

# **Prairie Analysis Seminar 2007**

Kansas State University

## **Fully non linear equations with integral diffusions**

**Luis Caffarelli** (Principal lecturer)  
**University of Texas at Austin**

Fully non linear equations arise in optimal control (Hamilton Jacobi equations) when we are allowed to choose in each step the best diffusion to fit our purpose, and in Game Theory (Isaac 's equation) when competing players alternatively choose their most convenient diffusion.

There is a rather complete theory in the case in which the underlying diffusions are continuous (Wiener processes). We will describe recent work (with L. Silvestre) where a parallel theory is developed for discontinuous (Levy) processes.

A preprint is available at [arXiv.org](http://arXiv.org)

## **Hele-Shaw flows and obstacle problems: Application to random homogenization**

**Antoine Mellet** (Invited Speaker)  
**University of British Columbia**

It is a well-known fact that Hele-Shaw and Stefan free boundary problems can be formally formulated as obstacle problems via an integration in time. Using the notion of viscosity solutions, we make this connection more rigorous. We then deduce some results on the periodic and random homogenization of Hele-Shaw type free boundary problems, when oscillations arise along the free boundary.

## **Modeling and stability for bent-core liquid crystal molecules**

**Daniel Phillips** (Invited speaker)  
**Purdue University**

We investigate a model for drawn fibers made up of cylindrical, smectic layers of banana-shaped liquid crystal molecules. The molecular shape induces both a spontaneous polarization and a distinctive effect on layer density. These features lead to a model that predicts stable fiber formation. We cast the model as a free boundary problem for the fiber's cross-section and analyze its solutions. This is joint work with P. Bauman.

## Contributed Talks

**Adamowicz, Tomasz** - Syracuse University

**Title:** On the geometry of p-harmonics

**Abstract:** We will discuss some geometric properties of p-harmonic functions and transformations. A map  $u = (u^1, \dots, u^m) : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  in Sobolev space  $W_{loc}^{1,p}(\Omega, \mathbb{R}^m)$  is called p-harmonic if it satisfies the following system of equations:

$$\operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty,$$

where  $Du$  is the Jacobi matrix of  $u$ . If  $m = 1$ , dimension of target domain, then this system reduces to a single p-harmonic equation. The methods of generating p-harmonic functions and mappings via compositions will be discussed. We will also present the relation between p-harmonic planar mappings and quasiregular transformations. Despite similarity of definition to scalar counterpart, the setting of mappings turns out to be much more difficult and challenging. Many questions remain open, we will address some of them.

**Demirkaya, Aslihan** - University of Kansas

**Title:** Existence of global attractor for a periodic Kuramoto-Sivashinsky type equation in space dimension two

**Abstract:** We consider a periodic Kuramoto-Sivashinsky type equation in space dimension two. We use a Lyapunov function argument to show the boundedness of the solution for large times. We prove the existence of global attractor in  $L^2([-L, L]^2)$  when the equation is subject to  $L^2$  external force.

**Hermi, Lotfi** - University of Arizona

**Title:** Improved Universal Weyl-Type Bounds for Eigenvalues of the Dirichlet Laplacian

**Abstract:** Trace identities of the type derived by Harrell-Stubbe, and later generalized by Levitin-Parnowski, proved to be a very efficient procedure to produce universal Yang-type bounds for eigenvalues of the Dirichlet Laplacian. In this talk we show how these identities can be used to produce new universal Weyl-type bounds for averages of eigenvalues, and provide alternative routes to the Berezin-Li-Yau inequality as viewed by Laptev and Weidl. This is joint work with Professor Evans Harrell of Georgia Tech. A preprint is available at <http://lanl.arxiv.org/abs/0705.3673>

**Jochmann, Frank** - Technische Universität Berlin, Institut für Mathematik

**Title:** A mathematical model for magnetized media

**Abstract:** This talk is concerned with the Landau-Lifschitz equation without exchange energy coupled with Maxwell's equations describing the electromagnetic field in a magnetized medium. In this framework, the Landau-Lifschitz equation is a first order ordinary differential equation for the magnetization field coupled to the hyperbolic equations for the electromagnetic field. The main subjects are the existence, uniqueness and asymptotic behavior of the solutions as well as certain small parameter limits in this model.

**Khanal, Netra** - Oklahoma State University

**Title:** The Kawahara equations in weighted Sobolev spaces

**Abstract:** The initial and boundary value problem for the Kawahara equation, a fifth order Kdv type equation, is studied in weighted Sobolev Spaces. This functional framework is based on Dual-Petrov-Galerkin algorithm, a numerical method proposed by Shen to solve third and higher odd-order partial differential equations. The theory presented includes the existence and uniqueness of a local mild solution and a global strong solution in these weighted spaces.

