§6 Padé Approximations

As pointed out in section 1, from the differential equation

$$y'' - xy = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

we can determine $y^{(n)}(0)$ for every n. Knowledge of these derivatives lets us write the Taylor polynomial approximations to the solution $y(x)$. But the Taylor polynomials are not necessarily the best approximations we can build with this information. Suppose instead of building a polynomial we set out to build a rational function, that is a function which is a fraction of two polynomials

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The Taylor polynomial of degree n about x_0 for a function $y(x)$ is the polynomial, $T(x)$, of degree n with $T^{(j)}(x_0) = y^{(j)}(x_0)$ for $j = 0, 1, \ldots, n$. A Padé approximation of degree n about x_0 for a function $y(x)$ is a rational function $r(x) = p(x)/q(x)$ with degree$(p(x)) = j$ and degree$(q(x)) = k$ with $j + k = n$ with $r^{(j)}(x_0) = y^{(j)}(x_0)$ for $j = 0, 1, \ldots, n$. Note that the Taylor polynomial is an example of a Padé approximation, with the degree of the denominator set to 0. While it takes more computational work to find a general Padé approximation than a Taylor polynomial, once found it is usually quicker and more accurate to evaluate a rational approximation. In this lab you will both find a few very simple Padé approximations and use the computer to compare the accuracy and speed of the different approximations. Get into matlab and enter the command **xc12**. This will bring up a window very similar to that in the previous section (lab12). Enter the equation

$$(x^2 + 2x + 1)y'' + (x + 1)y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

which was one of the problems from the last section. Experiment with different settings for the degree of the numerator and denominator in the rational approximation. Note both which approximations are the most accurate and which take the fewest flops (floating point operations) to compute. You can plot the true solution by entering the commands **x=-1:.01:1; y=cos(log(x+1)); plot(x,y)** in the Matlab command window if you want to compare the approximations to the true solution.

Exercises:

(1) Find a rational function $r(x) = p(x)/q(x)$ with degree$(p(x)) = 2$ and degree$(q(x)) = 1$ with $r(0) = 1$, $r'(0) = 0$, $r''(0) = -1$ and $r'''(0) = 0$.

(2) Find a rational function $r(x) = p(x)/q(x)$ with degree$(p(x)) = 2$ and degree$(q(x)) = 1$ with $r(0) = 0$, $r'(0) = 1$, $r''(0) = 0$ and $r'''(0) = -1$.

(3) Find a rational function $r(x) = p(x)/q(x)$ with degree$(p(x)) = 2$ and degree$(q(x)) = 1$ with $r(0) = 1$, $r'(0) = 1$, $r''(0) = 1$ and $r'''(0) = 1$.

(4) Find a rational function $r(x) = p(x)/q(x)$ with degree$(p(x)) = 1$ and degree$(q(x)) = 2$ with $r(0) = 1$, $r'(0) = 1$, $r''(0) = 1$ and $r'''(0) = 1$.

(5) Write a paragraph about the behavior of the Padé approximations to the solution $y = \cos(\log(x+1))$ of

$$(x^2 + 2x + 1)y'' + (x+1)y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

§7 Euler Equations

Discussion: We will now begin to work on solutions to differential equations near singular points. The simplest examples we can easily solve are called Euler equations. A differential equation is called an **Euler equation** if it can be written in the form $a_n x^n y^{(n)}(x) + \cdots + a_1 xy' + a_0 y = f(x)$. We will be interested primarily in second order homogeneous Euler equations. To find the general solution to such an equation, we need two linearly independent solutions. One way to find such solutions, which is very quick if it works, is to guess what they are. The key idea is to notice that if $y(x) = x^r$, then $x^n y^{(n)}(x) = r!/(r-n)!x^r$ for $r \geq n$. But then every term in the Euler equation will be a constant times x^r and we can hope to choose r so that they all cancel out. That is the strategy.

Paradigm: $x^2y'' + 4xy' + 2y = 0$

STEP 1: Guess $y(x) = x^r$ and plug into the equation.

$$\begin{aligned} x^2y'' + 4xy' + 2y &= x^2(r(r-1)x^{r-2}) + 4x(rx^{r-1}) + 2x^r \\ &= (r(r-1) + 4r + 2)x^r \\ &= (r^2 + 3r + 2)x^r \overset{\text{set}}{=} 0 \end{aligned}$$

STEP 2: Solve for r.

The roots of $r^2 + 3r + 2 = 0$ are $r = -1$ and $r = -2$. So two solutions are x^{-1} and x^{-2}.

STEP 3: If you have found two distinct real roots, r_1 and r_2, then the general solution is $y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

The general solution is therefore $y(x) = c_1 x^{-1} + c_2 x^{-2}$.

EXAMPLE: Find the general solution to

$$x^2 y'' + xy' - n^2 y = 0$$

(n is an integer constant. This equation arises in solving Laplace's Equation $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ on the disk. This should seem reasonable if you recall that in polar coordinates $\Delta = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial/\partial\theta^2$.)

