THE CAUCHY PROBLEM FOR THE TWO PHASE STEFAN PROBLEM

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ABSTRACT: On the upper half space, we consider the two-phase Stefan problem $u_t = \Delta\alpha(u)$ where $\alpha(u) = u + 1$ for $u < -1$, $\alpha(u) = 0$ for $-1 \leq u \leq 1$, and $\alpha(u) = u - 1$ for $u \geq 1$, taken in the sense of conservation laws. We show that the Cauchy problem is solvable for function and measure data which satisfy the proper growth condition at infinity.
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1. INTRODUCTION

In this paper we will discuss the Cauchy problem for the two-phase Stefan problem on the upper half space $\mathbb{R}^{n+1}_+ = \{(x,t) |\, x \in \mathbb{R}^n, t > 0\}$. Define $\alpha(u)$ on $\mathbb{R}$ by $\alpha(u) = 0$ if $-1 \leq u \leq 1$, $\alpha(u) = u-1$ for $u > 1$, and $\alpha(u) = u+1$ for $u < -1$. The two-phase Stefan problem can be stated classically as: Find u, defined on $\mathbb{R}^n \times [0,T)$, for some $T > 0$, where $u(x,0) = u_0(x)$ on $\mathbb{R}^n$ is given, such that u is piecewise smooth except for discontinuities along a finite number of smooth surfaces S_i so that away from the discontinuities we have

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta\alpha(u) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.1}$$

and along a surface S_i we have

$$(\nabla_x \alpha(u^+) - \nabla_x \alpha(u^-)) \cdot \nu_x = (u^+ - u^-) \cdot \nu_t, \tag{1.2}$$

where (ν_x, ν_t) is the normal to the surface and u^+ and u^- represent the values taken from different sides.

In this paper we consider solutions taken in the sense of conservation laws. Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with appropriate growth at infinity (defined precisely below), we say that u, which is integrable on bounded measurable subsets of $\mathbb{R}^n \times (0,T)$, satisfies the two-phase Stefan problem in the sense of conservation laws if

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx\, dt + \int_{\mathbb{R}^n} \varphi(x,0) u_0(x)\, dx = 0, \tag{1.3}$$

for every $\varphi \in C^\infty(\mathbb{R}^n \times (-\infty, T))$ with compact support. A solution which satisfies (1.1) and (1.2) satisfies (1.3). (See Ladyženskaja, Solonnikov, and

Ural'ceva [LSU], Chapter 5, section 9, or Bouillet [B].) In the one-phase case, that is, if $u_0 \geq 0$, existence of a solution in the sense of (1.3) was shown by Andreucci and Korten [AK]. The proof we give below for the two-phase case is somewhat similar to their proof, although in the two-phase case we cannot utilize the monotonicity arguments they are able to use.

2. THE CAUCHY PROBLEM FOR FUNCTION DATA

Definition 2.1. *We say that a function $f \in L^1_{loc}(\mathbb{R}^n)$ belongs to $\mathcal{G}_c$, $c > 0$ if*

$$\int_{\mathbb{R}^n} e^{-c|x|^2} |f(x)|\, dx < \infty.$$

We now come to our main result.

Theorem 2.2. *Suppose $u_0 \in \mathcal{G}_c$. Then there exists a solution u of (1.3) on $\mathbb{R}^n \times (0,T)$, where $T = \frac{1}{4c}$. Moreover, u is bounded on compact subsets of $\mathbb{R}^n \times (0,T)$.*

Before proving this, we state two results from Korten and Moore [KM]. Concerning the classical (1.1), (1.2) or the conservation laws formulation (1.3), in general, we do not expect any smoothness of solutions or any particularly good behavior. If, for example, $-1 < u_0(x) < 1$ for all $x \in \mathbb{R}^n$, then the solution is just $u(x,t) = u_0(x)$, so that if u_0 lacks smoothness, then $u(x,t)$ need not be any better. However, we can expect more from $\alpha(u)$, namely, it is continuous. This was shown in the one-phase case by Korten [K] and in the two-phase case in [KM]. The result in [KM] makes use of the two theorems below as well as regularity results of Caffarelli and Evans [CE], DiBenedetto [D], Sacks [S] and Ziemer [Z].

