 0$ on $B(x_0,r_1)$ and $\frac{\partial\phi}{\partial n} = c_1 > 0$ on $\partial B(x_0,r_1)$, where $n$ is the outward normal and $c_1$ is a constant depending only on $r_1$ and the dimension.

By Green's theorem we have

$$\begin{aligned}
&\int_{B(x_0,r_1)} \alpha_m(v_m)^2\Delta\phi dx \\
&= \int_{B(x_0,r_1)} \Delta(\alpha_m(v_m))^2\phi dx + \int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2\frac{\partial\phi}{\partial n}d\sigma - \int_{\partial B(x_0,r_1)} \phi\frac{\partial}{\partial n}[\alpha_m(v_m)]^2 d\sigma \\
&= 2\int_{B(x_0,r_1)} \alpha_m(v_m)\Delta\alpha_m(v_m)\phi dx + 2\int_{B(x_0,r_1)} |\nabla\alpha_m(v_m)|^2\phi dx + c_1\int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2 d\sigma \\
&\le 2\int_{B(x_0,r_1)} v_{m_t}\alpha_m(v_m)\phi dx + c_1\int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2 d\sigma \\
&= 2\frac{d}{dt}\int_{B(x_0,r_1)} A_m(v_m)\phi dx + c_1\int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2 d\sigma
\end{aligned}$$

where $A_m$ is an antiderivative of $\alpha_m$. Integrate from $a$ to $T_1$ to obtain

$$\int_a^{T_1}\int_{B(x_0,r_1)} \alpha_m(v_m)^2\Delta\phi dx\, dt \le$$

$$2\int_{B(x_0,r_1)} A_m(v_m(x,T_1))\phi dx - 2\int_{B(x_0,r_1)} A_m(v_m(x,a))\phi dx + c_1\int_a^{T_1}\int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2 d\sigma dt.$$

Now $0 \le A_m(x) \le x^2$, $\phi < 0$, $\Delta\phi = 1$ and recalling $\omega_2 = B(x_0,r_1)\times(a,T_1)$, this yields:

$$\begin{aligned}
\iint_{\omega_2} \alpha_m(v_m)^2 dx\, dt &\le 2\int_{B(x_0,r_1)} v_m(x,a)^2 dx + c_1\int_a^{T_1}\int_{\partial B(x_0,r_1)} \alpha_m(v_m)^2 d\sigma dt \\
&= 2\int_{B(x_0,r_1)} u_m(x,a)^2 dx + c_1\int_a^{T_1}\int_{\partial B(x_0,r_1)} w_m^2 d\sigma dt \qquad (2.8) \\
&\le C\iint_{\omega_3} u^2 dx\, dt
\end{aligned}$$

where for the last inequality we have used (2.6) and (2.7). Let $\psi(x)$ be a nonnegative $C_0^\infty(\mathbb{R}^n)$ function such that $\psi \equiv 1$ on $B(x_0, \frac{3r+R}{4})$, $\psi \equiv 0$ outside $B(x_0, \frac{r+R}{2})$. To simplify notation set $B(x_0, \frac{3r+R}{4}) = B_1$, $B(x_0, \frac{r+R}{2}) = B_2$. Then

$$\begin{aligned}
\int_{B_2} \psi\alpha_m(v_m)v_{mt} dx &= \int_{B_2} \psi\alpha_m(v_m)\Delta\alpha_m(v_m) dx \\
&= -\int_{B_2} \nabla\psi\cdot\nabla\alpha_m(v_m)\alpha_m(v_m) dx - \int_{B_2} \psi|\nabla\alpha_m(v_m)|^2 dx \\
&= \frac{1}{2}\int_{B_2} \Delta\psi\alpha_m(v_m)^2 dx - \int_{B_2} \psi|\nabla\alpha_m(v_m)|^2 dx.
\end{aligned}$$

