

**Low regularity solutions of the
Benjamin-Ono and the KP-I equations**

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Three models of initial-value problems:

The Korteweg-de Vries (KdV) equation on $\mathbb{R}_x \times \mathbb{R}_t$:

$$\begin{cases} \partial_t u + \partial_x^3 u + u \cdot \partial_x u = 0; \\ u(0) = \phi. \end{cases} \quad (0.1)$$

The Benjamin-Ono (BO) equation on $\mathbb{R}_x \times \mathbb{R}_t$:

$$\begin{cases} \partial_t u + \mathcal{H} \partial_x^2 u + u \cdot \partial_x u = 0; \\ u(0) = \phi. \end{cases} \quad (0.2)$$

Here \mathcal{H} denotes the Hilbert transform on \mathbb{R} defined by the Fourier multiplier $-i \operatorname{sgn}(\xi)$.

The Kadomtsev-Petviashvili I (KP-I) equation on $\mathbb{R}_{x,y}^2 \times \mathbb{R}_t$:

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1} \partial_y^2 u + u \cdot \partial_x u = 0; \\ u(0) = \phi. \end{cases} \quad (0.3)$$

These models are time-reversible, dispersive nonlinear equations. They are completely integrable (infinitely many conserved quantities). They are all models for propagation of water waves. The KdV equation is a model for one-dimensional propagation of nonlinear dispersive long waves. The Benjamin-Ono equation is a model for one-dimensional waves in deep stratified fluids. The KP-I equation is a model for two-dimensional propagation of dispersive long waves, with weak transverse effects.

All of these models are of the type

$$\text{Linear operator}(u) + u \cdot \partial_x u = 0.$$

In all of these models the nonlinearity is $\partial_x(u^2)/2$. This is referred to as a "derivative loss" in the nonlinearity. The dispersive character of the equations is related to the symbol of the linear operator in the formula above; these symbols are

$$\begin{cases} i(\tau - \xi^3) & \text{for the KdV;} \\ i(\tau + \xi|\xi|) & \text{for the BO;} \\ i(\tau - \xi^3 - \eta^2/\xi) & \text{for the KP-I.} \end{cases}$$

Well-posedness

It is known that smooth solutions to all of these equations exist: if we start with smooth real-valued data ϕ in L^2 then there is a unique solution, which stays smooth, real-valued, and in L^2 at all times. The question is what happens for data that is not smooth. More precisely, let

$$u = S^\infty \phi$$

denote the (nonlinear) operator that associates to any smooth data $\phi \in H_r^\infty$ the solution u . Does the nonlinear operator S^∞ extend (say continuously) to larger spaces of functions?

The mapping $\phi \rightarrow S^\infty \phi$ is nonlinear, so there could be blow-up in finite time for data ϕ sufficiently large.

Global well-posedness in some Banach space X : the mapping $\phi \rightarrow S^\infty \phi$ extends to a **continuous** mapping from the Banach space X to the Banach space $C([-T, T] : X)$ for any $T > 0$.

Local well-posedness in some Banach space X : for any $r > 0$ there is $T = T(r)$ with the property that the mapping $\phi \rightarrow S^\infty \phi$ extends to a **continuous** mapping from the ball $B_X(r)$ in X to the Banach space $C([-T(r), T(r)] : X)$.

Main conservation laws: conservation of L^2 norm and conservation of energy. For the three equations considered, the quantities

$$\int_{\mathbb{R}} u^2(., t) dx \text{ or } \int_{\mathbb{R}} u^2(., ., t) dx dy$$

are conserved. Conservation of energy (Hamiltonian):
for KdV:

$$E_1(t) = \int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{3} \int_{\mathbb{R}} u^3 dx$$

for BO:

$$E_1(t) = \int_{\mathbb{R}} u \cdot \mathcal{H}u_x dx - \frac{1}{3} \int_{\mathbb{R}} u^3 dx$$

for KP-I

$$E_1(t) = \int_{\mathbb{R}^2} (\partial_x u)^2 + (\partial_x^{-1} \partial_y u)^2 dx dy - \frac{1}{3} \int_{\mathbb{R}^2} u^3 dx dy$$

Most common well-posedness question: are these equations well-posed in the Sobolev spaces H^s , $s \in \mathbb{R}$. The energy spaces correspond to H^1 for KdV, $H^{1/2}$ for BO, Y^1 for KP-I.

