

Acceleration of Fourier Series

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Abstract

We discuss the effects of several sequence acceleration methods on the partial sums of Fourier series. For a large set of functions we show that these methods fail.

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1 Introduction

Because of the widespread applications of Fourier series, it is of interest to analyze their speed of convergence. Particularly enticing is the possibility of applying methods to accelerate this convergence. Various methods of acceleration of convergence of sequences have been applied with some success to the partial sums of Fourier series. In this paper, we discuss the application of some well-known methods of sequence acceleration to the partial sums of Fourier series.

For a function f which is integrable on $[-\pi, \pi]$, we define the Fourier coefficients by

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

for each integer n , and we define the n^{th} partial sum of the Fourier series as

$$S_n f(x) := \sum_{k=-n}^n \hat{f}(k) e^{ikx},$$

where n is a positive integer and x is in $[-\pi, \pi]$.

When f is square-integrable, $S_n f(x)$ converges to $f(x)$ in L^2 , that is, $\int_{-\pi}^{\pi} |S_n f(x) - f(x)|^2 dx \rightarrow 0$ as $n \rightarrow \infty$. Carleson [6] showed that for square-integrable f , $S_n f(x) \rightarrow f(x)$ at every point except for a set of zero Lebesgue measure. Earlier results due to Dini-Lipschitz, Lebesgue and Dirichlet-Jordan give conditions for pointwise convergence (see e.g., Zygmund [17] for these). A typical result of this type is the theorem of Dirichlet-Jordan: If f is of bounded variation over $[-\pi, \pi]$, then $S_n f(x)$ converges to $f(x)$ at a point of continuity of f .

Given a numerical sequence $\{s_n\}$ which has limit s , we say a transformation s_n^* of s_n accelerates convergence if there exists a k such that each s_n^* depends only on s_0, \dots, s_{n+k} and s_n^* converges to s faster than s_n . Many sequence transformations have been developed to speed convergence of numerical sequences which arise in many contexts (see e.g. Brezinski and Redivo-Zaglia [5], Delahaye [7], Sidi [13] or Wimp [16]).

Consider a sequence $\{a_n\}$, $n = 0, 1, 2, 3, \dots$, and set $s_n = \sum_{k=0}^n a_k$. The δ^2 process (or δ^2 transform) takes the sequence s_n and transforms it to

$$e_1(s_n) = t_n := \frac{s_{n+1}s_{n-1} - s_n^2}{s_{n+1} + s_{n-1} - 2s_n} = s_n - \frac{(s_{n+1} - s_n)(s_n - s_{n-1})}{(s_{n+1} - s_n) - (s_n - s_{n-1})}, \quad (1)$$

where we set $t_n = s_n$ if the denominator of the fraction is zero.

This transformation is usually attributed to Aitken [2], although the idea had appeared earlier in the works of other authors. Generalizations of this transformation, which transform a sequence $\{s_n\}$ into a family

of sequences $\{e_k(s_n)\}$, were studied extensively by Shanks [12] and for this reason these transformations are sometimes called Shanks transformations.

Let $\rho_n = a_{n+1}/a_n$. The Lubkin W-transform of s_n is

$$s_n^* := s_n + \frac{a_{n+1}(1 - \rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n\rho_{n+1}}. \quad (2)$$

In the case when the s_n are the partial sums of a geometric series, both of these transforms produce a constant sequence, in which each term is the sum of the series. Shanks [12] shows an acceleration of convergence using (1) for partial sums of series which are, in a sense made precise in [12], “nearly geometric”. Consider the function f on the interval $[-\pi, \pi]$ given by $f(x) = 1$ if $0 \leq x \leq \pi$, $f(x) = -1$ if $-\pi \leq x < 0$. After computation and simplification, its Fourier series is:

$$f \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}. \quad (3)$$

At $x = \frac{\pi}{2}$ we obtain the slowly convergent Leibniz series: $1 = \frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$. Shanks [12] applies the transformation (1) and iterations of this transform to the sequence of partial sums and the results are dramatic: t_8 is accurate to three significant figures, and iterating the transform four times gives a sequence whose fifth term is correct to eight significant figures. By contrast, Shanks notes it would take over 40,000 summands in the original series to obtain this accuracy.

Smith and Ford [15] used numerical tests to compare different methods of convergence acceleration on the partial sums of Fourier series. Using a set of five points they tested slowly and rapidly converging Fourier series and showed some improvement of convergence in some cases. Drummond [8] discusses many methods of convergence acceleration and includes discussion of their application to Fourier series.

Although with specific series it is possible to get better approximations – even dramatically better results at some points – neither the δ^2 process nor Lubkin W-transformation behave well in general. Consider a function which is 2π periodic and C^2 except for a single jump discontinuity, at say, the endpoints $-\pi$ and π . The following lemma gives the order of decay of its Fourier coefficients.

Lemma 1. *Let $f \in C^2([-\pi, \pi])$. Then for every integer n ,*

$$\hat{f}(n) = \frac{(-1)^{n+1}}{2in} \alpha + \frac{(-1)^n}{2n^2} \beta - \frac{\widehat{f''}(n)}{n^2},$$

where $\alpha = [f(\pi) - f(-\pi)]/\pi$ and $\beta = [f'(\pi) - f'(-\pi)]/\pi$.

This is easily shown using integration by parts. Notice also that $\frac{\widehat{f''}(n)}{n^2} = o(\frac{1}{n^2})$ as $|n| \rightarrow \infty$ by the Riemann-Lebesgue lemma. (See, e.g. Zygmund [17], pg. 45.)