Rearrange and integrate from $a$ to $T_1$ to obtain

$$\begin{aligned}\int_a^{T_1}\int_{B_2}\psi|\nabla\alpha_m(v_m)|^2dx\,dt &= \frac{1}{2}\int_a^{T_1}\int_{B_2}\alpha_m(v_m)^2\Delta\psi dx\,dt - \int_a^{T_1}\int_{B_2}\psi\frac{d}{dt}A_m(v_m)dx\,dt\\ &\le C\int_a^{T_1}\int_{B_2}\alpha_m(v_m)^2dx\,dt + \int_{B_2}\psi(x)A_m(v_m(x,a))dx\\ &\le C\int_a^{T_1}\int_{B_2}\alpha_m(v_m)^2dx\,dt + \int_{B_2}v_m(x,a)^2dx\\ &\le C\iint_{\omega_2}\alpha_m(v_m)^2dx\,dt + \int_{B(x_0,r_1)}v_m(x,a)^2dx\\ &\le C\iint_{\omega_3}u^2dx\,dt\end{aligned}\tag{2.9}$$

where we have used (2.8) and (2.6) and the definition of $v_m$ for the last inequality.

We now seek a similar estimate for the $t$ derivative. Let $\eta(x)$ be a nonnegative $C_0^\infty(\mathbb{R}^n)$ function such that $\eta\equiv 1$ on $B(x_0,r)$, $\eta\equiv 0$ outside $B_1$ and so that $\|\frac{\nabla\eta}{\sqrt{\eta}}\|_\infty<\infty$. Note that $\alpha_m(v_m)_t=\alpha_m'(v_m)v_{mt}$ and $0<\alpha_m'\le 1$ so that $(\alpha_m(v_m)_t)^2\le\alpha_m(v_m)_t v_{m_t}$. Then

$$\begin{aligned}\int_{B_1}\eta(\alpha_m(v_m)_t)^2dx &\le \int_{B_1}\eta\alpha_m(v_m)_t v_{m_t}dx\\ &= \int_{B_1}\eta\alpha_m(v_m)_t\Delta\alpha_m(v_m)dx\\ &= -\int_{B_1}\nabla\eta\cdot\nabla\alpha_m(v_m)\,\alpha_m(v_m)_tdx - \int_{B_1}\eta\nabla\alpha_m(v_m)_t\cdot\nabla\alpha_m(v_m)dx\\ &= -\int_{B_1}\nabla\eta\cdot\nabla\alpha_m(v_m)\,\alpha_m(v_m)_tdx - \frac{1}{2}\frac{d}{dt}\int_{B_1}\eta|\nabla\alpha_m(v_m)|^2dx\end{aligned}$$

Integrate from $c$ to $T_1$, where $c$ is to be chosen momentarily. We obtain

$$\begin{aligned}&\int_c^{T_1}\int_{B_1}\eta(\alpha_m(v_m)_t)^2dx\,dt\\ &\qquad\le\int_c^{T_1}\left|\int_{B_1}\sqrt{\eta}\frac{\nabla\eta}{\sqrt{\eta}}\nabla\alpha_m(v_m)\alpha_m(v_m)_tdx\right|dt + \frac{1}{2}\int_{B_1}\eta(x)|\nabla\alpha_m(v_m)(x,c)|^2dx\\ &\qquad\le\left\|\frac{\nabla\eta}{\sqrt{\eta}}\right\|_\infty\left(\int_c^{T_1}\int_{B_1}|\nabla\alpha_m(v_m)|^2dx\,dt\right)^{\frac{1}{2}}\left(\int_c^{T_1}\int_{B_1}\eta(\alpha_m(v_m)_t)^2dx\,dt\right)^{\frac{1}{2}}\\ &\qquad\qquad+\frac{1}{2}\int_{B_1}\eta(x)|\nabla\alpha_m(v_m)(x,c)|^2dx.\end{aligned}\tag{2.10}$$

Choose $c_m$ (depending on $m$), $\frac{T_0+3t_0}{4}<c_m<t_0$, so that

$$\int_{B_1}\eta(x)|\nabla\alpha_m(v_m)(x,c_m)|^2dx\le C\int_a^{T_1}\int_{B_2}\psi|\nabla\alpha_m(v_m)|^2dx\,dt.\tag{2.11}$$