The solution operator $\phi \rightarrow S^\infty \phi$ is nonlinear and there is no explicit (integral) representation for it. We discuss *the iteration method* and *the energy method*.

The iteration method: The equations can be written in the integral form

$$u(t) = W_t(\phi) - \int_0^t W_{t-s}(u \partial_x u(s)) ds,$$

where W_t is the free evolution (linear) operator. This is of the form

$$u = f(\text{ which is a known function }) + Q(u).$$

The nonlinear term $Q(u)$ is thought of as a perturbation. Such an equation can be studied by iteration, provided that

the nonlinear operator Q is “small”.

Indeed, let $u_0 = f$, $u_1 = f + Q(u_0) \dots$, and in general

$$u_{n+1} = f + Q(u_n).$$

This procedure converges (exponentially fast) if $Q(u) - Q(v)$ is “smaller” than $(u - v)/2$ in a suitable Banach space.

The energy method. The energy method is a uniqueness argument. The main issue is to prove that the weak solutions (which have to be constructed in a separate argument) are unique. Assume that $u_1, u_2 \in C([-T, T] : X)$ are *real-valued* solutions of the equation

$$\begin{cases} \partial_t u + L(u) + u \cdot \partial_x u = 0; \\ u(0) = \phi, \end{cases} \quad (0.4)$$

for the same initial data ϕ . We subtract the two equations, multiply by $(u_1 - u_2)$ and integrate in the space variables. We use the identity

$$\int L(u_1 - u_2) \cdot (u_1 - u_2) dx \equiv 0$$

to obtain

$$\begin{aligned} \partial_t \|(u_1 - u_2)(t)\|_{L^2}^2 &\leq C \left| \int \partial_x (u_1^2 - u_2^2) \cdot (u_1 - u_2) dx \right| \\ &\leq C \|(u_1 - u_2)(t)\|_{L^2}^2 \cdot (\|\partial_x u_1(t)\|_{L^\infty} + \|\partial_x u_2(t)\|_{L^\infty}). \end{aligned}$$

Using Gronwall's inequality and $u_1(0) = u_2(0) = \phi$, we have $u_1 \equiv u_2$ (the desired uniqueness), provided that

$$\int_{-T}^T \|\partial_x u(t)\|_{L_x^\infty} = \|\partial_x u\|_{L_t^1 L_x^\infty} < \infty. \quad (0.5)$$

So the question becomes: assume $u \in C([-T, T] : X)$ is a solution of (0.4). For what Banach spaces X can we have control of $\|\partial_x u\|_{L_t^1 L_x^\infty}$?

Classical energy method: the Sobolev imbedding theorem leads to control of $\|\partial_x u\|_{L_t^\infty L_x^\infty}$. It gives well-posedness in $H^{3/2+}$ for KdV and BO, and Y^{2+} for KP-I.

Refined energy methods: exploit the fact that we only need control of the L_t^1 norm rather than the L_t^∞ norm. One uses averaging techniques, such as Strichartz estimates, local smoothing estimates, and maximal function estimates (which are all adapted to the specific equation), to allow for larger spaces X . The best known results using these techniques are well-posedness in $H^{3/4+}$ for KdV, $H^{9/8+}$ for BO, and $Y^{3/2+}$ for KP-I.

The KdV equation: the initial-value problem is globally well-posed in the energy space H^1 (Kenig, Ponce, Vega 1991, using a refined form of the *energy method*).

Also, the smallness condition on the nonlinear part Q is satisfied in suitable spaces, and the initial value problem is well-posed in L^2 (Bourgain 1993, using the *iteration method* in a suitable Banach space). It is also well-posed (locally and globally) in the larger spaces of distributions H^s , $s > -3/4$ (Kenig, Ponce, Vega, Colliander, Keel, Stafillani, Takaoka, Tao, Christ and many others).

The main Banach spaces in which to perform the iteration method are called $X^{s,b}$ spaces. They are defined by space-time norms in the Fourier space, and are adapted to the geometry of the KdV equation:

$$\|g\|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} |\widehat{g}(\xi, \tau)|^2 (1 + \xi^2)^s (1 + |\tau - \xi^3|)^{2b} d\xi d\tau.$$

For $s > -3/4$ and suitable $b > 0$, one can show

$\phi \rightarrow W_t(\phi)$ is a bounded (linear) mapping $H^s \rightarrow X^{s,b}$

and

$$\|Q(u) - Q(v)\|_{X^{s,b}} \leq C \|u - v\|_{X^{s,b}} (\|u\|_{X^{s,b}} + \|v\|_{X^{s,b}}).$$

Thus, the iterative argument explained before produces a convergent sequence (at least when $\|\phi\|_{H^s}$ is small).

