A new regularity approach for weak solutions of degenerate parabolic equations

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Outline I

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Introduction

Let $E \subset \mathbb{R}^N$ be an open set and let E_T denote $E \times (0, T]$.

We are interested in solutions to quasi–linear degenerate parabolic equations of the type

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, Du) = 0 \quad \text{weakly in } \; E_T$$

where the function $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is only assumed to be measurable and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, Du) \cdot Du \geq C_o |Du|^p \\ |\mathbf{A}(x, t, u, Du)| \leq C_1 |Du|^{p-1} \end{cases} \qquad \text{a.e. in } \; E_T$$

where $p \in (1, \infty)$ and C_o and C_1 are given positive constants.

We could also consider the more general case

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,Du) = B(x,t,u,Du) \quad \text{weakly in } E_T$$

where the functions $\mathbf{A} : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B : E_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are only assumed to be measurable and subject to the structure conditions

$$\left\{ \begin{array}{l} \mathbf{A}(x,t,u,Du) \cdot Du \geq C_o |Du|^p - C^p \\ |\mathbf{A}(x,t,u,Du)| \leq C_1 |Du|^{p-1} + C^{p-1} \\ |B(x,t,u,Du)| \leq C |Du|^{p-1} + C^p \end{array} \right. \qquad \text{a.e. in } E_T \tag{1}$$

where $p \in (1, \infty)$ and C_o and C_1 are given positive constants, and C is a given non–negative constant.

However ... let us keep it simple here.

The Prototype

The model case we have in mind is the so called *parabolic p–Laplacian*

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0,$$

but we want to rely only on the previous structural hypotheses, and not on special properties of the prototype equation.

The Modulus of Ellipticity and the Range for p

Notice that when $p > 2$ and $|Du| \to 0$, the modulus of ellipticity $|Du|^{p-2} \to 0$ and the equation becomes degenerate. In other words, the time evolution dominates over the diffusion process, and the diffusion becomes slow.

Notice that when $1 < p < 2$ and $|Du| \to 0$, the modulus of ellipticity $|Du|^{p-2} \to +\infty$ and the equation tends to favour its elliptic component. In other words, the diffusion process dominates over the time evolution, and the diffusion becomes fast.

The fast diffusion case is more complicated than the slow diffusion one and the behaviour of the solution differs dramatically if we have

- either $\frac{2N}{N+1} < p < 2$: *super–critical range*.
- or $1 < p \leq \frac{2N}{N+1}$: *sub–critical range*.

There are three different approaches to prove Hölder regularity for solutions of elliptic and parabolic equations with L^∞ coefficients. and are respectively based on DeGiorgi's approach:

E. DeGiorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari,*Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat., 1957*

on Nash's approach:

J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math., 1958*

and on Moser's approach:

J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math., 1961*

Nash's approach

He proved that if u is a weak solution of the parabolic equation

$$u_t = D_i(a_{ij}(x,t)D_j u)$$

with a_{ij} assumed to be only measurable, bounded and uniformly elliptic, then u is Hölder continuous.

To prove such a result, he used a method based on deep estimates on the behaviour of the fundamental solutions of parabolic equations. This approach is indeed widely used, but up to now only in the context of linear operators.

De Giorgi's approach

The original result was proved only for elliptic equations. The extension to parabolic equations is due to

O.A. Ladyzhenskaja, V.A. Solonnikov and N.N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, AMS Transl. Math. Mono., 1968.

The case of the p-laplacian ($p > 2$, degenerate case) was faced in the eighties by several authors:

E. DiBenedetto, Regularity results for the porous media equation, *Ann. Mat. Pura Appl., 1979*

D. G. Aronson and P. Bénilan, Régularité des solutions de l'équation des milieux poreux dans $\mathbb{R}^N$, *C. R. Acad. Sci. Paris Sér., 1979*

L. A. Caffarelli and A. Friedman, Continuity of the density of a gas flow in a porous medium, *Trans. Amer. Math. Soc., 1979*

W. P. Ziemer, Interior and boundary continuity of weak solutions of degenerate parabolic equations, *Trans. Amer. Math. Soc., 1982*

E. DiBenedetto, Local regularity of weak solutions of degenerate elliptic equations, *Non-Linear Anal. TMA, 1983*

E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, *Indiana Univ. Math. J., 1983*