Thus, for functions which are well-behaved except for a single jump discontinuity, the Fourier coefficients decay like $O(\frac{1}{n})$, so such series converge very slowly, exactly like the example (3) given above. It would be particularly fruitful if the convergence of such series could be accelerated by application of either the δ^2 process or the Lubkin W-transform. Unfortunately, either of these work well at all, in fact, the results are quite bad. Not only is convergence not accelerated, it is completely destroyed.

Theorem 1. (Abebe, Graber, Moore [1]). *Suppose that $f \in C^2([-\pi, \pi])$ and that $f(-\pi) \neq f(\pi)$. Consider the sequence $S_n^* f(x)$ formed by applying the δ^2 process (1) to the sequence $S_n f(x)$. Then $S_n^* f(x)$ fails to converge to $f(x)$ at every x of the form $x = 2\pi a$, where $a \in (-\frac{1}{2}, \frac{1}{2})$ is irrational.*

Theorem 2. (Boggess, Bunch, Moore [4]). *Suppose that $f \in C^2([-\pi, \pi])$ and that $f(-\pi) \neq f(\pi)$. Consider the sequence $S_n^* f(x)$ formed by applying the Lubkin W-transform (2) to the sequence $S_n f(x)$. Then $S_n^* f(x)$ fails to converge to $f(x)$ at every x of the form $x = 2\pi a$, where $a \in (-\frac{1}{4}, \frac{1}{4})$ is irrational.*

Remarks. Both theorems consider f with a jump discontinuity at the endpoints of the interval. These results remain valid for any 2π periodic function f which is C^2 except for a single jump occurring in the interval $[-\pi, \pi]$.

At a jump discontinuity of f , the partial sums of the Fourier series exhibit the Gibbs phenomenon and many authors report difficulties with acceleration methods near such a discontinuity. These theorems show that an application of the δ^2 process to the partial sums causes difficulties just about everywhere, and that if one applies the Lubkin W-transform to the partial sums of a Fourier series with a jump discontinuity, then curiously, difficulties appear on a dense set away from the discontinuity.

We should mention a few results on convergence. Instead of the partial sums $S_n f$, consider separately the series $\frac{1}{2}\hat{f}(0) + \sum_{k=1}^{\infty} \hat{f}(k)e^{ikx}$ and $\frac{1}{2}\hat{f}(0) + \sum_{k=1}^{\infty} \hat{f}(-k)e^{-ikx}$. A result of Sidi [14] shows that if, say, f is smooth enough with a single jump discontinuity, then the Lubkin transform can be used to approximate these accurately, and consequently, an approximation of f is obtained upon adding these two approximations. Brezinski [3] used a similar idea in a study of the effect of the ε algorithm (similar to the Lubkin transform) on the Gibbs phenomenon. Given a Fourier series add to it its conjugate series as an imaginary part. Applying the ε algorithm to the resulting power series is then equivalent to the computation of Padé approximants and by taking real parts one obtains an approximation of the original function with the Gibbs phenomenon reduced.

2 The proofs of the Theorems

In this section we give a sketch of the proofs of the Theorems. Proofs can be found in [1] and [4], although the proofs we outline here are a little more efficient. If $S_n f(x)$ denotes the sequence of partial sums of the Fourier series of f , applying the δ^2 process (1) results in the sequence of functions:

$$S_n^* f(x) = S_n f(x) - \frac{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right) \left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)}{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right) - \left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)}. \quad (4)$$

Applying the Lubkin transform (2) results in the sequence of functions:

$$S_n^* f(x) = S_n f(x) + \frac{a_{n+1}(1 - \rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n \rho_{n+1}} \quad (5)$$

where

$$a_n = S_n f(x) - S_{n-1} f(x) = \hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx} \quad (6)$$

and

$$\rho_n = \frac{a_{n+1}}{a_n} = \frac{\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}}{\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}} \quad (7)$$

Since the functions we are considering have bounded variation, the Dirichlet-Jordan theorem applies so that $S_n f(x)$ converges to $f(x)$ at every point of the interval $(-\pi, \pi)$. Therefore, Theorem 1 will be shown if we show that the expression

$$\frac{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right) \left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)}{\left(\hat{f}(-(n+1))e^{-i(n+1)x} + \hat{f}(n+1)e^{i(n+1)x}\right) - \left(\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx}\right)} \quad (8)$$

has a subsequence which stays away from 0 at all x of the form $x = 2\pi a$, $a \in (-\frac{1}{2}, \frac{1}{2})$, a irrational. Likewise, to show Theorem 2, it suffices to show that the expression (with a_n as in (6) and ρ_n as in (7))

$$\frac{a_{n+1}(1 - \rho_{n+1})}{1 - 2\rho_{n+1} + \rho_n \rho_{n+1}} \quad (9)$$

has a subsequence which stays away from 0 at all x of the form $x = 2\pi a$, $a \in (-\frac{1}{4}, \frac{1}{4})$, a irrational.

Lemma 1 gives

$$\begin{aligned}
a_n &= \hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx} \\
&= \left(\frac{(-1)^{n+1}}{2in} \alpha + \frac{(-1)^n}{2n^2} \beta - \frac{\widehat{f}''(n)}{n^2} \right) e^{inx} + \left(-\frac{(-1)^{(-n+1)}}{2in} \alpha + \frac{(-1)^{-n}}{2n^2} \beta - \frac{\widehat{f}''(-n)}{n^2} \right) e^{-inx} \\
&= \frac{\alpha(-1)^{n+1}}{n} \sin(nx) + \frac{\beta(-1)^n}{n^2} \cos(nx) - \frac{\widehat{f}''(-n)}{n^2} e^{-inx} - \frac{\widehat{f}''(n)}{n^2} e^{inx} \\
&= \frac{\alpha(-1)^{n+1}}{n} \sin(nx) + \varepsilon_n
\end{aligned} \tag{10}$$

where $\varepsilon_n = \frac{\beta(-1)^n}{n^2} \cos(nx) - \frac{\widehat{f}''(-n)}{n^2} e^{-inx} - \frac{\widehat{f}''(n)}{n^2} e^{inx}$ satisfies $|\varepsilon_n| = O(\frac{1}{n^2})$.