Step 1:

$$\begin{aligned} x^2 y'' + xy' - n^2 y &= x^2 r(r-1)x^{r-2} + xrx^{r-1} - n^2 x^r \\ &= r^2 x^r - n^2 x^r \\ &= (r^2 - n^2)x^r \overset{\text{set}}{=} 0 \end{aligned}$$

Step 2: The roots of $r^2 - n^2 = 0$ are $r = \pm n$. So the two solutions are x^n and x^{-n}.

Step 3: The general solution is $y(x) = c_1 x^n + c_2 x^{-n}$.

You can check using the Wronskian that x^{r_1} and x^{r_2} are linearly independent if $r_1 \neq r_2$. If the equation is of higher order than second, this paradigm still works, you just need n distinct roots of the equation to get n linearly independent functions to build the general solution of an n^{th} order equation. Now what if the roots are not real and distinct? In the case where the roots are complex conjugates we can treat them as we have always treated complex roots. We find a complex solution and take its real and imaginary parts to obtain two real solutions. Of course, then we have to work with quantities like x^i. But this isn't as bad as it seems, we just use $x^i = e^{i \log x}$. (Actually, taking a logarithm in the complex plane can be a dangerous thing, but we won't worry about that now. You should take the introductory complex variables class sometime though and learn all about it.) But what if the roots are repeated? In that case we have no obvious guess for how to find a second root. There are several ways to deal with this. The way we will use is to make a change of variables to reduce the original equation to a constant coefficient equation. This technique will work on any Euler equation, not just those with repeated roots. But the first paradigm is more efficient when it works.

Paradigm: (Take 2) $x^2y'' + 7xy' + 9y = 0$.

STEP 1: Make the change of variables $z = \log x$ or equivalently $x = e^z$.

The crucial calculations here are how the derivatives of y with respect to x convert to derivatives of y with respect to z. We first compute

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

where we have used the chain rule. So $xdy/dx = dy/dz$. Next we tackle the second derivative. Here we must remember we want to write the derivative with respect to x of the derivative with respect to x in terms of the derivative with respect to z of the derivative with respect to z.

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{1}{x}\frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)\\ &= \frac{1}{x}\left(\frac{d(1/x)}{dz}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\right)\\ &= \frac{1}{x}\left(-\frac{1}{x}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\right)\\ &= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\end{aligned}$$

where we used the product rule in the third line. So $x^2d^2y/dx^2 = d^2y/dz^2 - dy/dz$. Applying this rule, along with our computation for dy/dx we find the equation becomes

$$\begin{aligned}&\frac{d^2y}{dz^2} - \frac{dy}{dz} + 7\frac{dy}{dz} + 9y\\ =\ &\frac{d^2y}{dz^2} + 6\frac{dy}{dz} + 9y = 0.\end{aligned}$$

STEP 2: Solve the transformed equation.

Since $D^2 + 6D + 9 = 0$ has a double root of -3, the solution is $c_1e^{-3z} + c_2ze^{-3z}$.

STEP 3: Back transform to find the answer in terms of the original variable.

$$\begin{aligned}c_1e^{-3z} + c_2ze^{-3z} &= c_1e^{-3\log x} + c_2(\log x)e^{-3\log x}\\ &= c_1x^{-3} + c_2(\log x)x^{-3}\end{aligned}$$

The crucial step to note here is that while $xdy/dx = dy/dz$, $x^2d^2y/dx^2 = d^2y/dz^2 - dy/dz$. So the leading coefficient and the constant coefficient will remain unchanged when the equation is transformed, but the coefficient of dy/dx will be changed because it picks up not only the dy/dx term but also part of the d^2y/dx^2 term. The two easiest errors to make in solving Euler equations using the second paradigm is either to forget to change the middle coefficient in transforming the equation or to forget to back substitute to get the answer in terms of the original variable.

Of course we have only dealt with the homogeneous case in these paradigms. To handle the inhomogeneous equation, we can either use the second paradigm, make the change of variables on the right hand side as well and hope we can apply undetermined coefficients (which isn't all that likely), or we can just use variation of parameters.

EXAMPLE: $x^2y'' + 5xy' + 3y = e^x$

We will use the first paradigm and variation of parameters.

Step 1: $y(x) = x^r$, so $xy'(x) = rx^r$ and $x^2y''(x) = r(r-1)x^r$. Plugging these into the equation we obtain $(r^2 + 4r + 3)x^r = 0$

Step 2: The roots of $r^2 + 4r + 3 = 0$ are $r = -3$ and $r = -1$.

Step 3: Two linearly independent solutions of the homogeneous equation are x^{-3} and x^{-1}.