E. DiBenedetto, On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1986*

The case of the singular p–laplacian ($1 < p < 2$) was faced and solved in

Y.Z. Chen and E. DiBenedetto, Hölder estimates of solutions of singular parabolic equations with measurable coefficients, *Arch. Rational Mech. Anal., 1992.*

The Hölder continuity of solutions is a consequence of the by–now classical method of the reduction of oscillation in a set of nested and shrinking cylinders.
Let $Q_R(x_o, t_o)$ be a parabolic cylinder, namely

$$Q_R(x_o, t_o) \equiv [(x_o, t_o) + B_R \times \{-R^2, 0\}] \subset E_T,$$

and denote with

$$\omega_R = \operatorname*{ess\,osc}_{Q_R} u.$$

The Hölder continuity will follow if one proves that there exists $\eta \in (0, 1)$ such that, for any R and for any cylinder $Q_R \subset E_T$

$$\omega_{\frac{R}{2}} \leq (1 - \eta)\omega_R.$$

The DeGiorgi's method is based on the following energy estimates (or Caccioppoli estimates).

$$\sup_{t_o-(\sigma\rho)^2\le t\le t_o} \|(u-k)_\pm(\cdot,t)\|^2_{2,B_{\sigma\rho}} + \|\nabla(u-k)_\pm\|^2_{2,Q_{\sigma\rho}} \le \frac{\gamma}{(1-\sigma)^2\rho^2}\|(u-k)_\pm\|^2_{2,Q_\rho} + \frac{\gamma}{(1-\sigma)\rho^2}\|(u-k)_\pm\|^2_{2,Q_\rho}.$$

If a function *u* satisfies a Caccioppoli estimates (for any *k*, for any $\rho > 0$ and for any point of the domain) then is Hölder continuous.
Notice that the fact that the function *u* solves an equation is used only at the first and preliminary step, when one shows that a weak solution of a parabolic equation satisfies the previous Caccioppoli estimates.

In the case of the p–laplacian (with p different from 2) the Caccioppoli estimates are

$$\sup_{t_o-(\sigma\rho)^p\omega^{p-2}\le t\le t_o}\|(u-k)_\pm(\cdot,t)\|^2_{2,B_{\sigma\rho}}+\|\nabla(u-k)_\pm\|^p_{p,Q_{\sigma\rho}}$$
$$\le\frac{\gamma}{(1-\sigma)^p\rho^p}\|(u-k)_\pm\|^p_{p,Q_\rho}+\frac{\gamma\omega^{p-2}}{(1-\sigma)\rho^p}\|(u-k)_\pm\|^2_{2,Q_\rho},$$

but in this case, the cylinder Q_ρ is an intrinsic cylinder, namely

$$Q_\rho=B_\rho(x_o)\times\{t_o-\rho^p\omega^{2-p},t_o\}\subset E_T.$$

This seems contradictory.

Let us be more precise: the goal is the reduction of the oscillation.
If the nested cylinder is not contained in the first one, then it means that the oscillation has been reduced.
Otherwise, the geometry we built is OK.

In this case, in order to face the lack of homogeneity of the estimate, it is necessary to operate in an intrinsic geometry. This regularity approach requires an argument based on several alternatives. For a detailed description of this method see

E. DiBenedetto, *Degenerate Parabolic Equations,* Springer Verlag, Series Universitext, New York, 1993

J.M. Urbano, *The Method of Intrinsic Scaling,* Lecture Notes in Mathematics 1930, Heidelberg, 2008

Moser's approach

In his fundamental paper he proved that if u is a non–negative solution of the elliptic equation

$$D_i(a_{ij}(x)D_ju) = 0$$

in $B_R(x_o) \subset \mathbb{R}^N$ with a_{ij} only measurable, bounded and uniformly elliptic, then $\forall r < R$ and $x \in B_r(x_o)$ we have

$$c(r, R)u(x_o) \leq u(x) \leq c(r, R)^{-1}u(x_o)$$

It is easy to check, that this estimate implies the Hölder regularity of the solution.

Such a result was generalized to the elliptic quasilinear case (i.e. elliptic p–Laplace equation) in

J. Serrin, Local behavior of solutions of quasi–linear equations, *Acta Math., 111, 1964,*

and also in

N. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, *Comm. Pure Appl. Math., 20, 1967.*

The elliptic form of the Harnack inequality we saw before does not hold in the parabolic case.