**Kim, Young-Heon** - University of Toronto

**Title:** Curvature and continuity of optimal transport

**Abstract:** We will discuss continuity of optimal transport maps, in view of a pseudo-Riemannian structure which we have formulated recently. A necessary condition for the continuity is given as some non-negativity condition on the curvature of this pseudo-Riemannian metric. This result gives a natural geometric framework and new perspectives for the regularity theory of Ma, Trudinger, Wang and Loeper on the potential functions of optimal transport; it also yields some extensions of previous results and new examples. This is joint work with Robert McCann (University of Toronto).

**Laurent, Thomas** - UCLA

**Title:** Local existence, global existence and blow-up for an aggregation equation

**Abstract:** An aggregation equation is a non-linear, non-local partial differential equation that is a regularization of a backward diffusion process. The non-locality arises via convolution with a potential. Depending on how regular the potential is we prove either local or global existence for the solutions. In the case of a potential with a Lipschitz point at the origin, e.g.  $\exp(-|x|)$ , we prove finite-time blow-up of solutions. Aggregation equations arise in a number of context of recent interest in the physics and biology literature.

**Neelon, Tejinder** - California State University San Marcos

**Title:** Restrictions of formal power series and infinitely differentiable functions to algebraic surfaces

**Abstract:** We prove the analogs of the following classical theorems in the setting of certain classes of infinitely differentiable functions: the Zorn-Lelong theorem- if a double power series  $F(x, y)$  converges on a family of lines with slope set of positive capacity then  $F(x, y)$  is convergent; the Abhyankar-Moh-Sathaye theorem- the transfinite diameter of the set all directions in which a given divergent series can converge is zero, the Levenberg-Molzon theorem which extends the Abhyankar-Moh-Sathaye theorem with the lines replaced by the planes, and, the Bochnak-Siciak theorem- a infinitely differentiable function on  $\mathbb{R}^n$  that is real analytic on every line is real analytic. The methods applied here also yield new proofs of the above mentioned theorems.

**Nguyen, Phuc Cong** - Purdue University

**Title:** Quasilinear and Hessian equations with power source terms and rough data

**Abstract:** We give complete characterizations for the solvability of the following quasilinear and Hessian equations:

$$-\Delta_p u = u^q + \omega, \quad F_k[-u] = u^q + \omega, \quad u \geq 0$$

on a domain  $\Omega \subset \mathbb{R}^n$ . Here  $\Delta_p$  is the  $p$ -Laplacian,  $F_k[u]$  is the  $k$ -Hessian, and  $\omega$  is a given nonnegative measurable function (or measure) on  $\Omega$ . Our results give a complete answer to a problem posed by Bidaut-Véron, and extend earlier results due to Adams and Pierre, Kalton and Verbitsky to nonlinear operators. This talk is based on joint work with Igor E. Verbitsky.

**Okoudjou, Kasso** - University of Maryland, College Park

**Title:** Local well-posedness of nonlinear dispersive equations on modulation spaces

**Abstract:** In this talk, we shall present some new time-frequency estimates for certain Fourier multipliers. As an application, we obtain some improved local well-posedness results for the NLS, NLW and NLKG equations with Cauchy data in modulation spaces.

**Radu, Petronela** - University of Nebraska-Lincoln

**Title:** Decay rates for wave equations with linear damping and variable coefficients

**Abstract:** The authors establish weighted decay rates for the energy associated with the Cauchy problem

$$\begin{cases} u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t = 0, & x \in \mathbf{R}^n, t > 0 \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbf{R}^n. \end{cases}$$

Such a system appears in models for traveling waves in a non-homogeneous gas with damping that changes with the position. This problem has been studied intensively for the homogeneous medium (when  $b$  is constant), but the results are scarce for the variable coefficient case. In fact, to the authors' knowledge, the results of this paper are the first to be obtained for wave equations with variable coefficients on the entire space  $\mathbf{R}^n$  when damping is present. The proof is based on the multiplier method where the multiplier is cleverly engineered from an asymptotic profile of solutions. These results were obtained in collaboration with G. Todorova and B. Yordanov from University of Nebraska-Lincoln.

**Raynor, Sarah** - Wake Forest University

**Title:** Neumann fixed boundary regularity for an elliptic free boundary problem

**Abstract:** We examine the regularity properties of solutions to an elliptic free boundary problem near a Neumann fixed boundary. Consider a nonnegative function  $u$ , defined variationally, which is harmonic where it is positive and satisfies a gradient jump condition weakly along the free boundary  $\partial\{u > 0\}$ . Our main result is that  $u$  is Lipschitz continuous. Additionally, we prove various basic properties of such a minimizer near a portion of the fixed boundary on which  $\frac{\partial u}{\partial \nu} = 0$  weakly. Our results include up-to-the boundary gradient estimates on harmonic functions with Neumann boundary conditions on convex domains, which have independent interest.