Now that we have two linearly independent homogeneous solutions, we want to use the variation of parameters formula to find the general solution. First we note that the formula on p. 100 only applies to equations where the coefficient of the leading term is 1. So we divide through our Euler equation by x^2 to put it in the correct form,

$$y'' + 5x^{-1}y' + 3x^{-2}y = x^{-2}e^x.$$

Our linearly independent solutions of the homogeneous equation, as noted in step 3, are $y_1(x) = x^{-3}$ and $y_2(x) = x^{-1}$. We compute $W(x^{-3}, x^{-1})(x) = -x^{-5} + 3x^{-5} = 2x^{-5}$. The right hand side is $g(x) = x^{-2}e^x$. From our formula for variation of parameters we then find the general solution is

$$\begin{aligned}
y(x) &= -x^{-3}\int_0^x \frac{s^{-1}s^{-2}e^s}{2s^{-5}}\,ds + x^{-1}\int_0^x \frac{s^{-3}s^{-2}e^s}{2s^{-5}}\,ds + C_1x^{-3} + C_2x^{-1} \\
&= \frac{-x^{-3}}{2}\int_0^x s^2e^s\,ds + \frac{x^{-1}}{2}\int_0^x e^s\,ds + C_1x^{-3} + C_2x^{-1} \\
&= \frac{-x^{-3}}{2}(x^2e^x - 2xe^x + 2e^x - 2) + \frac{x^{-1}}{2}(e^x - 1) + C_1x^{-3} + C_2x^{-1} \\
&= \frac{-x^{-1}}{2}e^x + x^{-2}e^x - x^{-3}e^x + \frac{x^{-1}}{2}e^x + (C_1+2)x^{-3} + (C_2 - 1/2)x^{-1} \\
&= x^{-2}e^x - x^{-3}e^x + c_1x^{-3} + c_2x^{-1}
\end{aligned}$$

where $c_1 = C_1 + 2$ and $c_2 = C_2 - 1/2$.

Exercises:

(1) $x^2y'' + 3xy' - 8y = 0$

(2) $x^2y'' + 3xy' - y = 0$

(3) $x^2y'' - 2y = 0$

(4) $x^2y'' - xy' + y = 0$

(5) $2x^2y'' + 5xy' + y = 0$

(6) $x^2y'' + xy' - y = 0$

(7) $x^2y'' + xy' - 3y = 0$

(8) $x^2y'' + 8xy' + 3y = 0$

(9) $x^2y'' + 2xy' - 12y = 0$

(10) $3x^2y'' + 13xy' + 3y = 0$

(11) $x^2y'' + 5xy' + 4y = 0$

(12) $2x^2y'' + xy' - 1y = 0$

(13) $x^2y'' + 7xy' + 8y = 0, \quad y(1) = 1, \quad y'(1) = 0$

(14) $x^2y'' - 2xy' + 2y = 0, \quad y(2) = 0, \quad y'(2) = 4$

(15) $x^2y'' + xy' - 4y = 0, \quad y(-1) = 1, \quad y'(-1) = 0$

(16) $x^2y'' + 4xy' - 10y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$

(17) $x^2y'' - xy' - 3y = x^2$

(18) $x^2y'' - xy' - 3y = e^{-x}$

(19) $x^2y'' + xy' - y = x$

(20) $x^2y'' + xy - y = e^x$

§8 Regular Singular Points

Discussion: We have seen earlier in this chapter how to find series solutions about ordinary points. While those techniques don't work in general around singular points, there are different techniques that will work around some singular points. The key is whether the differential equation looks like an Euler equation near the singular point. If it does, then we can guess the correct form of the series solution by considering an Euler equation. If it doesn't look like an Euler equation, we are probably out of luck. There are no general techniques for that case.

Obviously, it is important to determine whether an equation looks like an Euler equation in the neighborhood of a singular point. If it does, we say the singular point is a regular singular point. Otherwise the singular point is an irregular singular point. The precise definition is the following.

Definition. *A point x_0 is a* **regular singular point** *for the linear differential equation $p(x)y'' + q(x)y' + r(x)y = 0$ if $p(x)$, $q(x)$ and $r(x)$ are all analytic in a neighborhood of x_0 and the following limits exist:*

$$\lim_{x \to x_0} \frac{(x - x_0)q(x)}{p(x)} \stackrel{\text{def}}{=} a_0 \qquad \text{and} \qquad \lim_{x \to x_0} \frac{(x - x_0)^2 r(x)}{p(x)} \stackrel{\text{def}}{=} b_0$$

If either limit fails to exist, x_0 is an **irregular singular point**.

As before, don't worry about functions being "analytic." Any polynomial is analytic as is any rational, exponential, logarithmic or trigonometric function away from their singularities. The $\stackrel{\text{def}}{=}$ symbol means that a_0 and b_0 are defined to be the values of those limits (if they exist). We will use a_0 and b_0 in what follows.