The right form for the heat equation was found, independently, by

J. Hadamard, Extension a l'equation de la chaleur d'un theoreme de A. Harnack, *Rend. Circ. Mat. Palermo (2), 3, (1955)*

and

B. Pini, Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico, *Rend. Sem. Mat. Univ. Padova, 23, 1954.*

They proved that if u is a non–negative solution of the heat equation in $B_R(x_o) \times [-R^2, R^2]$, then there exists a positive parameter a, such that, for any $r < R$, if one defines

$$P^+(r) = \{(x,t) : 0 < t < r^2, |x - x_o|^2 < at\},$$

and

$$P^-(r) = \{(x,t) : -r^2 < t < 0, |x - x_o|^2 < a|t|\},$$

we have

$$c(r,R) \sup_{(x,t)\in P^-(r))} u(x,t) \le u(x_o,0) \le c(r,R)^{-1} \inf_{(x,t)\in P^+(r)} u(x,t)$$

A breakthrough due to Moser

In

J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math., 17, 1964,*

Moser proved that if u is a non–negative solution of the parabolic equation

$$u_t = D_i(a_{ij}(x)D_j u), \quad a_{ij} \in L^\infty, \quad \text{the matrix } [a] \text{ positive definite},$$

then there exists a positive parameter a such that for any $r < R$, we have

$$c(r,R) \sup_{(x,t)\in P^-(r)} u(x,t) \le u(x_o) \le c(r,R)^{-1} \inf_{(x,t)\in P^+(r)} u(x,t)$$

Moreover this form of parabolic Harnack inequality implies the Hölder regularity of solutions, as in the elliptic case.

Generalizations of these parabolic results hold for quasilinear equations with quadratic growth or for doubly nonlinear equations of the type

$$(|u|^{p-2}u)_t = \mathrm{div}(|Du|^{p-2}Du)$$

Under this point of view see

D. G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, *Arch. Rational Mech. Anal. 1967,*

A. V. Ivanov, The Harnack inequality for generalized solutions of second order quasilinear parabolic equations, *Trudy Matematicheskogo Instituta imeni V. A. Steklova 1967,*

N. S. Trudinger, Pointwise estimates and quasilinear parabolic equations, *Comm. Pure Appl. Math., 1968.*

On the other hand the result is false for the p–Laplacian, for p other than 2, as can be easily seen thanks to the explicit Barenblatt solution, which for $p > 2$ is

$$B_p(x,t) = \frac{kR^N}{S^{\frac{N}{\lambda}}(t)} \left\{ 1 - b(p) \left(\frac{|x|}{S^{\frac{1}{\lambda}}(t)} \right)^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{p-2}}$$

where $k, R, t > 0$,

$$b(p) = \lambda^{-\frac{1}{p-1}} \frac{p-1}{p},$$

$$\lambda = N(p-2) + p$$

and

$$S(t) = k^{p-2} R^{N(p-2)} t + R^{\lambda}.$$

The case $p \neq 2$ was dealt with by DiBenedetto in two papers

E. DiBenedetto, Intrinsic Harnack type inequalities for solutions of certain degenerate parabolic equations, *Arch. Rational Mech. Anal., 1988.*

E. DiBenedetto and Y.C. Kwong, Intrinsic Harnack estimates and extinction profile for certain singular parabolic equations, *Trans. Amer. Math. Soc., 1992.*

Let $P_o \equiv (x_o, t_o)$, assume $p > \frac{2N}{N+1}$, and set

$$P^+(r) = \{(x,t) : t_o < t < t_o + r^p, |x - x_o|^p < a(t - t_o)u(P_o)^{p-2}\}.$$

If u is a non–negative solution of the parabolic equation

$$u_t = \mathrm{div}(|Du|^{p-2} Du),$$

then

$$u(P_o) \leq \gamma \inf_{(x,t)\in P^+(r)} u(x,t)$$

The general case (when $p > 2$) was proved in

E. DiBenedetto, U. Gianazza and V. V., Harnack estimates for quasi–linear degenerate parabolic differential equations, *Acta Math., 2008.*

The case $1 < p < 2$ was dealt with in

E. DiBenedetto, U. Gianazza and V. V., Forward, Backward and Elliptic Harnack Inequalities for Non–Negative Solutions to Certain Singular Parabolic Partial Differential Equations, *Annali Scuola Normale Sup. Pisa (to appear).*

It is important to underline that the results are remarkably different if $p > \frac{2N}{N+1} = p_*$ (super–critical case) or if $1 < p \leq p_*$ (sub–critical case).