**Shen, Chun-Yen** - Indiana University

**Title:** An explicit sum product estimate in  $F_p$

**Abstract:** Let  $A$  be a subset of  $F_p$ , the field of  $p$  elements with  $p$  prime.

We let

$$A + A = \{a + b : a \in A, b \in A\},$$

and

$$AA = \{ab : a \in A, b \in A\}.$$

Bourgain, Katz and Tao have shown that if  $|A| < p^{1-\delta}$ , where  $\delta > 0$ , then one has the sum product estimate

$$\max(|A + A|, |AA|) \geq |A|^{1+\epsilon}$$

Recently, Garaev showed that when  $|A| < p^{\frac{1}{2}}$  one has the estimate

$$\max(|A + A|, |AA|) \geq |A|^{\frac{15}{14}}.$$

By using Plunnecke's inequality in a slightly more sophisticated way, we improve this exponent to  $\frac{14}{13}$ . We believe that further improvements might be possible through aggressive use of Ruzsa covering.

**Silvestre, Luis** - Courant Institute of Mathematical Sciences

**Title:** Lower dimensional obstacle problems and the fractional Laplacian

**Abstract:** We use the “thin” obstacle problem to obtain optimal regularity results for the obstacle problem for the fractional Laplacian. This is a joint work with Luis Caffarelli and Sandro Salsa.

**Stephens, Benjamin** - University of Toronto

**Title:** Thread-wire surfaces: Near-wire minimizers and topological finiteness

**Abstract:** Alt's thread problem asks for least-area surfaces bounding a fixed “wire” curve and a movable “thread” curve of length  $L$ . Near-wire minimizers are important to a finiteness conjecture. For near-wire minimizers on a generic wire we show  $C^1$ -regularity where the thread and wire join at cusp-corners. Our result confirms a prediction of the normal vector limit at such points. It is an example of the local geometry dominating other influences in a free boundary problem. Finally, we use a weighted isoperimetric inequality to prove that all minimizers are near-wire minimizers when the thread is near the wire length. For physical examples of the thread-wire system, feel free to visit <http://www.bkstephens.net>

**Teixeira, Eduardo** - Rutgers University

**Title:** Optimal design problems in rough inhomogeneous media

**Abstract:** Well known for modeling important problems in applied mathematics, respected for the challenging mathematical questions they give rise to and admired for their intrinsic beauty, optimization problems with volume constraints have received an overwhelming attention in the past few decades. In this talk we will report recent advances in the theory of non-linear optimal design problems governed by a rather general class of non-linear degenerate elliptic equations.

**Torres, Mónica** - Purdue University

**Title:** Removable singularities of divergence free vector fields and related equations with measure data

**Abstract:** We study the solvability of the equation  $\operatorname{div} F = \mu$ , with non-negative measure data  $\mu$ , in the class of continuous or  $L^p$  vector fields  $F$ , where  $1 \leq p \leq \infty$ . We obtain explicit characterizations in terms of densities of  $\mu$  for continuous and bounded vector fields, and in terms of potential energies of  $\mu$  for  $L^p$  vector fields. These results allow us to characterize the removable singularities of the corresponding homogeneous equation  $\operatorname{div} F = 0$ . Joint work with Nguyen Cong Phuc.

**Treinen, Ray** - Kansas State University

**Title:** Numerical Results for Floating Drops

**Abstract:** Numerical results are presented for various configurations of axi-symmetric floating drops. These drops can be seen as a drop of fluid floating on a reservoir of another fluid with, say, air above. In each case the drops are constructed by matching numerical solutions of the differential equations for the axi-symmetric capillary surfaces to solve a free boundary problem. These include both “heavy” and “light” drops in which the density of the fluid in the drop is either larger or smaller than that of the reservoir on which it rests. The reservoir may be either infinite or contained in a finite container. Of particular interest are heavy drops which have multiple necks, which lead to nonuniqueness for the Euler equations.

**Trokhimtchouk, Maxim** - University of California at Berkeley

**Title:** Everywhere regularity of certain nonlinear diffusion systems

**Abstract:** I will talk about nonlinear parabolic systems that are generalizations of scalar diffusion equations. More precisely, I consider systems of the form

$$u_t - \Delta[\nabla\Phi(u)] = 0,$$

where  $\Phi(z)$  is a strictly convex function. I will show that when  $\Phi$  is a function only of the norm of  $u$ , then bounded weak solutions of these parabolic systems are everywhere Hölder continuous and thus everywhere smooth. I will also show that the method used to prove this result can be easily adopted to simplify the proof of the result due to Wiegner on everywhere regularity of bounded weak solutions of strongly coupled parabolic systems.