The rationale behind this definition (other than that it is the one that makes the theorems we have later true) is the following. Divide through the equation $p(x)y''+q(x)y'+r(x)y = 0$ by $p(x)$ and multiply by $(x - x_0)^2$ and you obtain

$$(x - x_0)^2 y'' + (x - x_0) \left(\frac{(x - x_0)q(x)}{p(x)} \right) y' + \left(\frac{(x - x_0)^2 r(x)}{p(x)} \right) y = 0.$$

Now assuming x_0 is a regular singular point, we make the substitution $u = x - x_0$ and then can approximate this equation by the associated Euler equation

$$u^2 y'' + a_0 u y' + b_0 y = 0.$$

So the differential equation does indeed look like an Euler equation in the neighborhood of a regular singular point as promised. We will discuss how to use this information to obtain a series solution about a regular singular point in the next section.

Paradigm: Find and classify the singular points of $(x^3 - 3x^2)y'' + y' + 2y = 0$.

STEP 1: Find the singular points.

Since all the coefficients are polynomials, the singular points are just the roots of the leading coefficient, in this case $x^3 - 3x^2$. So the singular points are 0 and 3.

STEP 2: For each singular point, classify it as regular or irregular by computing the appropriate limits.

For $x = 0$,

$$\lim_{x\to 0} \frac{(x-0)1}{x^3 - 3x^2} = \lim_{x\to 0} \frac{1}{x^2 - 3x} \text{ is undefined.}$$

so $x = 0$ is an irregular singular point.

For $x = 3$,

$$\lim_{x\to 3} \frac{(x-3)1}{x^3 - 3x^2} = \lim_{t\to 3} \frac{1}{x^2} = \frac{1}{9}$$
$$\lim_{x\to 3} \frac{(x-3)^2 2}{x^3 - 3x^2} = \lim_{x\to 3} \frac{2(x-3)}{x^2} = 0$$

so both limits exist and $x = 3$ is a regular singular point.

Exercises: Find and classify the singular points of the following equations.

(1) $x^2y'' + 4xy' + 3y = 0$

(2) $(x^2 - 2x - 3)y'' + (x^2 + 4x)y' + 7y = 0$

(3) $y'' + 3x^2y' - 2xy = 0$

(4) $(x^3 + 3x^2 + 3x + 1)y'' + (x^3 + 2x^2 + x)y' + (x^2 - 1)y = 0$

(5) $(x^4 - 2x^2 + 1)y'' + (x^2 + 3x + 2)y' + (x^3 + 1)y = 0$

(6) $(x^2 - 6x + 9)y'' + (x^2 - 9)y' - xy = 0$

(7) $(x^3 + 7x^2 + 15x + 9)y'' + (x^2 + 4x + 4)y' + (x + 3)y = 0$

(8) $(x^2 - 4x + 4)y'' + (x^2 - 4)y' + xy = 0$

(9) $(x^3 + x^2 - 8x - 12)y'' + (x^2 - x - 2)y' + 3xy = 0$

(10) $(x^3 + x)y'' + 3y' - xy = 0$

(11) $(4x^3 + 16x^2 + 21x + 9)y'' + (2x^2 + 5x + 3)y' + (x^2 - 1)y = 0$

(12) $(x^2 + 2x + 1)y'' - (x^2 - 1)y' + y = 0$

(13) $x^3y'' + x^2y' + xy = 0$

(14) $x^3y'' + x^2y' + y = 0$

(15) $y'' + (4/x)y' + (x/4)y = 0$

(16) $\cos(x)y'' + \sin(x)y' + e^x y = 0$

(17) $(\cos(x) - 1)y'' + \sin(x)y' + e^x y = 0$

(18) $(\cos(x) - 1)y'' + xy' + e^x y = 0$

(19) $\cosh(x)y'' + y' + y = 0$

(20) $e^x y'' + \sin(x)y' + \cos(x)y = 0$

§9 Series Solutions about Regular Singular Points

Discussion: We are finally ready to work out series solutions about regular singular points. About ordinary points we guess the form of the solution is $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, plug that into the equation and solve for the a_n. About regular singular points we do pretty much the same thing, except we must pick a different form for the solution. We pick the form in analogy with Euler equations. If x_0 is a regular singular point for the equation $p(x)y'' + q(x)y' + r(x)y = 0$ with $\lim_{x\to x_0}(x - x_0)q(x)/p(x) = a_0$ and $\lim_{x\to x_0}(x - x_0)^2 r(x)/p(x) = b_0$, then near the point x_0 the equation $p(x)y'' + q(x)y' + r(x)y = 0$ should behave similarly to the equation $(x - x_0)^2y'' + a_0(x - x_0)y' + b_0y = 0$, called the associated Euler equation. We can solve this Euler equation and find it has solutions that look like cx^r where r is a root of the **indicial equation** $r^2 + (a_0 - 1)r + b_0 = 0$ and c is a constant (remember that r as used in this sentence is a constant, not to be confused with the function $r(x)$ that appears in the original equation). So near the point x_0, the solution to $p(x)y'' + q(x)y' + r(x)y = 0$ should look like cx^r. Now the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ looks like the constant a_0 near x_0, since all the other terms drop out. If we want to make it look like a constant times x^r, the obvious guess is to multiply through by x^r. And that is exactly the right guess to make. In the paradigm that follows we will assume that the roots of the associated Euler equation are real. If they are complex, we can still find a solution, but the situation becomes messier than I care to think about. If you are curious about the complex valued case, stop by my office and I'll go over it with you.