Note that $C$ depends only on the original domains $\omega = \omega_1$ and $\tilde{\omega} = \omega_3$. Put $c = c_m$ in (2.10). Notice that $a < c_m$ for all $m$. Then recalling that $\psi \equiv 1$ on $B_1$, and using (2.11) and (2.9) we have

$$\begin{aligned}
&\int_{c_m}^{T_1} \int_{B_1} \eta(\alpha_m(v_m)_t)^2 dx\,dt \\
&\leq \left\| \frac{\nabla\eta}{\sqrt{\eta}} \right\|_\infty \left( \int_{c_m}^{T_1} \int_{B_2} \psi |\nabla\alpha_m(v_m)|^2 dx\,dt \right)^{\frac{1}{2}} \left( \int_{c_m}^{T_1} \int_{B_1} \eta(\alpha_m(v_m)_t)^2 dx\,dt \right)^{\frac{1}{2}} + C \iint_{\omega_3} u^2 dx \\
&\leq \left\| \frac{\nabla\eta}{\sqrt{\eta}} \right\|_\infty \left( \iint_{\omega_3} u^2 dx\,dt \right)^{\frac{1}{2}} \left( \int_{c_m}^{T_1} \int_{B_1} \eta(\alpha_m(v_m)_t)^2 dx\,dt \right)^{\frac{1}{2}} + C \iint_{\omega_3} u^2 dx\,dt
\end{aligned}$$

from which it follows that

$$\int_{c_m}^{T_1} \int_{B_1} \eta(\alpha_m(v_m)_t)^2 dx\,dt \leq C \iint_{\omega_3} u^2 dx\,dt$$

and consequently

$$\iint_{\omega_1} (\alpha_m(v_m)_t)^2 dx\,dt \leq C \iint_{\omega_3} u^2 dx\,dt. \tag{2.12}$$

Thus, recalling $\omega = \omega_1$, $\tilde{\omega} = \omega_3$, (2.9) and (2.12) give

$$\iint_{\omega} |\nabla\alpha_m(v_m)|^2 dx\,dt \leq C \iint_{\tilde{\omega}} u^2 dx\,dt \quad \text{and} \quad \iint_{\omega} (\alpha_m(v_m)_t)^2 dx\,dt \leq C \iint_{\tilde{\omega}} u^2 dx\,dt. \tag{2.13}$$

To obtain (2.1) and (2.2) we will need to take limits. We first remark that with more care, similar estimates could be obtained with any compact set $K \subset \omega_2$ replacing $\omega = \omega_1$ on the left hand side of the inequalities in (2.13); naturally, the constants on the right hand side depend on the position of $K$ within $\omega_2$. Thus, from (2.8) and this observation, we have:

$$\begin{gathered}
\iint_{\omega_2} \alpha_m(v_m)^2 dx\,dt \leq C \iint_{\tilde{\omega}} u^2 dx\,dt, \qquad \iint_K |\nabla\alpha_m(v_m)|^2 dx\,dt \leq C(K) \iint_{\tilde{\omega}} u^2 dx\,dt \\
\text{and} \quad \iint_K \left| \frac{\partial}{\partial t}\alpha_m(v_m) \right|^2 dx\,dt \leq C(K) \iint_{\tilde{\omega}} u^2 dx\,dt
\end{gathered} \tag{2.14}$$

for every compact $K \subset \omega_2$.

By Rellich-Kondrachov there exists a subsequence $\{\alpha_{m_k}(v_{m_k})\}$ of $\{\alpha_m(v_m)\}$ (which we still write as $\{\alpha_m(v_m)\}$) and $h \in L^2(\omega_2)$ such that $\alpha_m(v_m) \to h$ in $L^2(K)$ for every compact set $K \subset \omega_2$. By taking subsequences, if necessary, we also may assume this convergence is a.e. By weak compactness, and again, by taking subsequences, we may assume that $\alpha_m(v_m) \to h$ weakly in $L^2(\omega_2)$. Equation (2.14) implies that the $L^2(\omega_2)$ norms of the $v_m$ are uniformly bounded, hence there exists a subsequence, (still denoted by $v_m$) such that $v_m \to v \in L^2(\omega_2)$ weakly.