Theorem

Let u be a non–negative, weak solution to our equation for p in the super–critical range. There exist positive constants δ_* and c, depending only upon the data, such that for all $(x_o, t_o) \in E_T$ and all cylinders of the type $Q_{8\rho}(x_o, t_o) \subset E_T$

$$c\, u(x_o, t_o) \le \inf_{B_\rho(x_o)} u(\cdot, t)$$

for all times

$$t_o - \delta_* [u(P_o)]^{2-p} \rho^p \le t \le t_o + \delta_* [u(P_o)]^{2-p} \rho^p.$$

The sub-critical case was considered in a very recent paper:

E. DiBenedetto, U. Gianazza and V. V., Harnack estimates and Hölder continuity for solutions to singular parabolic partial differential equations in the sub–critical range, *preprint IMATI 2009*

These intrinsic Harnack inequalities suggested us a new and more geometric approach to prove regularity for solutions to quasilinear parabolic equations.

E. DiBenedetto, U. Gianazza and V. V., A geometric approach to the Hölder continuity of solutions to certain singular parabolic partial differential equations, *preprint IMATI 2009*, (when $1 < p < 2$),

U. Gianazza, M. Surnachev and V. V., A new proof of the Hölder continuity of solutions to p–Laplace type parabolic equations, *preprint IMATI 2009*, (when $p > 2$).

The degenerate case, i.e. $p > 2$.

We recall that to prove the Hölder continuity of the solution, we have to show that the oscillation of the function u reduces in a fixed and quantitative way, going from one cylinder to its nested half.

Moreover the cylinders are intrinsic, i.e. their size depends on the solution itself.

Without loss of generality we may assume that $(x_o, t_o) = (0, 0)$, $\rho = 1$, $Q_1 = B_1 \times \{-A-1, 0\}$ where A is a constant to be fixed later, $0 < u < 1$ in Q_1 and $\omega = 1$ in Q_1.

Three steps are necessary to prove the reduction of oscillation, namely

- Cluster the positivity in the cylinder $Q_o = B_1 \times \{-A-1, -A\}$
- Time–expansion of positivity
- Space–expansion of positivity

Clustering the positivity

Assume that

$$|[(x,t) \in Q_o : u(x,t) \geq \frac{1}{2}]| \geq \frac{1}{2},$$

otherwise we work with the function $v = 1 - u$. We want to apply the following

Theorem

Let u be a weak solution to our equation. There exists a positive constant $\eta < 1$ such that, if there is a level time t_o, a point x_o and a radius $\rho > 0$ with the property that $|[x \in B_\rho(x_o) : u(x,t_o) \leq \alpha_o]| \leq \eta\rho^N$ where α_o is a positive number, then $u \geq K_o\alpha_o$ everywhere in $B_{\frac{\rho}{2}}(x_o) \times \{t_o + K\rho^p\alpha_o^{2-p}, t_o + 2K\rho^p\alpha_o^{2-p}\}$ where K and K_o are positive constants depending only upon the data.

Severini-Egoroff Quantitative Estimate

Therefore we have to find a time level where the positivity is clustering. In

E. DiBenedetto, U. Gianazza and V. V., Local Clustering of the Non Zero Set of Functions in $W^{1,1}_{loc}(E)$, *Rend. Accad. Naz. Lincei, 2006*

we proved the following

Theorem

Let u be a $W^{1,1}$ *function in* B_r*, such that* $\int_{B_r} |Du|\,dx \le \gamma r^{N-1}$.

Assume that

$$|[u > \lambda] \cap B_r| \ge \alpha r^N,$$

for some $\alpha \in (0,1)$*. Then for any* $\eta > 0$*, and* $\theta < \lambda$*, there exists* $x_o \in B_r$ *and* $\varepsilon \in (0,1)$ *(quantitatively stated), such that*

$$|[u > \theta] \cap B_{\varepsilon r}| > \omega_N (1-\eta)(\varepsilon r)^N.$$

By assumption in Q_o, the measure where $u \geq \frac{1}{2}$ is large.
Moreover, thanks to the Caccioppoli estimates, in Q_o we can estimate the $W^{1,p}$ norm of the solution.
Therefore we find a time level where we can apply the Severini-Egoroff Quantitative Estimates and we get that

$$u(x, t_o + K\rho^p(\frac{1}{4})^{2-p}) \geq \frac{K_o}{4}$$

for every $x \in B_{\frac{\rho}{2}}(x_o)$

Time–expansion of positivity

Here we use a result proved in

E. DiBenedetto, U. Gianazza and V. V., Harnack estimates for quasi–linear degenerate parabolic differential equations, *Acta Math., 2008.*