Paradigm: Solve $2x^2y'' - xy' + (x+1)y = 0$ about $x_0 = 0$.

STEP 1: Check that x_0 is a regular singular point. (If it isn't, give up and call a mathematician for help.)

$$\lim_{x\to 0}\frac{(x-0)(-x)}{2x^2} = -\frac{1}{2}$$
$$\lim_{x\to 0}\frac{(x-0)^2(x+1)}{2x^2} = \frac{1}{2}$$

Both limits exist so $x_0 = 0$ is a regular singular point.

STEP 2: Find the roots of the indicial equation, $r^2 + (a_0 - 1)r + b_0 = 0$.

The indicial equation in this case is $r^2 - 3/2r + 1/2 = 0$ and it has roots $r = 1$ and $r = 1/2$.

STEP 3: Guess that a solution takes the form $(x-x_0)^r \sum_{n=0}^{\infty} a_n(x-x_0)^n$ with $a_0 = 1$ using the *larger* r found in step 2. (This assumes the roots are real). Then plug in and solve for the a_n.

The process of carrying out step 3 is very similar to the process of finding the series solution about an ordinary point.

SubStep 1: Make the guess and compute all the pieces.

$$y = x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y' = \sum_{n=0}^{\infty}(n+1)a_n x^n$$
$$y'' = \sum_{n=0}^{\infty}(n+1)n a_n x^{n-1}$$

and from these we get

$$(x+1)y = \sum_{n=0}^{\infty} a_n x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$-xy' = \sum_{n=0}^{\infty} -(n+1)a_n x^{n+1}$$
$$2x^2y'' = \sum_{n=0}^{\infty} 2(n+1)n a_n x^{n+1}$$

SubStep 2: Change all terms to the original form of y.

In this case the original form involves x^{n+1}. So we want all the pieces to be sums of x^{n+1} or x^{j+1} or some such term. So let $j = n+1$. Then we have

$$\begin{aligned} (x+1)y &= \sum_{j=1}^{\infty} a_{j-1}x^{j+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ -xy' &= \sum_{n=0}^{\infty} -(n+1)a_n x^{n+1} \\ 2x^2y'' &= \sum_{n=0}^{\infty} 2(n+1)na_n x^{n+1} \end{aligned}$$

SubStep 3: Change all indicies to the same letter (I use m) and plug into the equation.

$$2x^2y'' - xy' + (x+1)y = \begin{matrix} \sum_{m=0}^{\infty} 2(m+1)ma_m x^{m+1} + \sum_{m=0}^{\infty} -(m+1)a_m x^{m+1} \\ + \sum_{m=1}^{\infty} a_{m-1}x^{m+1} + \sum_{m=0}^{\infty} a_m x^{m+1} \end{matrix} = 0$$

SubStep 4: Collect like terms.

$$(0a_0 - 1a_0 + 1a_0)x^1 + \sum_{m=1}^{\infty} (2(m+1)ma_m - (m+1)a_m + a_{m-1} + a_m)x^{m+1} = 0$$

SubStep 5: Equate coefficients to 0 and solve for the recurrence relation.

$$\begin{aligned} 0 &= 0 \qquad (m = 0 \text{ term}) \\ (2m^2 + m)a_m + a_{m-1} &= 0 \qquad (\text{general term}) \end{aligned}$$

Solving the last equation for a_m, the highest order term, we find the recurrence relation

$$a_m = -\frac{a_{m-1}}{2m^2 + m}.$$

SubStep 6: Plug in $a_0 = 1$ to find the solution.

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -\frac{1}{3} \\ a_2 &= -\frac{-1/3}{10} = \frac{1}{30} \\ a_3 &= -\frac{1/30}{21} = -\frac{1}{630} \\ &\vdots \end{aligned}$$

So the solution is

$$y_1(x) = x - \frac{1}{3}x^2 + \frac{1}{30}x^3 - \frac{1}{630}x^4 + \ldots$$

STEP 4: If the smaller root of the indicial equation doesn't differ from the larger root by an integer, find a second solution using the smaller root. (If it does differ by an integer, the process of finding a second solution is much harder. Stop by my office for details if you are interested. Fortunately, in applications the solution corresponding to the larger root is usually more interesting if the two roots differ by an integer. There is a good reason for this, and again you can stop by my office if you are curious.)

The substeps here are the same as for step 3 and so we will just run through them quickly and without comment.

