MATH 240 - Extra credit problems solutions - Summer 2009

If you have any questions, e-mail me at ostapyuk@math.ksu.edu.

1 (4 pts) (Section 1.6)

Find all solutions to the equation: $y' = 1 - x^2 + 2xy - y^2$
(Hint: You need to use a substitution other than those in the book).

Solution:
$y' = 1 - (y-x)^2$, make a substitution $v = y - x$, so $y' = v' + 1$. Then

$$v' + 1 = 1 - v^2 \qquad v' = -v^2 \qquad \frac{dv}{dx} = -v^2 \qquad -\frac{dv}{v^2} = dx$$

Inegrate both sides:

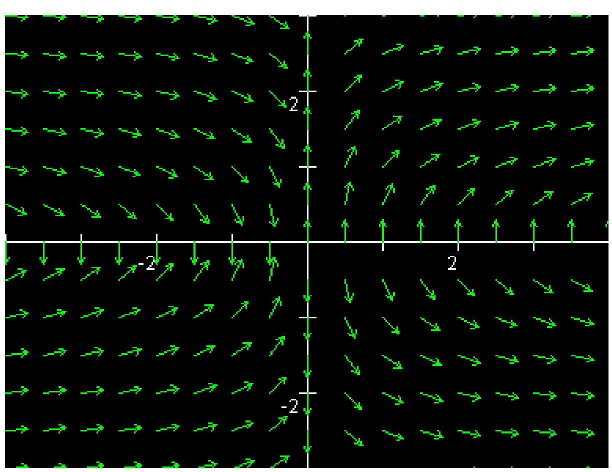
$$\frac{1}{v} = x + C \qquad v = \frac{1}{x+C} \qquad y = \frac{1}{x+C} + x$$

Singular solution: $v = 0$, i.e. $y = x$

Answer: $y = \frac{1}{x+C} + x$, $y = x$.

2 (2 pts) (Section 1.7 and Lab 1)

Below is the slope field for $y = f(x, y)$. Where is the function $f(x, y)$ positive, negative, and zero? Is $f(x, y)$ a function of x alone, y alone, or a function of both variables together? Find a function $f(x, y)$ whose slope field looks like this?



Solution:
$f(x,y)$ is positive in the 1^{st} and 3^{d} quadrants, negative in the 2^{d} and 4^{th}, never zero. $f(x,y)$ is a function of both variables. Note that on both axes arrows are vertical, so function is not defined when $x = 0$ and when $y = 0$.

Answer: $f(x,y) = \frac{1}{xy}$

3 (4 pts) (Section 2.11)

A mass of $1kg$ is attached to a spring causing it to stretch $5cm$. There is damping, but it is unknown. The spring is then set into free motion. What is the range of the possible values for the quasiperiod? Use $g = 9.8m/sec^2$. (The quasiperiod was defined in problem 11 on the page 112 in the textbook.)

Solution:
$m = 1kg$, $s = 0.05m$, so $1 \cdot 9.8 = k \cdot 0.05$ and $k = 196$. The differential equation is

$$x'' + cx' + 196x = 0$$

As in problem 11, quasiperiod is $p = \dfrac{2\pi}{\omega}$, where $\omega = \dfrac{\sqrt{4mk - c^2}}{2m}$, i.e. the quasiperiod is

$$p = \frac{4\pi}{\sqrt{4 \cdot 196 - c^2}}$$

Since c is positive

$$p > \frac{4\pi}{\sqrt{4 \cdot 196}} = \frac{4\pi}{28} = \frac{\pi}{7}$$

Answer: $(\frac{\pi}{7}, \infty)$

4 (2 pts) (Section 3.1)

Find the Laplace transform: $\mathcal{L}\{\sin t \cos t\} =$

Solution:

$$\mathcal{L}\{\sin t \cos t\} = \mathcal{L}\{\frac{1}{2}\sin(2t)\} = \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}$$

5 (2 pts) (Section 3.5)

Simplify the following: $u(t-2)u(t-3)u(t-7) =$

Solution:
Note that when $t < 7$ then $u(t-7) = 0$ and $u(t-2)u(t-3)u(t-7) = 0$. When $t > 7$ then $u(t-7) = u(t-3) = u(t-2) = 1$ and $u(t-2)u(t-3)u(t-7) = 1$. So

$$u(t-2)u(t-3)u(t-7) = \begin{cases} 0, & t < 7 \\ 1, & t > 7 \end{cases} = u(t-7)$$

6 (4 pts) (Section 3.6)

Do problem 10 on the page 162 from the texbook.

Solution:
Apply Laplace transform to both sides of the equation, then use convolutions:

$$\mathcal{L}\{x\} = \frac{1}{s^2 + 4\pi^2}\mathcal{L}\{(-1)^{[t]}\}$$

$$x(t) = \frac{1}{2\pi}\sin(2\pi t) * \left((-1)^{[t]}\right) = \frac{1}{2\pi}\int_0^t \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau$$

The most difficult part is to compute the integral. Note that

$$(-1)^{[\tau]} = \begin{cases} 1, & 2n \le \tau < 2n+1 \\ -1, & 2n+1 \le \tau < 2n+2 \end{cases}$$

The integral can be represented as a sum:

$$\int_0^t \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau = \int_0^1 \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau +$$

$$\int_1^2 \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau + \ldots + \int_{[t]}^t \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau$$

Note that $\sin(2\pi t)$ has a period 1, so

$$\int_n^{n+1} \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau = 0 \qquad \forall n$$

The only nonzero integral is

$$\int_{[t]}^t \sin(2\pi(t-\tau))\left((-1)^{[\tau]}\right) d\tau = \left((-1)^{[t]}\right)\int_{[t]}^t \sin(2\pi(t-\tau))d\tau =$$

$$\left((-1)^{[t]}\right)\frac{1}{2\pi}\cos\left(2\pi(t-\tau)\right)\Big|_{[t]}^{t}=\left((-1)^{[t]}\right)\frac{1}{2\pi}\left(1-\cos\left(2\pi(t-[t])\right)\right)$$

So, the **solution** is

$$x(t)=\left((-1)^{[t]}\right)\frac{1}{4\pi^2}\left(1-\cos\left(2\pi(t-[t])\right)\right)$$