Theorem

Let u be a nonnegative weak solution to our equation. Assume that there exist a point $(x_o, t_o) \in E_T$*, a parameter* $r > 0$*, and a constant* $K_2 > 0$ *such that*

$$u(x, t_o) \geq K_2 \quad \textit{for any} \quad x \in B_r(x_o).$$

Then for any (x, t) *such that* $x \in B_{\frac{r}{2}}$ *and* $t > t_o$*, we have*

$$u(x, t) \geq \gamma K_2 (t - t_o + 1)^{-\frac{1}{p-2}},$$

where γ *is a positive constant that depends only upon the data.*

Summarising, up to now in the cylinder $B_1 \times \{-A, 0\}$ we have found a smaller cylinder centered at x_o of radius $\frac{\rho}{4}$ where

$$u(x,t) \geq \gamma \frac{K_o}{4} (A + t + 2)^{-\frac{1}{p-2}}.$$

Space–expansion of positivity

The classical expansion of positivity says the following:

Theorem

Let u be a weak solution to our equation ($0 \leq u \leq 1$)*. Assume that there exist constants* γ *and* η *such that in the cylinder* $Q_\gamma = B_r \times \{-r^p K \gamma^{2-p}, 0\}$ *(where K is a constant depending only upon the data and upon* η*) we have that for any time level* $\tau \in [-r^p \gamma^{2-p}, 0]$*,*

$$|[u(x,\tau) \geq \gamma]| \geq \eta r^N.$$

Then there exists a constant β *(depending only upon the data and upon* η*) such that* $u(x,t) \geq \beta\gamma$ *in the half cylinder.*

If one could apply the previous estimate, the proof would be finished. Actually the oscillation in the half cylinder is reduced of $(1 - \beta\gamma)$

Unfortunately we cannot apply the previous theorem, because we lack a uniform–in–time estimate. The solution satisfies

$$u \geq \gamma \frac{K_o}{4} (2)^{-\frac{1}{p-2}}$$

at the bottom of the thin cylinder, but at the top we have

$$u \geq \gamma \frac{K_o}{4} (A+2)^{-\frac{1}{p-2}}.$$

So the longer we take the cylinder, the smaller the solution is, which forces to take a longer cylinder ...

To overcome this difficulty, a new idea is necessary

If $u(x,t)$ is a solution of the p–Laplacian ($p>2$), then $w(x,\tau)=u(x,e^{\tau})e^{\frac{\tau}{p-2}}$ is a supersolution of the same equation.

So $w \geq \gamma\frac{K_o}{4}(2)^{-\frac{1}{p-2}}$ uniformly in time in the thin cylinder. The proof is over if we choose A large enough to apply the previous expansion theorem. Therefore we get

$$w \geq \beta\gamma\frac{K_o}{4}(2)^{-\frac{1}{p-2}}$$

in the half cylinder, which implies

$$u \geq A^{p-2}\beta\gamma\frac{K_o}{4}(2)^{-\frac{1}{p-2}}$$

in the corresponding half cylinder.

The singular case, i.e. $1 < p < 2$.

Without loss of generality we may assume that $(x_o, t_o) = (0, 0)$, $\rho = 1$, $Q_1 = B_1 \times \{-A, 0\}$ where A is a constant to be fixed later, $0 < u < 1$ in Q_1 and $\omega = 1$ in Q_1.
Three steps are necessary to prove the reduction of oscillation, namely

- Positivity in time
- Space–expansion of positivity
- Localization of the expansion of positivity

Positivity in time

We use the following weak backward inequality:

Theorem (L^1_{loc}–L^1_{loc} Harnack–Type Estimates)

Let u be a non–negative, weak solution for $1 < p < 2$. *There exists a positive constant* γ *depending only upon the data, such that for all cylinders* $B_{2\rho}(y) \times [s, t] \subset E_T$ *we have*

$$\left(\sup_{s<\tau<t} \int_{B_\rho(y)} u(x,\tau)dx \right) \le \gamma \inf_{s<\tau<t} \int_{B_{2\rho}(y)} u(x,\tau)dx + \gamma \left(\frac{t-s}{\rho^\lambda} \right)^{\frac{1}{2-p}}$$

where

$$\lambda \overset{def}{=} N(p-2) + p.$$

Starting from B_ρ, the solution u is required to exist in a larger neighborhood $B_{2\rho}$.