SubStep 1:

$$\begin{aligned} y &= x^{1/2}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1/2} \\ y' &= \sum_{n=0}^{\infty}(n+1/2)a_n x^{n-1/2} \\ y'' &= \sum_{n=0}^{\infty}(n+1/2)(n-1/2)a_n x^{n-3/2} \end{aligned}$$

So we get

$$\begin{aligned} (x+1)y &= \sum_{n=0}^{\infty} a_n x^{n+3/2} + \sum_{n=0}^{\infty} a_n x^{n+1/2} \\ -xy' &= \sum_{n=0}^{\infty} -(n+1/2)a_n x^{n+1/2} \\ 2x^2y'' &= \sum_{n=0}^{\infty} 2(n+1/2)(n-1/2)a_n x^{n+1/2} \end{aligned}$$

SubStep 2: Let $j = n + 1$.

$$\begin{aligned}
(x+1)y &= \sum_{j=1}^{\infty} a_{j-1} x^{j+1/2} + \sum_{n=0}^{\infty} a_n x^{n+1/2} \\
-xy' &= \sum_{n=0}^{\infty} -(n+1/2) a_n x^{n+1/2} \\
2x^2 y'' &= \sum_{n=0}^{\infty} 2(n+1/2)(n-1/2) a_n x^{n+1/2}
\end{aligned}$$

SubStep 3:

$$\begin{aligned}
x^2 y'' - xy' + (x+1)y = &\sum_{m=0}^{\infty} 2(m+1/2)(m-1/2) a_m x^{m+1/2} \\
&+ \sum_{m=0}^{\infty} -(m+1/2) a_m x^{m+1/2} \\
&+ \sum_{m=1}^{\infty} a_{m-1} x^{m+1/2} \\
&+ \sum_{m=0}^{\infty} a_m x^{m+1/2} \\
= 0
\end{aligned}$$

SubStep 4:

$$\begin{aligned}
&(-1/2a_0 - 1/2a_0 + a_0) x^{1/2} \\
&+ \sum_{m=1}^{\infty} (2(m+1/2)(m-1/2)a_m - (m+1/2)a_m + a_{m-1} + a_m) x^{m+1/2}
\end{aligned} = 0$$

SubStep 5:

$$\begin{aligned}
0 &= 0 \qquad (m = 0 \text{ term}) \\
(2m^2 - m) a_m + a_{m-1} &= 0 \qquad (\text{general term})
\end{aligned}$$

We solve the last equation for the recurrence relation

$$a_m = -\frac{a_{m-1}}{2m^2 - m}$$

SubStep 6:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= -\frac{1}{2-1} = -1 \\
a_2 &= -\frac{-1}{8-2} = \frac{1}{6} \\
a_3 &= -\frac{1/6}{18-3} = -\frac{1}{90} \\
&\vdots
\end{aligned}$$

So the solution is

$$y_2(x) = x^{1/2} - x^{3/2} + \frac{1}{6}x^{5/2} - \frac{1}{90}x^{7/2} + \dots$$

STEP 5: The general solution is $c_1 y_1(x) + c_2 y_2(x)$.

EXAMPLE: Find one solution of Bessel's equation of order 0, $x^2y'' + xy' + x^2y = 0$ about $x_0 = 0$.

Step 1:

$$\begin{aligned}
\lim_{x\to 0} \frac{x \cdot x}{x^2} &= 1 = a_0 \\
\lim_{x\to 0} \frac{x^2 \cdot x^2}{x^2} &= 0 = b_0
\end{aligned}$$

So $x_0 = 0$ is a regular point.

Step 2: $r^2 + (1-1)r + 0 = 0$ has a double root at $r = 0$.

Step 3: Since the largest (and only) root of the indicial equation is 0, we try $y = x^0 \sum_{n=0}^{\infty} a_n x^n$.

SubStep 1:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.
\end{aligned}$$

Plugging these in we get

$$x^2y = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$
$$x^2y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n.$$

SubStep 2: Let $j = n + 2$ so

$$x^2y = \sum_{j=2}^{\infty} a_{j-2} x^j$$
$$xy' = \sum_{n=0}^{\infty} n a_n x^n$$
$$x^2y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n.$$

SubStep 3:

$$x^2y'' + xy' + x^2y = \sum_{m=0}^{\infty} m(m-1)a_m x^m + \sum_{m=0}^{\infty} m a_m x^m + \sum_{m=2}^{\infty} a_{m-2} x^m = 0$$

SubStep 4:

$$0a_0x^0 + 1a_1x^1 + \sum_{m=2}^{\infty} \big(m(m-1)a_m + m a_m + a_{m-2}\big) x^m = 0$$

SubStep 5:

$$0 = 0$$
$$a_1 = 0$$
$$m^2 a_m + a_{m-2} = 0, \qquad m \geq 2$$

The recurrence relation is

$$a_m = -\frac{a_{m-2}}{m^2}, \qquad m \geq 2$$

SubStep 6:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= -\frac{1}{4} \\ a_3 &= 0 \\ a_4 &= \frac{1}{64} \\ &\vdots \end{aligned}$$

$$J_0(x) = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots$$

Step 4: Since the indicial equation has a double root, the two roots differ by an integer, 0, and so we stop here.