Using this notation, we may write the fraction (8) from the right hand side of (4) as:

$$\begin{aligned}
&\frac{\left(\frac{(-1)^{n+2}}{n+1} \alpha \sin(n+1)x + \varepsilon_{n+1} \right) \left(\frac{(-1)^{n+1}}{n} \alpha \sin nx + \varepsilon_n \right)}{\left(\frac{(-1)^{n+2}}{n+1} \alpha \sin(n+1)x + \varepsilon_{n+1} \right) - \left(\frac{(-1)^{n+1}}{n} \alpha \sin nx + \varepsilon_n \right)} \\
&= \frac{(-1)^n \left(\alpha \sin(n+1)x + (-1)^n(n+1)\varepsilon_{n+1} \right) \left(-\alpha \sin nx + (-1)^n n \varepsilon_n \right)}{\alpha \left(n \sin(n+1)x + (n+1) \sin nx \right) + n(n+1)(-1)^n \left(\varepsilon_{n+1} - \varepsilon_n \right)}.
\end{aligned} \tag{11}$$

We temporarily set, for typographical convenience, $s(n) = \frac{(-1)^{n+1}}{n} \alpha \sin nx$. Then the fraction (9) from the right hand side of (5) can then be rewritten using (10) and simplified to become:

$$\frac{n(n+1)(s(n) + \varepsilon_n)(s(n+1) + \varepsilon_{n+1}) \left(s(n+1) + \varepsilon_{n+1} - s(n+2) - \varepsilon_{n+2} \right)}{n(n+1) \left((s(n) + \varepsilon_n)(s(n+1) + \varepsilon_{n+1}) - 2(s(n) + \varepsilon_n)(s(n+2) + \varepsilon_{n+2}) + (s(n+1) + \varepsilon_{n+1})(s(n+2) + \varepsilon_{n+2}) \right)} \tag{12}$$

To finish the proof of Theorem 1, use the identities $\sin nx = \sin(n+1/2)x \cos \frac{x}{2} - \sin \frac{x}{2} \cos(n+1/2)x$ and $\sin(n+1)x = \sin(n+1/2)x \cos \frac{x}{2} + \sin \frac{x}{2} \cos(n+1/2)x$ to rewrite (11) as

$$(-1)^n \frac{\alpha^2 \left(\sin^2 \frac{x}{2} \cos^2(n+\frac{1}{2})x - \sin^2(n+\frac{1}{2})x \cos^2 \frac{x}{2} \right) + (-1)^n \alpha \left(n \varepsilon_n \sin(n+1)x - (n+1)\varepsilon_{n+1} \sin nx \right) + \varepsilon_n \varepsilon_{n+1}}{\alpha \left(\sin nx + 2n \cos \frac{x}{2} \sin(n+\frac{1}{2})x \right) + n(n+1)(-1)^n (\varepsilon_{n+1} - \varepsilon_n)} \tag{13}$$

We claim: Given $x = 2\pi a$, a irrational, there exists integers $m_n \rightarrow \infty$ such that $|\sin(m_n + 1/2)x| < \frac{\pi}{2m_n}$. Assuming this, then also $\sin^2(m_n + 1/2)x \rightarrow 0$ and hence $\cos^2(m_n + 1/2)x \rightarrow 1$ as $m_n \rightarrow \infty$. Recalling that $\varepsilon_n = O(\frac{1}{n^2})$, we notice that along the sequence m_n the denominator of (13) is bounded,

$$\begin{aligned}
&\left| \alpha \left(\sin m_n x + 2m_n \cos \frac{x}{2} \sin(m_n + \frac{1}{2})x \right) + m_n(m_n + 1)(-1)^{m_n} (\varepsilon_{m_n+1} - \varepsilon_{m_n}) \right| \\
&\leq C + |2\alpha m_n \sin(m_n + \frac{1}{2})x| < C + \alpha\pi,
\end{aligned}$$

and that as $n \rightarrow \infty$ along the sequence m_n , the numerator of (13) is asymptotically $(-1)^{m_n} \alpha^2 \sin^2 \frac{x}{2}$. Thus, along the subsequence m_n , the expression (13) fails to converge to 0, and thus $S_n^* f$ fails to converge to f .

It remains to substantiate the claim. This follows from the following lemma whose proof is similar to the proof of Theorem 7.11 in Niven and Zuckerman [11].

Lemma 2. *Let a be an irrational number. Then there exist infinitely many rational numbers $\frac{h}{k}$ with k odd such that $\left| a - \frac{h}{k} \right| < \frac{1}{k^2}$.*

The lemma then shows the claim: Let $\{m_n\}$ and $\{l_n\}$ be sequences of integers such that $\{m_n\}$ is strictly increasing, each m_n is positive, and, for every n , $\left|a - \frac{l_n}{2m_n+1}\right| < \frac{1}{(2m_n+1)^2} < \frac{1}{4m_n^2+2m_n}$. Thus, for each n ,

$$\begin{aligned} \left|\sin\left(m_n + \frac{1}{2}\right)x\right| &= \left|\sin\left(\left(m_n + \frac{1}{2}\right)x - l_n\pi\right)\right| \leq \left|\left(m_n + \frac{1}{2}\right)x - l_n\pi\right| \\ &= \left|(2m_n + 1)\pi a - l_n\pi\right| = \left|(2m_n + 1)\pi\right| \left|a - \frac{l_n}{2m_n+1}\right| < (2m_n + 1)\pi \frac{1}{4m_n^2 + 2m_n} = \frac{\pi}{2m_n} \end{aligned}$$

which establishes the claim and completes the proof of Theorem 1.