Assume that

$$\int_{B_{\frac{1}{2}}} u(x,0)dx \geq \frac{1}{4}$$

(otherwise one can consider the function $v = 1 - u$), and choose

$$A = \varepsilon \int_{B_{\frac{1}{2}}} u(x,0)dx$$

with ε so small that

$$\frac{1}{4} \leq \int_{B_{\frac{1}{2}}} u(x,0)dx$$

$$\leq \sup_{-A<\tau<0} \int_{B_{\frac{1}{2}}} u(x,\tau)dx \leq \gamma \inf_{-A<\tau<0} \int_{B_1} u(x,\tau)dx$$

with γ a positive constant. Hence for any $\tau \in (-A,0)$ we have

$$\int_{B_1} u(x,\tau)dx \geq \frac{1}{4}\gamma^{-1}.$$

As $u \leq 1$, from the previous inequality we deduce that there exists a positive number ε_o such that for any $\tau \in (-A, 0)$

$$|[x \in B_1 : u(x,\tau) \geq \varepsilon_o]| \geq \varepsilon_o.$$

Space–expansion in space of positivity

The classical expansion of positivity says the following:

Theorem

Let u be a weak solution to our equation ($0 \le u \le 1$). Assume that there exist constants γ and η such that in the cylinder $Q_\gamma = B_r \times \{-r^p K \gamma^{2-p}, 0\}$ (where K is a constant depending only upon the data and upon η) we have that for any time level $\tau \in [-r^p \gamma^{2-p}, 0]$,

$$|[u(x,\tau) \ge \gamma]| \ge \eta r^N.$$

Then there exists a constant β (depending only upon the data and upon η) such that

$$u(x,t) \ge \beta\gamma$$

in a smaller cylinder $B_{\frac{1}{2}r} \times \{s, s + r^p \varepsilon K(\beta\gamma)^{2-p}, 0\}$ where $s \in [-r^p K \gamma^{2-p}, 0]$.

Notice that this Theorem does not say where the expansion of positivity is located, so we cannot deduce the reduction of oscillation of the function u.

Localization of the expansion of positivity

We want to show that the expansion of positivity occurs for any $-\frac{A}{2} \leq \tau \leq 0$. Without loss of generality let us show that the expansion occurs when $\tau = 0$.
As in the degenerate case, we use the following Proposition concerning the time–expansion of positivity:

Theorem

Let u be a weak solution to our equation. If there exist a time level t_o and a positive number α_o, such that $u(x, t_o) \geq \alpha_o$ for any $x \in B_{\frac{1}{2}}$, then

$$u \geq K_o \alpha_o$$

everywhere in $B_{\frac{1}{4}} \times \{t_o, t_o + K\alpha_o^{2-p}\}$, where K and K_o are positive constants depending only upon the data.

Therefore if the expansion of positivity occurs near the time level $\tau = 0$, we have finished the proof.
Otherwise, to overcome this difficulty, it is necessary to make use of the same idea of the degenerate case:

If $u(x,t)$ is a solution of the p–Laplacian ($1 < p < 2$), then $w(x,\tau) = u(x,-e^{\tau})e^{\frac{\tau}{2-p}}$ is a supersolution of the same equation.

With this change of variables, the function w is defined in the stripe $B_{\frac{1}{2}} \times \{0,+\infty\}$.

Splitting this stripe in infinitely many cylinders with the *right* size and applying in each of them the expansion of positivity, we get that there exists a sequence of times $t_n \to +\infty$ where the expansion of positivity occurs for the function w.

Rephrasing this result for the function u, we have that there exists a sequence of times $t_n \to 0^-$ where the expansion of positivity occurs for the function u, i.e. the expansion of positivity occurs near the time level 0 and we have finished the proof.

Final remarks

- This new approach gives exactly the same results obtained by DiBenedetto and Chen-DiBenedetto, using a pure DeGiorgi–type approach
- All the previous results hold also for porous medium type equations and for doubly nonlinear type equations (M. Sosio, *in preparation*).
- This approach works not only in the case of the interior regularity, but also in the case of boundary regularity with Dirichlet, Neumann and mixed conditions (P. Candito and R. Livrea, *in preparation*).