Theorem 2.3. *Suppose $\Omega \subseteq \mathbb{R}^n \times (0,T)$ and $u \in L^2_{loc}(\Omega)$ satisfies*

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx\, dt = 0$$

for every $\varphi \in C^\infty$ which is compactly supported in Ω. Suppose $r < R$, $T_0 < t_0 < T_1$, set $\omega = (t_0, T_1) \times B(x_0, r)$ and $\tilde{\omega} = (T_0, T_1) \times B(x_0, R)$, and suppose the closure of $\tilde{\omega}$ is contained in Ω. Then $\nabla\alpha(u)$, $\alpha(u)_t$ exist in $L^2(\omega)$ and there exists a constant C, depending only on ω and $\tilde{\omega}$ such that

$$\iint_\omega |\nabla\alpha(u)|^2 \, dx\, dt \leq C \iint_{\tilde{\omega}} u^2 \, dx\, dt$$

and

$$\iint_\omega \left|\frac{\partial}{\partial t}\alpha(u)\right|^2 \, dx\, dt \leq C \iint_{\tilde{\omega}} u^2 \, dx\, dt.$$

Theorem 2.4. *Suppose Ω is as in the previous theorem. Then $|\alpha(u)|$ is weakly subcaloric, that is, it satisfies*

$$\int_\Omega -\nabla|\alpha(u)|\nabla\eta + |\alpha(u)|\eta_t \, dx\, dt \geq 0$$

for any nonnegative η which is C^∞ and compactly supported in Ω.

We can now prove Theorem 2.2.

Proof. For $m = 1, 2, 3, \dots$ define

$$u_0^m(x) = \begin{cases} 0 & \text{if } |x| > m \\ u_0(x) & \text{if } |x| \leq m \end{cases}$$

Let ρ be a nonnegative C^∞ function on $\mathbb{R}^n$, supported in the unit ball, radially symmetric and with integral 1. For $m = 1, 2, 3, \dots$ set $\rho_m(x) = m\rho(xm)$ and $u_{0,m} = \rho_m * u_0^m$.

Then for each m, $u_{0,m}$ is C^∞ and of compact support on $\mathbb{R}^n$. Let $u_m(x,t)$ be the solution to (1.3) with initial data u_0^m. (See Ladyženskaja, Solonnikov, and Ural'ceva [LSU], Chapter 5, section 9, for the existence of such solutions.) Furthermore, $u_m(x,t)$ is defined on all of $\mathbb{R}^n \times (0,\infty)$.

Let $v_m(x,t)$ be the solution to the heat equation with initial data $|\alpha(u_{0,m})|$:

$$\Delta v_m = \frac{\partial v_m}{\partial t}, \quad \text{on } \mathbb{R}^n \times (0,\infty)$$
$$v_m(x,0) = |\alpha(u_{0,m}(x))| \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then $v_m = g_t * |\alpha(u_{0,m})|$, where $g_t(y) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|y|^2}{4t})$ is the Gauss kernel.

Since $|\alpha(u_m(x,t))|$ has initial values $|\alpha(u_{0,m}(x))|$ and is subcaloric, then

$$|\alpha(u_m(x,t))| \leq v_m(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Consequently, for $(x,t) \in \mathbb{R}^n \times (0,T)$, where $T = \frac{1}{4c}$,

$$\begin{aligned} |\alpha(u_m(x,t))| &\leq v_m(x,t) \\ &= \int_{\mathbb{R}^n} g_t(x-y)\, |\alpha(u_{0,m})(y)|\, dy \\ &\leq \int_{\mathbb{R}^n} g_t(x-y)(|u_{0,m}(y)| + 1)\, dy \\ &\leq 1 + \int_{\mathbb{R}^n} g_t(x-y)(\rho_m * |u_0^m|)(y)\, dy \\ &\leq 1 + \int_{\mathbb{R}^n} g_t(x-y)(\rho_m * |u_0|)(y)\, dy \\ &= 1 + \rho_m * g_t * |u_0|(x) \end{aligned} \tag{2.1}$$

Note that the last quantity is finite because $|u_0| \in \mathcal{G}_c$.

Consider any cylinder $Q = B(x_0, r)\times(t_0, t_1)$ contained in $\mathbb{R}^n\times(0, T)$. Let $\tilde{Q}$ be a slightly larger cylinder whose closure is still contained in $\mathbb{R}^n \times (0, T)$. Then on $\tilde{Q}$, $g_t * |u_0(x)|$ as well as all its derivatives are bounded. (Of course, the bound depends on $\tilde{Q}$.) It follows then from (2.1) that the $|\alpha(u_m(x, t))|$ are uniformly bounded on Q for sufficiently large m. Hence also, the $|u_m|$ are uniformly bounded on Q. This holds true on any cube slightly larger than Q so that then the energy estimates, Theorem 2.3, imply that the L^2 norms of the derivatives of the $\alpha(u_m)$ are uniformly bounded on Q.