The end of the proof of Theorem 2 is similar but a little more technical so we merely provide a sketch of the proof. The details can be found in [4]. We examine both the numerator and denominator of (12). Because $\varepsilon_n = O\left(\frac{1}{n^2}\right)$, the denominator behaves asymptotically like

$$n(n+1) \left(s(n)s(n+1) - 2s(n)s(n+2) + s(n+1)s(n+2) \right) + O\left(\frac{1}{n}\right).$$

Recalling that $s(n) = \frac{(-1)^{n+1}}{n} \alpha \sin nx$ and simplifying, this becomes:

$$-\alpha^2 \left(\sin nx \sin((n+1)x) + 2\frac{n+1}{n+2} \sin nx \sin((n+2)x) + \frac{n}{n+2} \sin((n+1)x) \sin((n+2)x) \right) + O\left(\frac{1}{n}\right).$$

Lemma 3. *Given $x = 2\pi a$, $a \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, a irrational, there exists an infinite number of n_k such that*

$$\left| \sin n_k x \sin((n_k+1)x) + 2\frac{n_k+1}{n_k+2} \sin n_k x \sin((n_k+2)x) + \frac{n_k}{n_k+2} \sin((n_k+1)x) \sin((n_k+2)x) \right| < \frac{4+48\pi}{n_k}$$

Again, as in the case of Theorem 1, this reduces to a fact from number theory on the approximation of irrationals by rationals. In this case the relevant Theorem is an approximation due to Chebyshev (see A. Ya. Khinchin [10], Theorem 24, p. 39).

Thus, for x as in the hypotheses of Theorem 2, along the sequence n_k , the denominator of (12) is bounded by $\frac{c}{n_k}$.

Because $\varepsilon_n = O\left(\frac{1}{n^2}\right)$, along the sequence n_k the numerator of (12) behaves asymptotically like

$$n_k(n_k+1)s(n_k)s(n_k+1) \left(s(n_k+1) - s(n_k+2) \right) + O\left(\frac{1}{n_k^2}\right)$$

which, after using the definition of $s(n)$ and simplifying becomes:

$$\alpha^3 (-1)^{n_k+1} \sin((n_k)x) \sin((n_k+1)x) \left(\frac{\sin((n_k+1)x)}{n_k+1} + \frac{\sin((n_k+2)x)}{n_k+2} \right) + O\left(\frac{1}{n_k^2}\right). \quad (14)$$

The following lemma will then give the desired estimate of the numerator. Its proof involves some trigonometric identities.

Lemma 4. *For x and $\{n_k\}$ as in Lemma 3, the expression*

$$\alpha^3 (-1)^{n_k+1} \sin((n_k)x) \sin((n_k+1)x) \left(\frac{\sin((n_k+1)x)}{n_k+1} + \frac{\sin((n_k+2)x)}{n_k+2} \right)$$

is bounded below, for n_k large, by $\frac{c}{n_k}$ for some constant c (which depends on x but not n_k).

From this and (14), we conclude that along the subsequence n_k the numerator of (12) is bounded below by $\frac{c}{n_k}$ for n_k large. The estimates for the numerator and denominator of (12) then give a subsequence along which the fraction (9) appearing on the right hand side of (5) fails to converge to 0 and consequently shows Theorem 2.

3 Conditions under which sequence acceleration fails

Ideally, we would like to be able to determine necessary and sufficient conditions which would insure that an application of the δ^2 process or Lubkin transform does not destroy convergence. Even better, we would like to know necessary and sufficient conditions which would insure these transforms actually accelerate convergence. In this section we will make little progress on these questions; the results we give will show just how difficult these questions are. We will focus on a discussion of the δ^2 process. Similar analysis could be given for the Lubkin transform, but we will not discuss this.

For functions with a single jump discontinuity, we have seen that the Fourier coefficients decay like $\frac{1}{|n|}$ as n goes to infinity or negative infinity. This might lead to the conjecture that the rate of decay of the Fourier coefficients plays a role. This isn't necessarily true.

Proposition 1. *Let $s_n(x) = \sum_{k=1}^n a_k e^{ikx}$, $n = 1, 2, \dots$ be the partial sums of the series $\sum_{k=1}^{\infty} a_k e^{ikx}$ where the a_k are real, $a_k \neq 0$ for every k , and $a_k \rightarrow 0$ as $k \rightarrow \infty$. Let $t_n(x)$ be the sequence of functions obtained by applying the δ^2 process (1) to the sequence $s_n(x)$. Then at any $x \in [-\pi, \pi]$, $x \neq 0, -\pi, \pi$ at which $\lim_{n \rightarrow \infty} s_n(x)$ exists, then so does $\lim_{n \rightarrow \infty} t_n(x)$ and the two limits are equal.*

Remarks. 1. In particular, if the $s_n(x)$ are the partial sums of the Fourier series of an $f \in L^1[-\pi, \pi]$ then by the Riemann-Lebesgue lemma $a_n = \hat{f}(n) \rightarrow 0$ as $n \rightarrow \infty$. Thus, at points where the Fourier series of f converges (except possibly at $0, -\pi, \pi$) the transformed partial sums will converge to the same sum. For any $f \in L^p[-\pi, \pi]$, $p > 1$, the Carleson-Hunt theorem shows that $s_n(x) \rightarrow f(x)$ a.e. and hence at each point of convergence (except possibly $0, -\pi, \pi$), the transformed partial sums will converge also to $f(x)$. However, note that here we require that the $a_n = \hat{f}(n)$ are real so this is a very restricted class of functions.