The BO equation: The natural definition of the $X^{s,b}$ spaces would be

$$\|g\|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} |\widehat{g}(\xi, \tau)|^2 (1 + \xi^2)^s (1 + |\tau + \xi|\xi|)^{2b} d\xi d\tau,$$

since the symbol of the linear part is $i(\tau + \xi|\xi|)$. For this definition, however, the bilinear estimate

$$\|Q(u) - Q(v)\|_{X^{s,b}} \leq C\|u - v\|_{X^{s,b}}(\|u\|_{X^{s,b}} + \|v\|_{X^{s,b}}),$$

fails very badly (it fails by $1/2$ of a derivative, in the sense that one could only control a nonlinearity of the form $J_x^{-1/2}\partial_x(u^2)/2$).

Even worse, the smallness condition for the nonlinear operator Q cannot be satisfied for **any choice** of Banach spaces (Molinet, Saut, Tzvetkov 2001, using the fact that the flow mapping cannot be real-analytic).

The energy method gives well-posedness in H^s , $s > 3/2$. Refinements of the energy method (Ponce,

Koch-Tzvetkov, Kenig-Koenig) lead to local well-posedness in H^s , $s > 9/8$, still far from L^2 or the energy space $H^{1/2}$.

Gauge transformation. Tao (2004) constructed a suitable gauge transformation for the BO equation and proved global well-posedness in the “second energy” space H^1 . The idea of the gauge transformation is to modify the equation in such a way that part of the nonlinear term can be handled together with the linear component of the equation, while the remaining part is significantly smaller.

Ionescu-Kenig (2005) proved global well-posedness both in L^2 and the energy space $H^{1/2}$, using first a gauge transformation to weaken the nonlinearity, and then a variant of the *iteration method* in a suitable Banach space.

Burq-Planchon (2005) also proved global well-posedness in the energy space $H^{1/2}$, as well as local well-posedness in $H^{1/4+}$, using a gauge transformation.

The KP-I equation: The natural definition of the $X^{s,b}$ spaces would be

$$\|g\|_{X^{s,b}}^2 = \int_{\mathbb{R}^2} |\widehat{g}(\xi, \tau)|^2 (1+\xi^2)^s (1+|\tau-\xi^3-\eta^2/\xi|)^{2b} d\xi d\tau,$$

since the symbol of the linear part is $i(\tau - \xi^3 - \eta^2/\xi)$. For this definition, however, the bilinear estimate

$$\|Q(u) - Q(v)\|_{X^{s,b}} \leq C\|u - v\|_{X^{s,b}}(\|u\|_{X^{s,b}} + \|v\|_{X^{s,b}}),$$

fails very badly (as in the case of the BO equation, it fails by $1/2$ of a derivative). As in the case of the BO equation, the smallness condition for the nonlinear operator Q cannot be satisfied for **any choice** of Banach spaces (Molinet, Saut, Tzvetkov 2002, using the fact that the flow mapping cannot be real-analytic).

The energy method gives well-posedness in Y^s , $s > 2$ (Iorio-Nunes 1998, there are however a number of technical issues concerning the definition of the operator $\partial_x^{-1}\partial_y^2$). A recent refinement (Kenig 2004) gives local well-posedness in Y^s , $s > 3/2$, and global well-posedness in the “second energy” space Y^2 .

Open questions: global well-posedness for the KP-I equation in L^2 (with suitable low-frequency condition) or in the energy space Y^1 .

Informal computation for the dispersion: Heuristically, the “size” of the dispersion of the linear part of the equation is determined by a lower bound on

$$|\nabla_{\text{frequency variables}}(\text{symbol})|.$$

For the KdV equation, this is

$$|\nabla_{\xi}(\xi^3)| \approx |\xi|^2, \quad |\xi| \gg 1.$$

For the BO equation, this is

$$|\nabla_{\xi}(-\xi|\xi|)| \approx |\xi|, \quad |\xi| \gg 1.$$

For the KP-I equation, this is

$$\begin{cases} |\nabla_{\xi,\eta}(\xi^3 + \eta^2/\xi)| \approx |(3\xi^2 - \eta^2/\xi^2, 2\eta/\xi)| \approx |\xi| \\ \text{if } \xi \gg 1 \text{ and } |\eta \pm \sqrt{3}\xi^2| \leq 1. \end{cases}$$

Heuristically, the difficulty in using the *iteration method* for the BO and the KP-I equation is due to the lower amount of dispersion.