We claim that $\alpha(v) = h$. First note that $\|\alpha_m - \alpha\|_\infty \to 0$, so that for a.e $(x,t) \in \omega_2$, $\alpha(v_m) \to h$. Consider the set where $h > 0$. Then for a.e $(x,t)$ in this set, $\alpha(v_m(x,t)) \to h(x,t) > 0$, and hence $v_m(x,t) \to h(x,t) + 1$. Thus, $v(x,t) = h(x,t) + 1$ for a.e $(x,t)$ in the set where $h > 0$. Similarly, on $h < -1$, $v(x,t) = h(x,t) - 1$ a.e. On the set $h = 0$ we must have $-1 \le \liminf v_m(x,t) \le \limsup v_m(x,t) \le 1$ a.e. To see this consider an $(x,t)$ at which there exists a subsequence $v_{m_k}(x,t)$ which converges to $y_0 \notin [-1,1]$. Then for this $(x,t)$, $\alpha(v_{m_k}(x,t)) \to \alpha(y_0) \neq 0$ which implies $\alpha(v_m(x,t)) \nrightarrow h(x,t)$. Thus, $-1 \le \liminf v_m(x,t) \le \limsup v_m(x,t) \le 1$ a.e. on $h = 0$, and hence $-1 \le v \le 1$ a.e. on $h = 0$. We conclude that $\alpha(v) = h$ a.e.

Summarizing, we have $v_m \to v$ weakly in $L^2(\omega_2)$ and $\alpha_m(v_m) \to \alpha(v)$ weakly in $L^2(\omega_2)$, a.e. on $\omega_2$ and in $L^2(K)$ for every compact subset $K$ of $\omega_2$. To finish the proof we show that $\alpha(u) = \alpha(v)$ a.e. on $\omega_2$. By (2.3), $\frac{\partial}{\partial t}u_m - \Delta w_m = 0$ on $B(x_0,r_1) \times (a, T_1 - \varepsilon)$ (for fixed $\varepsilon$ and $m$ sufficiently large depending on $\varepsilon$). For convenience in the following computations set $\omega_2' = B(x_0,r_1) \times (a, T_1 - \varepsilon) = \omega_2 \cap \{t < T_1 - \varepsilon\}$. Then, using that $v_{mt} = \Delta\alpha_m(v_m)$ on $\omega_2$, $u_{mt} = \Delta w_m$ on $\omega_2$, and $\alpha_m(v_m) = w_m$ on $\partial B(x_0,r_1) \times (a, T_1)$ yields

$$\begin{aligned}
&\iint_{\omega_2'} (v_m - u_m)(\alpha_m(v_m) - w_m)dx\,dt \\
&\qquad = \int_a^{T_1-\varepsilon} \int_{B(x_0,r_1)} \int_a^t v_{mt}(x,\tau) - u_{mt}(x,\tau)d\tau(\alpha_m(v_m(x,t)) - w_m(x,t))dx\,dt \\
&\qquad = -\int_a^{T_1-\varepsilon} \int_{B(x_0,r_1)} \int_a^t \nabla(\alpha_m(v_m) - w_m)(x,\tau)d\tau \nabla(\alpha_m(v_m) - w_m)(x,t)dx\,dt \\
&\qquad = -\frac{1}{2}\int_a^{T_1-\varepsilon} \int_{B(x_0,r_1)} \frac{d}{dt}\left|\int_a^t \nabla(\alpha_m(v_m) - w_m)d\tau\right|^2 dx\,dt \\
&\qquad = -\frac{1}{2}\int_{B(x_0,r_1)} \left|\int_a^{T_1-\varepsilon} \nabla(\alpha_m(v_m) - w_m)d\tau\right|^2 dx \le 0.
\end{aligned} \tag{2.15}$$