2. The conclusion of the proposition is only that in this case an application of the δ^2 process does not destroy convergence. We do not know whether or not convergence is accelerated.

Proof. Explicitly

$$t_n(x) = s_n(x) - \frac{a_{n+1}e^{i(n+1)x}a_n e^{inx}}{a_{n+1}e^{i(n+1)x} - a_n e^{inx}}$$

The limit of $t_n(x)$ is the same as the limit of $s_n(x)$ if and only if the fraction on the right hand side of this equation goes to 0 as $n \rightarrow \infty$. We have

$$\left| \frac{a_{n+1}e^{i(n+1)x}a_n e^{inx}}{a_{n+1}e^{i(n+1)x} - a_n e^{inx}} \right| = \frac{|a_{n+1}||a_n|}{|a_{n+1}e^{ix} - a_n|} = \frac{|a_{n+1}||a_n|}{|a_{n+1} \cos x - a_n + ia_{n+1} \sin x|} \leq \frac{|a_{n+1}||a_n|}{|a_{n+1}| |\sin x|} = \frac{|a_n|}{|\sin x|}.$$

If $x \neq 0, -\pi, \pi$ this last expression tends to 0 as $n \rightarrow \infty$. □

We conclude that it doesn't seem to be the decay of the coefficients which determines if an application of the δ^2 process will destroy convergence. Recall that Theorem 1 applies to a function with a single jump discontinuity. Is it the lack of continuity which leads to the problems with the application of the δ^2 process? As we will see, the answer is no. We will see that things can go bad even with continuous functions. We introduce the Hardy-Littlewood series

$$\sum_{n=1}^{\infty} e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2} + \alpha}}$$

where we suppose that α is real and c is positive. This was studied by Hardy and Littlewood [9]; see also Zygmund [17], Chapter 5 for a discussion of this series. The fact we need is that if $\alpha > 0$ then the partial sums of the series are uniformly convergent. Hence, in this case, the series is the Fourier series of a continuous function.

Theorem 3. Suppose $c > 0$, $0 < \alpha \leq \frac{1}{2}$. Consider the partial sums

$$s_n(x) = \sum_{k=1}^n e^{ick \log k} \frac{e^{ikx}}{k^{\frac{1}{2}+\alpha}}$$

of the Hardy-Littlewood series and let $t_n(x)$ be the sequence of functions which results from an application of the δ^2 process (1) to the sequence $s_n(x)$. Then at every x , $t_n(x)$ fails to converge to the same limit as $s_n(x)$. In fact, if $0 < \alpha < \frac{1}{2}$, the sequence $t_n(x)$ has subsequences which become unbounded.

Proof.

$$t_n(x) = s_n(x) - \frac{e^{ic(n+1) \log(n+1)} \frac{e^{i(n+1)x}}{(n+1)^{\frac{1}{2}+\alpha}} e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2}+\alpha}}}{e^{ic(n+1) \log(n+1)} \frac{e^{i(n+1)x}}{(n+1)^{\frac{1}{2}+\alpha}} - e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2}+\alpha}}}$$

As in previous proofs, the conclusion will follow by exhibiting the bad behavior (as $n \rightarrow \infty$) of the fraction on the right. Taking absolute value and simplifying we obtain

$$\left| \frac{e^{ic(n+1) \log(n+1)} \frac{e^{i(n+1)x}}{(n+1)^{\frac{1}{2}+\alpha}} e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2}+\alpha}}}{e^{ic(n+1) \log(n+1)} \frac{e^{i(n+1)x}}{(n+1)^{\frac{1}{2}+\alpha}} - e^{icn \log n} \frac{e^{inx}}{n^{\frac{1}{2}+\alpha}}} \right| = \frac{1}{\left| n^{\frac{1}{2}+\alpha} e^{ic(n+1) \log(n+1)} e^{ix} - (n+1)^{\frac{1}{2}+\alpha} e^{icn \log n} \right|} \quad (15)$$

We estimate the denominator of this expression.

$$\begin{aligned} \left| n^{\frac{1}{2}+\alpha} e^{ic(n+1) \log(n+1)} e^{ix} - (n+1)^{\frac{1}{2}+\alpha} e^{icn \log n} \right| &\leq \\ \left| n^{\frac{1}{2}+\alpha} e^{ic(n+1) \log(n+1)} e^{ix} - n^{\frac{1}{2}+\alpha} e^{icn \log n} \right| &+ \left| n^{\frac{1}{2}+\alpha} e^{icn \log n} - (n+1)^{\frac{1}{2}+\alpha} e^{icn \log n} \right| = I + II \end{aligned}$$

each of which we estimate separately.