Noticing that $\mathbb{R}^n \times (0, T)$ can be exhausted by a countable number of such cylinders Q, by Rellich-Kondrachov there exists an $h \in L^2_{\text{loc}}(\mathbb{R}^n\times(0, T))$ and a subsequence $\alpha(u_{m_k})$ of $\alpha(u_m)$ such that $\alpha(u_{m_k}) \to h$ in $L^2(K)$ for every compact set $K \subset \mathbb{R}^n \times (0, T)$. By taking subsequences, if necessary, we can assume that $\alpha(u_{m_k}) \to h$ a.e. Since the u_m are uniformly bounded on each cylinder Q, we may further assume, again by taking subsequences, that there exists a locally bounded function u such that $u_{m_k} \to u$ weakly in $L^2(K)$ for any compact $K \subset \mathbb{R}^n \times (0, T)$.

We first claim $h = \alpha(u)$. On the set where $h > 0$, $h = \lim \alpha(u_{m_k})$ a.e. so that $h + 1 = \lim u_{m_k}$ a.e. on this set. Let F be a compact subset of the set $h > 0$. Then $\iint_F h + 1\, dx\, dt = \lim \int_F u_{m_k}\, dx\, dt = \int_F u\, dx\, dt$. This is true for every such F and so $h+1 = u$, that is, $\alpha(u) = h$ a.e. on the set where $h > 0$. Similarly, $\alpha(u) = h$ a.e. on the set where $h < 0$. On the set where $h = 0$, $\alpha(u_{m_k}) \to 0$ a.e. , and hence $-1 \le \liminf u_{m_k}(x, t) \le \limsup u_{m_k}(x, t) \le 1$ a.e. there. Let $F \subset \{h = 0\}$ be compact. Then $\lim \iint_F u_{m_k}(x, t)\, dx\, dt = \iint_F u(x)\, dx\, dt$ and hence $-|F| \le \iint_F u(x, t)\, dx\, dt \le |F|$. Consequently, $-1 \le u(x, t) \le 1$ a.e. on the set where $h = 0$ and thus $h = \alpha(u)$ a.e. there.

It remains to show that $u(x, t)$ satisfies (1.3). Let $\varphi \in C^\infty$ have compact support in $\mathbb{R}^n \times (-\infty, T)$. Then for each m_k,

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u_{m_k})\Delta\varphi + u_{m_k}\varphi_t\, dx\, dt + \int_{\mathbb{R}^n} \varphi(x, 0)u_{0,m_k}(x)\, dx = 0 \tag{2.2}$$

We want to take limits in this expression. On compact sets of $\mathbb{R}^n$, $u_{0,m_k} \to u_0$ in L^1 so

$$\int_{\mathbb{R}^n} \varphi(x, 0)u_{0,m_k}(x)\, dx \to \int_{\mathbb{R}^n} \varphi(x, 0)u_0(x)\, dx. \tag{2.3}$$

Now let B be a ball in $\mathbb{R}^n$ such that $B \times (-\infty, T)$ contains the support of φ. First we observe that $|u_0| \in \mathcal{G}_c$ implies that for $d > c$, $\rho_m * |u_0| \in \mathcal{G}_d$ for every m. To see this, note that for $|s - y| \le 1$, elementary computations show that $|s|^2 \le 1 + 1/\gamma^2 + (1 + \gamma^2)|y|^2$ for any $\gamma > 0$ and thus, choosing γ

so that $(1+\gamma^2)c = d$, gives $\exp(-d|y|^2) \le C(\gamma)\exp(-c|s|^2)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \rho_m * |u_0|(y) e^{-d|y|^2}\,dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_m(y-s)|u_0|(s)e^{-d|y|^2}\,ds\,dy \\ &\le C(\gamma)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_m(y-s)|u_0|(s)e^{-c|s|^2}\,ds\,dy \\ &= C(\gamma)\int_{\mathbb{R}^n} |u_0|(s)e^{-c|s|^2} ds\end{aligned} \tag{2.4}$$