We need to let $m \to \infty$ in this inequality. Write

$$\begin{aligned}
\iint_{\omega_2'} (v_m - u_m)(\alpha_m(v_m) - w_m)dx\,dt &= \iint_{\omega_2'} v_m\alpha_m(v_m)dx\,dt + \iint_{\omega_2'} v_m(-w_m)dx\,dt \\
&+ \iint_{\omega_2'} (-u_m)(\alpha_m(v_m))dx\,dt + \iint_{\omega_2'} u_m w_m dx\,dt = I + II + III + IV
\end{aligned}$$

Since $u_m \to u$ a.e. and in $L^2(\omega_2)$, $w_m \to \alpha(u)$ a.e. and in $L^2(\omega_2)$, $v_m \to v$ weakly, and $\alpha_m(v_m) \to \alpha(v)$ weakly, we conclude

$$II \to \iint_{\omega_2'} v(-\alpha(u))dx\,dt, \qquad III \to \iint_{\omega_2'} (-u)(\alpha(v))dx\,dt,$$

$$\text{and}\ \ IV \to \iint_{\omega_2'} u\alpha(u)dx\,dt.$$

Expand out

$$\iint_{\omega_2'} (\alpha_m(v_m) - \alpha_m(v))(v_m - v)dx \geq 0,$$

take $m \to \infty$ (make use of the fact that $\|\alpha_m - \alpha\|_\infty \to 0$) to conclude

$$\liminf_{m\to\infty} \iint_{\omega_2'} \alpha_m(v_m)v_m dx \geq \iint_{\omega_2'} \alpha(v)v dx.$$

This combined with the estimates for II-IV and (2.15) yields

$$\iint_{\omega_2'} (v-u)(\alpha(v) - \alpha(u))dx\,dt \leq 0.$$

Since the integrand of this is nonnegative, we conclude $\alpha(u) = \alpha(v)$ a.e. on $\omega_2'$. Taking $\varepsilon \to 0$ shows $\alpha(u) = \alpha(v)$ a.e. on $\omega_2$. Since $\omega = \omega_1 \subset \omega_2$, this completes the proof of the theorem.

□

# 3 $|\alpha(u)|$ is subcaloric

**Theorem 3.1.** *$\alpha(u)^+$ is weakly subcaloric, that is, it satisfies*

$$\int_\Omega -\nabla\alpha(u)^+ \cdot \nabla\eta + \alpha(u)^+\eta_t dx\,dt \geq 0$$

*for any nonnegative $\eta \in W_0^{1,2}(\Omega)$. Furthermore, $\alpha(u)^-$ and $|\alpha(u)|$ are subcaloric.*

Here $\alpha(u)^+ = \max\{\alpha(u), 0\}$, $\alpha(u)^- = -\min\{\alpha(u), 0\}$.

*Proof.* Let $0 \leq \eta \in W_0^{1,2}(\Omega)$. For $h > 0$ set

$$\phi_h(x) = \begin{cases} 1 & \text{if } x > h \\ \frac{2}{h}x - 1 & \text{if } \frac{h}{2} < x < h \\ 0 & \text{otherwise.} \end{cases}$$

Then $\eta\phi_h(\alpha(u))$ is supported in $\left\{\alpha(u) > \frac{h}{2}\right\}$ and thus, $\iint (u-\alpha(u))[\eta\phi_h(\alpha(u))]_t dx\,dt = 0$ since $u - \alpha(u) = 1$ on this support. Then

$$\begin{aligned} 0 &= \iint \alpha(u)[\eta\phi_h(\alpha(u))]_t - \nabla\alpha(u)\cdot\nabla[\eta\phi_h(\alpha(u))]dx\,dt \\ &= \iint \alpha(u)\eta_t\phi_h(\alpha(u))dx\,dt + \iint \alpha(u)\eta\phi_h'(\alpha(u))\alpha(u)_t dx\,dt \\ &\quad - \iint \nabla\alpha(u)\cdot\nabla\eta\,\phi_h(\alpha(u))dx\,dt - \iint \nabla\alpha(u)\cdot\nabla\alpha(u)\,\eta\phi_h'(\alpha(u))dx\,dt \\ &= I + II + III + IV. \end{aligned} \tag{3.1}$$