By the mean value theorem, there is an n^* between n and $n+1$ so that

$$II = \left| n^{\frac{1}{2}+\alpha} - (n+1)^{\frac{1}{2}+\alpha} \right| = \left(\frac{1}{2} + \alpha \right) (n^*)^{-\frac{1}{2}+\alpha} \leq \left(\frac{1}{2} + \alpha \right) \frac{n^\alpha}{\sqrt{n}}$$

Let k be a positive integer. Then

$$I = n^{\frac{1}{2}+\alpha} \left| e^{ic(n+1) \log(n+1)+ix} - e^{icn \log n} \right| = n^{\frac{1}{2}+\alpha} \left| e^{ic(n+1) \log(n+1)+ix-2k\pi i} - e^{icn \log n} \right|$$

Using the fact that $|e^{ia} - e^{ib}| \leq |a - b|$ for all real numbers a and b , we estimate that for any positive integer k ,

$$\begin{aligned} I &\leq n^{\frac{1}{2}+\alpha} \left| c(n+1) \log(n+1) + x - 2k\pi - cn \log n \right| = n^{\frac{1}{2}+\alpha} \left| c \log\left(1 + \frac{1}{n}\right) + c \log(n+1) + x - 2k\pi \right| \\ &\leq n^{\frac{1}{2}+\alpha} c \left| n \log\left(1 + \frac{1}{n}\right) - 1 \right| + n^{\frac{1}{2}+\alpha} \left| c \log(n+1) + x + c - 2k\pi \right| = I_1 + I_2 \end{aligned}$$

To estimate I_1 we will use the fact that $\left| \frac{1}{\eta} \log(1 + \eta) - 1 \right| \leq |\eta|$ for $|\eta| \leq \frac{1}{2}$. This is easily shown using Taylor series. Using this estimate with $\eta = \frac{1}{n}$ yields

$$I_1 = n^{\frac{1}{2}+\alpha} c \left| n \log\left(1 + \frac{1}{n}\right) - 1 \right| \leq c \frac{n^\alpha}{\sqrt{n}}.$$

We now examine I_2 . Fix a positive integer k , assume k is large. Then, given $x \in [-\pi, \pi]$, there exists an integer n_k such that

$$c \log n_k + x + c < 2k\pi \quad \text{and} \quad c \log(n_k + 1) + x + c \geq 2k\pi.$$

But then

$$|c \log(n_k + 1) + x + c - 2k\pi| \leq c \log(n_k + 1) + x + c - (c \log n_k + x + c) = c \log\left(1 + \frac{1}{n_k}\right) \leq \frac{c}{n_k}.$$

Here we have used the fact the $\log(1 + \eta) \leq \eta$ for $\eta \geq 0$. Consequently, for each integer k sufficiently large, we can find an integer n_k such that

$$I_2 = n_k^{\frac{1}{2} + \alpha} |c \log(n_k + 1) + x + c - 2k\pi| \leq c \frac{n_k^\alpha}{\sqrt{n_k}}.$$

Combining our estimates for I_1 , I_2 and II we conclude that given x , there exists a subsequence n_k (which depends on x) such that

$$\left| n_k^{\frac{1}{2} + \alpha} e^{ic(n_k+1) \log(n_k+1)} e^{ix} - (n_k + 1)^{\frac{1}{2} + \alpha} e^{icn_k \log n_k} \right| \leq C \frac{n_k^\alpha}{\sqrt{n_k}}$$

for an absolute constant C . If $0 < \alpha < \frac{1}{2}$, we conclude that for each x there exists a subsequence n_k (depending on x) along which the denominator of (15) goes to 0. Thus, along this subsequence $|t_{n_k}(x)| \rightarrow \infty$. If $\alpha = \frac{1}{2}$ we conclude that there is a subsequence n_k (depending on x) at which the denominator of (15) remains bounded by an absolute constant C . Then, along this subsequence $|t_{n_k}(x) - s_{n_k}(x)| \geq \frac{1}{C}$. \square

Remarks. Consider the real and imaginary parts of the partial sums of the Hardy-Littlewood series

$$\sum_{k=1}^n \frac{\cos(ck \log k + kx)}{k^{\frac{1}{2} + \alpha}} \quad \text{and} \quad \sum_{k=1}^n \frac{\sin(ck \log k + kx)}{k^{\frac{1}{2} + \alpha}} \quad (16)$$

Because the δ^2 process is a nonlinear transformation, we do not expect the real and imaginary parts of the transformed complex Hardy-Littlewood series to be the transformed real and imaginary parts of the complex Hardy-Littlewood series. However, we can say something about the transformations of the two series (16). Consider the transformation of each of these series. To analyze the behavior of the transformed cosine series involves examining the fractions

$$\begin{aligned} & \left| \frac{\frac{\cos(c(n+1) \log(n+1) + (n+1)x)}{(n+1)^{\frac{1}{2} + \alpha}} \frac{\cos(cn \log n + nx)}{n^{\frac{1}{2} + \alpha}}}{\frac{\cos(c(n+1) \log(n+1) + (n+1)x)}{(n+1)^{\frac{1}{2} + \alpha}} - \frac{\cos(cn \log n + nx)}{n^{\frac{1}{2} + \alpha}}} \right| \\ &= \frac{|\cos(c(n+1) \log(n+1) + (n+1)x) \cos(cn \log n + nx)|}{|n^{\frac{1}{2} + \alpha} \cos(c(n+1) \log(n+1) + (n+1)x) - (n+1)^{\frac{1}{2} + \alpha} \cos(cn \log n + nx)|}. \end{aligned}$$

Similarly, the behavior of the transformed sine series in (16) is analyzed using the same expression with cosine replaced by sine. Notice that the denominator of this last expression is less than the denominator of the expression in (15). Fix an x . Then corresponding to this x there is a sequence n_k along which the denominator is bounded by $c \frac{n_k^\alpha}{\sqrt{n_k}}$. This is also true for the same expression when cosine is replaced by sine.