No, you can't claim 1 point for a misprint for my writing $J_0(x)$ instead of $y_1(x)$ in the last example. The equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ is called Bessel's equation of order ν. It arises in solving Laplace's equation in cylindrical coordinates. $J_0(x)$ is the standard notation for the Bessel function of the first kind of order 0, which is the solution we just computed. A second linearly independent solution is $Y_0(x)$, the Bessel function of the second kind of degree 0. Since when we have a double root in an Euler equation we tack on a log term, it shouldn't be too surprising that $Y_0(x)$ has a logarithmic singularity at $x = 0$. Accordingly, if we want a bounded solution near $x = 0$, we want J_0 which is why it is accorded the privilege of being of the *first* kind. Bessel functions fall just after the trigonometric functions in their importance in practical applications. (They've even made a short appearance on "Star Trek, The Next Generation.") You can read much much more about Bessel functions in "A Treatise on Bessel Functions" by Watson.

Exercises: Find the roots of the indicial equations for the following problems.

(1) $x^2y'' + (x-2)y = 0$ about $x_0 = 0$.

(2) $(x^3 - x^2)y'' + (x^2 + x)y' + (2x - 4)y = 0$ about $x_0 = 0$.

(3) $(x^2 + 2x + 1)y'' + (x^2 - 1)y' + 4y = 0$ about $x_0 = -1$.

(4) $xy'' + 3y' + xy = 0$ about $x_0 = 0$.

(5) $\sin(x)y'' + \tan(x)y' - y = 0$ about $x_0 = 0$.

(6) $(\cos(x) - 1)y'' + \sin(x)y' + (x - 4)y = 0$ about $x_0 = 0$.

(7) $(x^2 + x)y'' - (x - 1)y' + (x + 1)y = 0$ about $x_0 = 0$.

(8) $(x^3 + 6x^2 + 12x + 8)y'' - (x^2 + 7x + 5)y' + (x - 1)y = 0$ about $x_0 = -2$.

(9) $(x^3 + 9x^2 + 24x + 16)y'' + (x^2 + 3x - 4)y' + (x^2 + 2x - 1)y = 0$ about $x_0 = -4$.

(10) $(x^3 + x^2 - 5x + 3)y'' + (x^2 + x - 2)y' + (x^2 + 4x - 4)y = 0$ about $x_0 = 1$.

Find the series solution to the following problems corresponding to the larger root of the indicial equation. If the smaller root of the indicial equation doesn't differ from the larger root by an integer, then find the series solution corresponding to the smaller root and the general solution.

(11) $x^2y'' + (x^2 + x)y' - y = 0$ about $x_0 = 0$.

(12) $x^2y'' + 3xy' + (2x + 1)y = 0$ about $x_0 = 0$.

(13) $x^2y'' + (x - 1/3)y' - 1/3y = 0$ about $x_0 = 0$.

(14) $8x^2y'' - 6xy' + (x^2 + 1)y = 0$ about $x_0 = 0$.

(15) $2x^2y'' + 5xy' - 3y = 0$ about $x_0 = 0$.

§10 Convergence About Regular Singular Points

Discussion: Convergence results near a regular singular point must be different from results near an ordinary point. After all, the series probably won't converge at the singular point. If a root of the indicial equation is negative, the leading term is $a_0(x - x_0)^r$ where $r < 0$ and so the leading term is undefined. By convergence about a regular singular point, we really mean "about" and not "at." While the series may diverge at the singular point itself, we can hope it converges in a "punctured neighborhood" of the singular point, excluding the singular point. Once we adjust our sights accordingly, it turns out that series solutions about regular singular points satisfy a convergence theorem very similar to series solutions about ordinary points.

THEOREM. *Suppose* $y(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is the series solution to the differential equation* $p(x)y'' + q(x)y' + r(x)y = 0$ *where* $p(x)$, $q(x)$ *and* $r(x)$ *are all polynomials and* x_0 *is a regular singular point. Then* $y(x)$ *converges in the regions* $0 < x - x_0 < a$ *and* $-a < x - x_0 < 0$ *where* a *is the distance from* x_0 *to the next nearest singular point of the equation.*

NOTE: We must consider complex x, even though the equation is real.

This theorem is exactly the same as the corresponding theorem for ordinary points, except that the series need not converge at the point x_0 and that we use the nearest singular point to x_0 excluding x_0 itself. The same points raised in the "More Discussion" part of that section apply here as well.

Paradigm: Where will the series solution to $(x^3 + x^2)y'' + xy' + y = 0$ about $x_0 = 0$ converge?

STEP 1: Check that x_0 is a regular singular point so the theorem applies.

Since all the coefficients are polynomials and the leading coefficient, $x^3 + x^2$ is 0 at $x_0 = 0$, x_0 is a singular point. Next we check the appropriate limits.

$$\lim_{x \to 0} \frac{(x-0)x}{x^3 + x^2} = \lim_{x \to 0} \frac{1}{x+1} = 1$$
$$\lim_{x \to 0} \frac{(x-0)^2}{x^3 + x^2} = \lim_{x \to 0} \frac{1}{x+1} = 1$$

Since both limits exist, x_0 is a regular singular point.