Next observe that for t small (say $t < \frac{1}{32c}$),

$$e^{2c|y|^2} \le e^{4c(|x-y|^2+|x|^2)} \le e^{\frac{|x-y|^2}{8t}} e^{4c|x|^2},$$

and consequently,

$$e^{2c|y|^2} g_t(x-y) \le 2^{\frac{n}{2}} e^{4c|x|^2} g_{2t}(x-y).$$

Then for all m, and $\delta < \frac{1}{32c}$,

$$\begin{aligned}&\int_0^\delta \int_B |\alpha(u_m)|\,dx\,dt \le \int_0^\delta\int_B \left(1+\int_{\mathbb{R}^n} g_t(x-y)(\rho_m * |u_0|)(y)\,dy\right)dx\,dt \\ &= |B|\delta + \int_0^\delta\int_B\int_{\mathbb{R}^n} e^{2c|y|^2} g_t(x-y)(\rho_m*|u_0|)(y)e^{-2c|y|^2}\,dy\,dx\,dt \\ &\le |B|\delta + 2^{\frac{n}{2}}\sup_{x\in B}\{e^{4c|x|^2}\}\int_0^\delta\int_{\mathbb{R}^n}\int_B g_{2t}(x-y)(\rho_m*|u_0|)(y)e^{-2c|y|^2}\,dx\,dy\,dt \\ &\le |B|\delta + C(n,c,B)\,\delta\end{aligned}$$

where $C(n,c,B)$ is a constant depending only on n, c and B. Here, the last inequality also uses the fact that $\rho_m * |u_0| \in \mathcal{G}_{2c}$.

Thus, given $\varepsilon > 0$, we can find $\delta > 0$ so that

$$\int_0^\delta\int_{\mathbb{R}^n} |\Delta\varphi\,\alpha(u_{m_k})|\,dx\,dt \le C\delta\|\Delta\varphi\|_\infty < \varepsilon$$

for every m_k. Similarly, since $|u| \le |\alpha(u)|+1$, the above computation also shows that we can assume δ is chosen so that

$$\int_0^\delta\int_{\mathbb{R}^n} |\varphi_t u_{m_k}|\,dx\,dt < \varepsilon.$$

Using these last two inequalities, (2.3), and the fact that $\alpha(u_{m_k}) \to \alpha(u)$ in $L^2(K)$ and $u_{m_k} \to u$ weakly in $L^2(K)$ for every compact set $K \subset \mathbb{R}^n \times (0,T)$, we may take $m_k \to \infty$ in (2.2) to obtain

$$\int_0^T\int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t\,dx\,dt + \int_{\mathbb{R}^n}\varphi(x,0)\,u_0(x)\,dx = 0.$$

□

3. THE CAUCHY PROBLEM FOR MEASURE DATA

Using essentially the same proof, we next show existence of solutions for measure data.

Definition 3.1. *We say a Radon measure $\mu \in \mathcal{G}_c$, $c > 0$, if*

$$\int_{\mathbb{R}^n} e^{-c|x|^2}\, d|\mu| < \infty$$

where $|\mu|$ is the total variation measure of μ.

Definition 3.2. *Given a Radon measure $\mu \in \mathcal{G}_c$ for some $c > 0$, we say that u, assumed integrable on any bounded measurable subset of $\mathbb{R}^n \times (0,T)$, solves the two phase Stefan problem with initial data μ in the sense of conservation laws if*

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx\, dt + \int_{\mathbb{R}^n} \varphi(x,0)\, d\mu(x) = 0 \tag{3.1}$$

for every $\varphi \in C^\infty(\mathbb{R}^n \times (-\infty, T))$ with compact support.

Theorem 3.3. *Suppose $\mu \in \mathcal{G}_c$. Then there exists a solution of* (3.1) *on $\mathbb{R}^n \times (0,T)$, where $T = \frac{1}{4c}$.*

Proof. The proof follows closely the proof of the previous theorem so we just give a sketch.

As before, let ρ be a nonnegative C^∞ function on $\mathbb{R}^n$, supported in the unit ball, radially symmetric and with integral 1, and for $m = 1, 2, 3, \ldots$ set $\rho_m(x) = m\rho(xm)$. Set $\mu_m(x) = \rho_m * \mu(x) = \int_{\mathbb{R}^n} \rho_m(x-y)\, d\mu(y)$.