Furthermore, by estimating as in the estimate of I in the proof above, we have that

$$|\cos(c(n_k + 1) \log(n_k + 1) + (n_k + 1)x) - \cos(cn_k \log n_k + n_k x)| \leq \frac{2c}{n_k},$$

with a similar estimate involving sine. Thus, for large n_k , $\cos(c(n_k + 1) \log(n_k + 1) + (n_k + 1)x) \approx \cos(cn_k \log n_k + n_k x)$ (and similarly for sine) so that for large n_k either

$$\begin{aligned} & |\cos(c(n_k + 1) \log(n_k + 1) + (n_k + 1)x) \cos(cn_k \log n_k + n_k x)| > \frac{1}{10} \quad \text{or} \\ & |\sin(c(n_k + 1) \log(n_k + 1) + (n_k + 1)x) \sin(cn_k \log n_k + n_k x)| > \frac{1}{10} \end{aligned}$$

(or both) hold. We conclude that given x , there is a sequence n_k along which either the transformed partial sums of the cosine series or the transformed partial sums of the sine series fail to converge (or both). Thus, for either the cosine series or sine series in (16) the transformed partial sums fail to converge on a set of positive measure. We suspect that both probably fail to converge at all x . In any case, we have an example of a real sine or cosine series whose partial sums converge to a continuous function, such that the transformed partial sums do not converge for a large set of x .

4 Remarks and further results

The transformation (4) also fails to produce convergence at points of the form $x = \frac{2\pi j}{k}$ where $\frac{j}{k}$ is in lowest terms and k is odd. See [1] for details. In both Theorems 1 and 2 it is quite likely that convergence fails at other x also but we have not been able to show this. It is possible to apply some of these methods to iterations of these transforms, to functions with more than one jump, and to some other slowly converging series. See [1] and [4] for these results.

Consider the function $f(x) = x$. Figure 1 shows the partial sums of its Fourier series, $S_n f(x)$, for $n = 25$ and 100 on the interval $[0, \pi]$. (This tells the whole story; f as well as all $S_n f$ are odd functions.) The transformed partial sums $S_n^* f(x)$ using the δ^2 process are shown for the same values of n in Figure 2. The graphs show the difficulties which occur at a dense set of values in $(-\pi, \pi)$. Figure 3 gives the transformed sums using the Lubkin W-transform. The graphs show the difficulties which occur at a dense set of values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Figure 4 gives both the real and imaginary parts of the 50th partial sum of the Hardy-Littlewood series with $c = 1$ and $\alpha = .25$, that is, Figure 4 gives the real and imaginary parts of $\sum_{n=1}^{50} e^{in \log n} \frac{e^{inx}}{n^{.75}}$. Applying the

δ^2 process to this results in a complex valued function on $[-\pi, \pi]$ whose real and imaginary parts are shown in Figure 5. To prove Theorem 3, we showed that given an x the transformed n th partial sum differs from the n th partial sum for values of $n = n_k$ which satisfy $c \log n_k + x + c < 2k\pi$ and $c \log(n_k + 1) + x + c > 2k\pi$ for some positive integer k . Since the sequence $\log n$ grows very slowly, for a given x , this doesn't happen very often, and so it's not surprising that the graphs in Figure 5 contain so few spikes. Examination of both graphs in Figure 5, and a careful examination of the data used to create Figure 5 reveal a noticeable spike in the data at about 1.37. We note that $\log(50) + 1.37 + 1 \approx 6.28 < 2\pi$ but $\log(51) + 1.37 + 1 \approx 6.30 > 2\pi$. At the point $x = 1.37$ it's not until $n = 26805$ that $\log(26805) + 1.37 + 1 < 4\pi$ but $\log(26806) + 1.37 + 1 > 4\pi$.

It is likely that the methods in this paper could be applied to show that other sequence acceleration transforms may give unreliable results when applied to Fourier series. We suspect that similar difficulties can be proven to exist for other transforms. Currently, each transform considered seems to require a different analysis so it would also be of interest to develop more general methods for these type of theorems. These are topics for future study.

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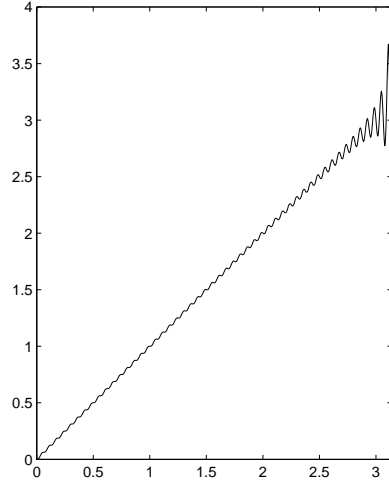
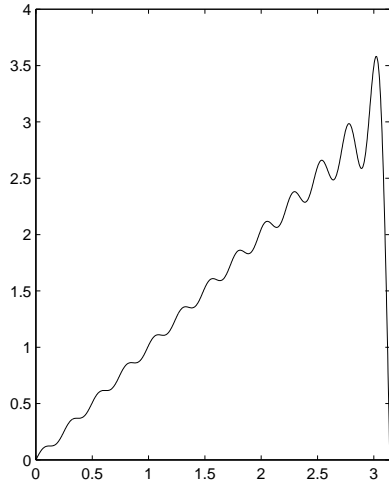