**Uriarte-Tuero, Ignacio** - University of Missouri, Columbia

**Title:** Sharp nonremovability examples for BMO and Hölder quasiregular mappings

**Abstract:** A classical problem in complex analysis is to characterize the removable sets for various classes of analytic functions: Hölder, Lipschitz, BMO,

bounded (this last case gives rise to the analytic capacity and the Painlevé problem which has been recently solved by Tolsa.)

One can ask the same questions in the setting of  $K$ -quasiregular maps (since they are a  $K$ -quasiconformal map followed by an analytic map.) Most of the bounded case was dealt with in a joint paper with K. Astala, A. Clop, J. Mateu and J. Orobitg, [ACMOUT]. The BMO case was dealt with in [ACMOUT] except for a gap at the critical dimension (Question 4.2 in [ACMOUT].) I answered the question filling the gap in [UT].

The Lipschitz case was dealt with by A. Clop, as well as most of the Hölder case, where again a gap at the critical dimension was left. In a joint paper with A. Clop [CUT] we closed the gap.

I will summarize the results and give some ideas of the proofs in the above papers. The talk will be self-contained.

References:

[ACMOUT] Kari Astala, Albert Clop, Joan Mateu, Joan Orobitg and Ignacio Uriarte-Tuero. Distortion of Hausdorff measures and improved Painlevé removability for bounded quasiregular mappings. *Duke Math J.*, to appear.

[CUT] Albert Clop and Ignacio Uriarte-Tuero. Sharp Nonremovability Examples for Hölder continuous quasiregular mappings in the plane. Preprint.

[UT] Ignacio Uriarte-Tuero. Sharp Examples for Planar Quasiconformal Distortion of Hausdorff Measures and Removability. Submitted.

**Volok, Dan** - Kansas State University

**Title:** Rational solutions of the Schlesinger system and related tau-functions

**Abstract:** The system of PDEs

$$dQ_i = \sum_{j \neq i} [Q_j, Q_i] d \log(t_i - t_j), \quad i = 1, \dots, n,$$

where  $Q_i$  are  $m \times m$  matrices and  $d$  stands for the exterior differentiation with respect to  $t = (t_1, \dots, t_n)$ , was discovered by L. Schlesinger as the non-linear system governing isomonodromic deformations of Fuchsian systems of ODEs. In the special case  $m = 2, n = 4, t_1, t_2, t_3$  - fixed, the Schlesinger system reduces to the sixth transcendent of Painlevé; more generally, the Schlesinger system enjoys the Painlevé property: any solution is meromorphic on the universal covering of

$$\mathbb{C} \setminus \bigcup_{i,j:i \neq j} \{t : t_i = t_j\}$$

(B. Malgrange, T. Miwa, 1982). In this talk a class of rational solutions of the Schlesinger system will be described and explicit formulas for the related tau functions (as introduced by M. Sato, T. Miwa and M. Jimbo) will be given. This is joint work with V. Katsnelson.

**Wang, Lianwen** - University of Central Missouri

**Title:** Optimal Control of Delay Differential Inclusions in Infinite Dimensional Spaces

**Abstract:** In the past few decades optimal control of delay differential inclusions in finite dimensional spaces has been extensively investigated. In the framework of infinite dimensional spaces, some results of optimal control of differential inclusions without delay have been established recently. However, not much work has been done for optimal control of delay differential inclusions. This paper deals with optimal control for a class of nonconvex delay differential inclusions in Banach spaces with endpoints constraints. Necessary optimality conditions are obtained using the technique of discrete approximations. This is a joint work with Dong Wang from Fayetteville State University.

**Wang, Peiyong** - Wayne State University

**Title:** The normalized infinity Laplacian

**Abstract:** The theory of the infinity Laplacian has been developed recently by many mathematicians. I plan to first briefly introduce the theory about the infinity Laplacian, then to elaborate about some recent progress by me and collaborators. In particular, the theory about inhomogeneous normalized infinity Laplacian developed by me and Dr. Guozhen Lu will be covered.

**Wright, Matt** - University of Missouri, Columbia

**Title:** Mixed Boundary Value Problems for the Stokes System

**Abstract:** This talk will focus on proving the well-posedness of the mixed boundary value problem for the Stokes system in a class of Lipschitz domains. Using the method of layer potentials, we first reduce the problem to solving a boundary integral equation. Since we lack a boundary Korn inequality for solutions of the Stokes system, we are forced to derive a new Rellich-type estimate in order to show that the boundary integral operator is semi-Fredholm. Using recent results by Russell Brown and Irina Mitrea concerning the mixed boundary problem for the Lamé system, we are then able to utilize a deformation argument to conclude that the operator in question is an isomorphism.