Each $\mu_m(x)$ is a C^∞ function on $\mathbb{R}^n$ and arguing as in (2.4), we can assert that $|\mu_m|$, and hence $|\alpha(\mu_m)|$ belong to $\mathcal{G}_d$ for any $d > c$. By the previous theorem, there exists a solution, $u_m(x,t)$ to (1.3) on $\mathbb{R}^n \times (0,T)$ with initial data $\mu_m(x)$. Let v_m be the solution to the Cauchy problem for the heat equation

$$\begin{aligned} \Delta v_m &= \frac{\partial v_m}{\partial t} \quad \text{on} \quad \mathbb{R}^n \times (0,\infty) \\ v_m(x,0) &= |\alpha(\mu_m(x))| \quad \text{for} \quad x \in \mathbb{R}^n. \end{aligned}$$

Then, since $|\alpha(u_m(x,t))|$ is subcaloric, we have $|\alpha(u_m)| \le v_m$ on $\mathbb{R}^n \times (0,T)$. Estimating as before yields

$$|\alpha(u_m(x,t))| \le 1 + g_t * |\mu_m(x)| \le 1 + \rho_m * g_t * |\mu|(x).$$

Again, $g_t * |\mu|(x)$ as well as all its derivatives are bounded in any cylinder strictly contained in $\mathbb{R}^n \times (0,T)$, and we may argue as in the previous theorem to produce a subsequence u_{m_k} and a function u such that $u_{m_k} \to u$ weakly in $L^2(K)$ and $\alpha(u_{m_k}) \to \alpha(u)$ in $L^2(K)$ and a.e. for any compact set $K \subset \mathbb{R}^n \times (0,T)$. For each m_k and $\varphi \in C^\infty(\mathbb{R}^n \times (-\infty, T))$ with compact support,

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u_{m_k})\Delta\varphi + u_{m_k}\varphi_t \, dx \, dt + \int_{\mathbb{R}^n} \varphi(x,0) u_{0,m_k}(x) \, dx = 0 \tag{3.2}$$

Again, we want to take limits in this expression. As before, given $\varepsilon > 0$, we may choose δ small so that

$$\int_0^\delta \int_{\mathbb{R}^n} |\Delta\varphi\alpha(u_{m_k})| \, dx \, dt \leq C\delta\|\Delta\varphi\|_\infty < \varepsilon \quad \text{and} \quad \int_0^\delta \int_{\mathbb{R}^n} |\varphi_t u_{m_k}| \, dx \, dt < \varepsilon.$$

Thus, as before,

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u_{m_k})\Delta\varphi + u_{m_k}\varphi_t \, dx \, dt \to \int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx \, dt.$$

Furthermore, straightforward computations show that $\int_{\mathbb{R}^n} \varphi(x,0)\mu_{m_k} dx \to \int_{\mathbb{R}^n} \varphi(x,0) \, d\mu$. Therefore we may take limits in (3.2) to complete the proof.

□

Remark. Instead of solving the Cauchy problem in the sense of conservation laws (3.1), one can also consider solutions $u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0,T))$ taken in the sense of distributions:

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx \, dt = 0 \\ &\qquad\qquad \text{for all compactly supported } \ \varphi \in C^\infty(\mathbb{R}^n \times (0,T)) \\ &\lim_{t\to 0} \int_{\mathbb{R}^n} \psi(x) u(x,t) \, dx = \int_{\mathbb{R}^n} \psi(x) d\mu \\ &\qquad\qquad \text{for all compactly supported } \ \psi \in C^\infty(\mathbb{R}^n). \end{aligned} \tag{3.3}$$

Suppose u solves the Cauchy problem with measure data $\mu \in \mathcal{G}_c$ in the conservation laws sense (3.1). We claim that then u is a solution in the sense of distributions. First note that, in particular, (3.1) gives $\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t \, dx \, dt = 0$ for φ compactly supported in $\mathbb{R}^n \times (0,T)$. Let $\eta(t)$ be a C^∞ function which is identically 1 for $t < -\frac{2}{5}$, is identically 0 for $t > \frac{2}{5}$ and decreasing and with $\eta(0) = \frac{1}{2}$. Fix $t_0 > 0$. For $m = 1, 2, \dots$ set $\eta_m(t) = \eta(mt)$. Let ψ be a C^∞ function of compact support on $\mathbb{R}^n$ and set $\varphi_m(x,t) = \psi(x)\eta_m(t - t_0)$. Then with this choice of φ, (3.1) becomes