Figure 1: Partial sums $S_n f$, where $f(x) = x$, for $n = 25$ and $n = 100$

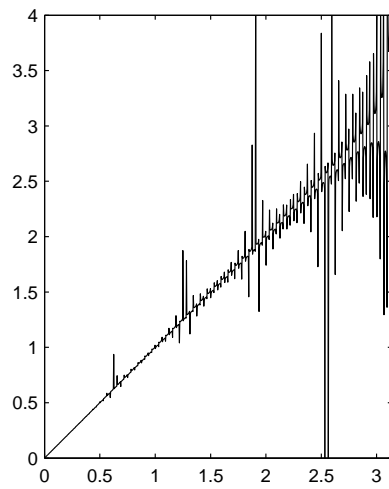
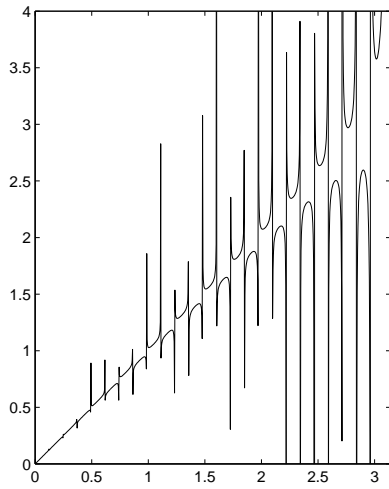


Figure 2: The δ^2 process applied to partial sums $S_n f$ for $n = 25$ and $n = 100$

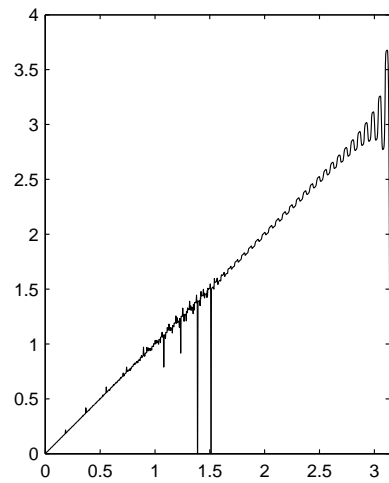
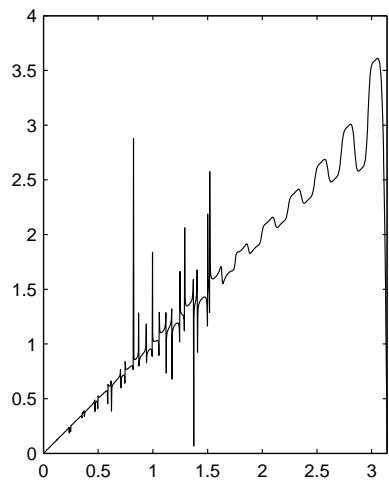


Figure 3: The Lubkin W-transform applied to partial sums $S_n f$ for $n = 25$ and $n = 100$

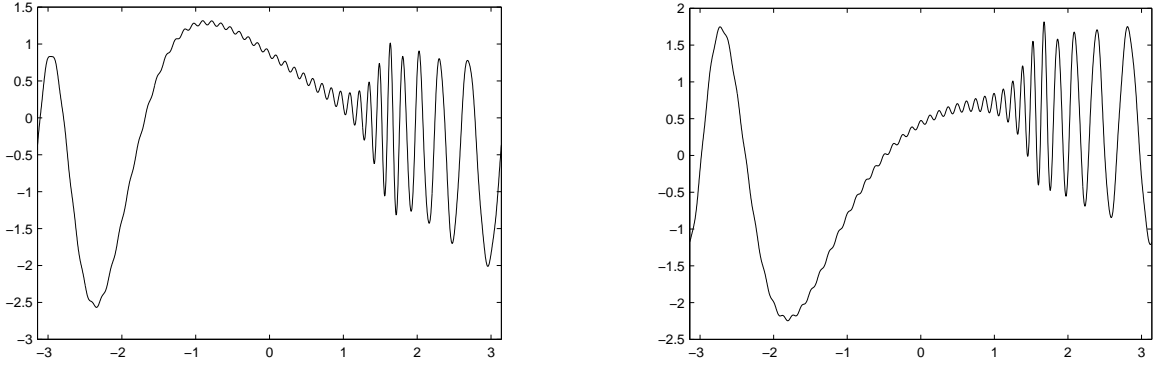


Figure 4: The real and imaginary parts of the 50th partial sum of the Hardy-Littlewood series

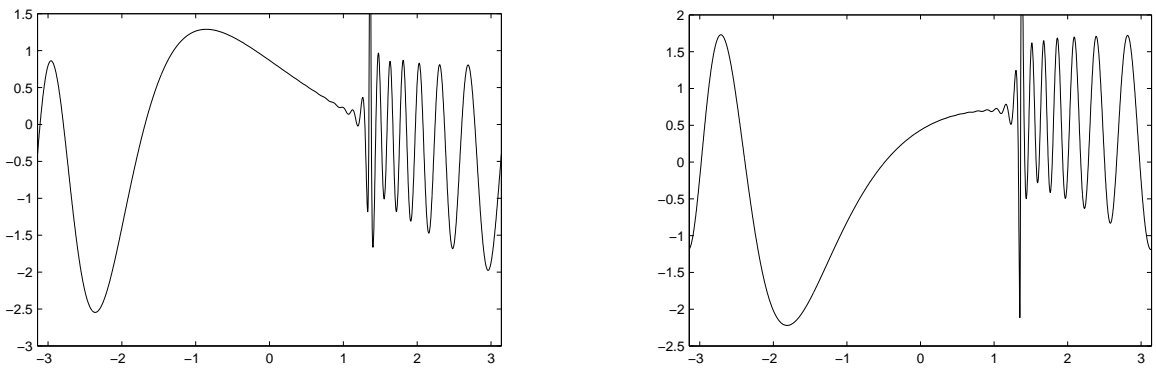


Figure 5: δ^2 applied to 50th partial sum of the Hardy-Littlewood series: real and imaginary parts