We investigate each of these as $h \to 0$. As $h \to 0$, $\alpha(u)\phi_h(\alpha(u)) \to \alpha(u)^+$ so that $I \to \iint \alpha(u)^+ n_t dx\, dt$. To estimate $II$, first note that

$$\phi_h'(\alpha(u)) = \frac{2}{h}\chi_{\left\{\frac{h}{2}<\alpha(u)<h\right\}}.$$

Then

$$II = \iint \alpha(u)\eta\frac{2}{h}\chi_{\{\frac{h}{2}<\alpha(u)<h\}}\frac{\partial}{\partial t}\alpha(u)dx\, dt$$

and consequently,

$$|II| \leq \iint |\alpha(u)|\eta\frac{2}{h}\chi_{\{\frac{h}{2}<\alpha(u)<h\}}\left|\frac{\partial}{\partial t}\alpha(u)\right| dx\, dt \leq 2\iint \eta\chi_{\{\frac{h}{2}<\alpha(u)<h\}}\left|\frac{\partial}{\partial t}(\alpha(u))\right| dx\, dt.$$

Since $\frac{\partial}{\partial t}\alpha(u) \in L^2_{loc}(\Omega)$, $II \to 0$ as $h \to 0$

To estimate $III$, note that when $\alpha(u) > h$, $\nabla\alpha(u)\phi_h(\alpha(u)) = \nabla\alpha(u)^+$. And when $\frac{h}{2} < \alpha(u) < h$,

$$|\nabla\alpha(u)\phi_h(\alpha(u))| \leq |\nabla\alpha(u)|\left(\frac{2}{h}|\alpha(u)| + 1\right) \leq 3|\nabla\alpha(u)|$$

Consequently,

$$\begin{aligned}\left|\iint_{\{\frac{h}{2}<\alpha(u)<h\}} \nabla\alpha(u)\cdot\nabla\eta\phi_h(\alpha(u))dx\, dt\right| &\leq \iint_{\{\frac{h}{2}<\alpha(u)<h\}} 3|\nabla\alpha(u)||\nabla\eta|dx\, dt \\ &\to 0 \text{ as } h \to 0 \text{ since } \nabla\alpha(u) \in L^2_{loc}(\Omega).\end{aligned}$$

Therefore, as $h \to 0$, $III \to -\iint \nabla\alpha(u)^+\cdot\nabla\eta dx\, dt$. Note that we can write $IV$ as

$$IV = \iint |\nabla\alpha(u)|^2\eta\ \phi_h'(\alpha(u))dx\, dt = \iint |\nabla\alpha(u)|^2\eta\frac{2}{h}\chi_{\left\{\frac{h}{2}<\alpha(u)<h\right\}}dx\, dt$$

Thus, letting $h \to 0$ in (3.1) yields:

$$0 = \iint \alpha(u)^+\eta_t dx\, dt - \iint \nabla\alpha(u)^+\cdot\nabla\eta dx\, dt - \lim_{h\to 0}\iint |\nabla\alpha(u)|^2\frac{2}{h}\chi_{\left\{\frac{h}{2}<\alpha(u)<h\right\}}\eta dx\, dt$$

and it follows that $\alpha(u)^+$ is weakly subcaloric.

Notice that $-u$ is a solution of $\Delta\alpha(u) = u_t$ if $u$ is, and that $\alpha(-u)^+ = \alpha(u)^-$. Consequently, $\alpha(u)^-$ is weakly subcaloric and thus, so is $|\alpha(u)| = \alpha(u)^+ + \alpha(u)^-$. $\square$

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