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \Delta\psi(x)\eta_m(t-t_0)\alpha(u(x,t)) + u(x,t)\psi(x)\eta_m'(t-t_0) \, dx \, dt \\ + \int_{\mathbb{R}^n} \psi(x)\eta_m(0 - t_0) d\mu = 0. \end{aligned} \tag{3.4}$$

Notice that for m sufficiently large, $\int_{\mathbb{R}^n} \psi(x)\eta_m(0-t_0)d\mu = \int_{\mathbb{R}^n} \psi(x)d\mu$. Also, $[0,T) \cap \operatorname{supp} \eta_m(t-t_0) \subset [0, t_0 + \frac{1}{2m}]$, so that for t_0 small and large m, $\int_0^T \int_{\mathbb{R}^n} \Delta\psi \eta_m(t-t_0)\alpha(u(x,t))\, dx\, dt$ is small. (Since, by assumption, $\alpha(u)$ is integrable on any bounded measurable subset of $\mathbb{R}^n \times (0,T)$.) As $m \to \infty$, for a.e. t_0, $\int_0^T \int_{\mathbb{R}^n} u(x,t)\psi(x)\eta_m'(t-t_0)\, dx\, dt \to \int_{\mathbb{R}^n} u(x,t_0)\psi(x)dx$. Consequently, (3.4) implies $\int_{\mathbb{R}^n} u(x,t)\psi(x)\, dx \to \int_{\mathbb{R}^n} \psi(x)\, d\mu$ as $t_0 \to 0$. Thus, if u solves the Cauchy problem in the sense of conservation laws, (3.1), then u solves the Cauchy problem in the sense of (3.3).

Conversely, if we assume $u \in L^1$ on bounded subsets of $\mathbb{R}^n \times (0,\infty)$, then a similar argument shows that a solution in the sense of (3.3) will be a solution in the sense of (3.1). This assumption is necessary for finiteness of the integrals in (3.1). In the definition (3.3) we only require that u be integrable on compact sets contained within $\mathbb{R}^n \times (0,T)$.

Remark. Regarding uniqueness of solutions, it would be desirable to show that if u and v were two solutions of (1.3) or (3.1) on $\mathbb{R}^n \times (0,T)$ then $u = v$ on $\mathbb{R}^n \times (0,T)$ if

$$\int_0^T \int_{\mathbb{R}^n} \alpha(u)\Delta\varphi + u\varphi_t\, dx\, dt = \int_0^T \int_{\mathbb{R}^n} \alpha(v)\Delta\varphi + v\varphi_t\, dx\, dt$$

for every compactly supported $\varphi \in C^\infty(\mathbb{R}^n \times (-\infty,T))$. Reasoning as in the previous remark, this is the same as the assumption that u and v are solutions to (1.3) or (3.1) which satisfy

$$\int_{\mathbb{R}^n} (u(x,t) - v(x,t))\psi(x)\, dx \to 0 \quad \text{as} \quad t \to 0$$

for every $\psi \in C^\infty(\mathbb{R}^n)$ with compact support. We have not been able to show uniqueness in this generality.

However, uniqueness results in this situation are known:

Theorem 3.4. *(Bouillet [B]) Let u and v be two solutions of* (3.3) *which belong to $\mathcal{G}_c$, for some $c > 0$. Then $u = v$ a.e. on $\mathbb{R}^n \times (0,T)$ provided*
(i) $\|(u-v)(\cdot,t)\|_{L^1_{loc}} \to 0$ *as* $t \to 0^+$ *or*
(ii) $n = 1$ *and* $\int_{\mathbb{R}^n} (u(x,t) - v(x,t))\psi(x)\, dx \to 0$ *as* $t \to 0$ *for every* $\psi \in C^\infty(\mathbb{R}^n)$ *with compact support.*

Consequently, (ii) gives uniqueness for the Cauchy problem with function data (1.3) or measure data (3.1) in the case $n = 1$. Unfortunately, the proof given there depends on Sobolev imbedding estimates which depend on dimension. If the initial data is nonnegative, then the solution to the Cauchy problem is nonnegative, and in this situation, for all n, $u = v$ a.e. if (ii) holds. (See Korten [K].) It seems likely then, that in the two-phase case, for all n we have uniqueness if (ii) holds, but we must leave this as an open problem.

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