Global well-posedness of the BO equation

(joint work with C. Kenig).

Theorem: The BO equation is globally well-posed in the Banach spaces H_r^σ , $\sigma \geq 0$, of real-valued Sobolev functions.

The starting point is the observation that the desired bilinear estimate

$$\|Q(u) - Q(v)\|_{X^{s,b}} \leq C\|u - v\|_{X^{s,b}}(\|u\|_{X^{s,b}} + \|v\|_{X^{s,b}}),$$

fails only **logarithmically** if the functions u and v do not have small frequencies. On the other hand, small frequencies correspond to smooth functions, for which global well-posedness is known.

1. Step 1: Decompose the data $\phi = \phi_{\text{low}} + \phi_{\text{high}}$. If u_0 is the smooth solution corresponding to the data ϕ_{low} , then $\tilde{u} = u - u_0$ satisfies the difference equation

$$\begin{cases} \partial_t \tilde{u} + \mathcal{H} \partial_x^2 \tilde{u} + \partial_x(u_0 \cdot \tilde{u}) + \tilde{u} \cdot \partial_x \tilde{u} = 0; \\ \tilde{u}(0) = \phi_{\text{high}}. \end{cases}$$

Step 2: Global gauge transformation $\tilde{u} = \text{Gauge}(\mathbf{w})$, for the purpose of “eliminating” the term $\partial_x(u_0 \cdot \tilde{u})$. The resulting equation is a vector-valued BO equation

$$\begin{cases} \partial_t \mathbf{w} + \mathcal{H} \partial_x^2 \mathbf{w} = \mathbf{E}(\mathbf{w}); \\ \mathbf{w}(0) = \Phi. \end{cases}$$

The benefit of the gauge transformation is that in the resulting system of equations we can essentially pretend that all the functions, including the initial data, do not have frequencies ≤ 1 .

Step 3: Removal of the logarithmic divergence. We work with spaces that have two components: an $X^{s,b}$ -type component in the frequency space + a normalized $L_x^1 L_t^2$ component in the physical space. Such spaces have been used in the context of wave maps by Tataru (1998), and Tao (2001). The space $L_x^1 L_t^2$ is related to the “local smoothing estimate”:

$$\|\partial_x u\|_{L_x^\infty L_t^2} \leq C \|(\partial_t + \mathcal{H} \partial_x^2) u\|_{L_x^1 L_t^2},$$

for any $u \in C_0^\infty(\mathbb{R} \times \mathbb{R})$.

Step 4: We prove linear and bilinear estimates in these spaces, as well as stability properties under the gauge transformation. Then we conclude the proof using an iterative argument.

The main difficulty is dealing with the action of the gauge transformation: assume $\phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth real-valued function, and let F^s denote the resolution spaces constructed in Step 3. Is it true that

$$\cdot e^{i\Phi} : F^s \rightarrow F^s \text{ as a bounded operator?}$$

In general, the answer is NO. However, let F_{high}^s denote the subspace of F^s of functions of high modulation $|\tau + \xi|\xi| + 1 \geq |\xi|$ and F_{low}^s denote the subspace of F^s of functions of low modulation $|\tau + \xi|\xi| + 1 \leq |\xi|$. Then, if $\partial_x \Phi \in L_x^2 L_t^\infty$,

$$\cdot e^{i\Phi} : F_{\text{high}}^s \rightarrow F^s \text{ as a bounded operator.} \quad (0.6)$$

For any function u let

$$u = u_{\text{high}} + u_{\text{low}}$$

denote the decomposition of u into high modulation + low modulation components. The main observation is

$$\left[u_{\text{low}} \cdot v_{\text{low}} \right]_{\text{low}} \equiv 0. \quad (0.7)$$

Then, (0.6) and (0.7), together with bilinear estimates, can be used to control expressions like

$$\partial_x(e^{i\Phi} \cdot u \cdot v),$$

which are the main contributions in the nonlinear part of the equation.