STEP 2: Find all singular points, including those in the complex plane.

Since all coefficients are polynomial, the singular points are just the roots of the leading coefficient, $x^3 + x^2$. The roots are -1 and 0.

STEP 3: Find the distances from x_0 to the other singular points.

The only singular point other than $x_0 = 0$ is -1 which is 1 unit away from x_0.

STEP 4: The series solution converges in the regions $0 < x - x_0 < a$ and $-a < x - x_0 < 0$ where a is the smallest distance found in step 3.

Since the only distance found was 1, the series solution converges in the regions $-1 < x < 0$ and $0 < x < 1$. Note that it may also converge outside the region as well; the theorem states a region where it converges but doesn't say anything about where it diverges.

EXAMPLE: Where does the series solution of Bessel's equation of order 1, $x^2y'' + xy' + (x^2 - 1)y = 0$ converge?

Step 1: $0^2 = 0$ so $x_0 = 0$ is singular.

$$\lim_{x\to 0} \frac{(x-0)x}{x^2} = 1$$
$$\lim_{x\to 0} \frac{(x-0)^2(x^2-1)}{x^2} = \lim_{x\to 0} x^2 - 1 = -1$$

So $x_0 = 0$ is a regular singular point.

Step 2: All coefficients are polynomials so the singular points are the roots of the leading coefficient, x^2, which is just 0.

Step 3: Since 0 is the only singular point, there are no other singular points to find the distance to.

Step 4: Since there is only the one singular point at $x_0 = 0$, the series converges in the regions $0 < x < \infty$ and $-\infty < x < 0$.

This last example illustrates why one might want to find a series solution about a singular point rather than an ordinary point. If a series solution for Bessel's equation is found about any ordinary point, we can only guarantee that the series will converge in a bounded interval because the singular point at 0 will be some finite distance away from the ordinary point (see problem 10 in section 3). But the expansion about the singular point 0 itself converges in two unbounded intervals. Actually, one solution to this particular equation will converge everywhere, including at the point 0. This is a special bonus to Bessel's equation and we won't be that lucky in general.

Exercises: State where you are guaranteed the series solutions about the given points converge. (If the given point is not an ordinary point or regular singular point, just indicate that fact and don't worry about convergence.)

(1) $x^2y'' + 3xy + 2y = 0,$ about $x_0 = 0$

(2) $x^3y'' + x^2y' + (x^3 + x^2)y = 0,$ about $x_0 = 0$

(3) $(x^3 + x)y'' + (3x - 4)y' + (x^2 - 1)y = 0,$ about $x_0 = 0$

(4) $(x^3 + 7x^2 + 14x + 8)y'' + (x^2 - 3x + 1)y' + (2x - 3)y = 0,$ about $x_0 = -2$

(5) $(x^2 + 2x + 1)y'' + (x^2 + 3x + 2)y' + y = 0,$ about $x_0 = -1$

(6) $(x^2 + 2x + 1)y'' + (x^2 + 3x + 2)y' + y = 0,$ about $x_0 = 0$

(7) $x^3y'' + (x^2 + x)y' + (x + 1)y = 0,$ about $x_0 = 0$

(8) $(x^3 + 2x^2)y'' + (x - 5)y' + 3y = 0,$ about $x_0 = -2$

(9) $(x^3 + 2x^2)y'' + (x - 5)y' + 3y = 0,$ about $x_0 = -3/2$

(10) $(x^3 + 2x^2)y'' + (x - 5)y' + 3y = 0,$ about $x_0 = 0$

(11) $(x^3 - 3x^2 + 4)y'' + (x^2 - 2x)y' + (x - 1)y = 0,$ about $x_0 = 2$

(12) $(x^3 - 3x^2 + 4)y'' + (x^2 - 2x)y' + (x - 1)y = 0,$ about $x_0 = 0$

(13) $(x^2 + 4x - 21)y'' + (x^2 + 3x - 2)y' + 4y = 0,$ about $x_0 = 1$

(14) $(x^2 + 4x - 21)y'' + (x^2 + 3x - 2)y' + 4y = 0,$ about $x_0 = 0$

(15) $(x^3 + 8)y'' + (x^2 + 8)y' + (x + 8)y = 0,$ about $x_0 = 1$

(16) $(x^3 + 8)y'' + (x^2 + 8)y' + (x + 8)y = 0,$ about $x_0 = -2$

(17) $(x^2 + 4x + 4)y'' + (x^2 - 4)y' + xy = 0,$ about $x_0 = -2$

(18) $(x^2 + 4x + 4)y'' + (x^2 - 4)y' + xy = 0,$ about $x_0 = 1$

(19) $(x^3 + 5x^2 + 8x + 4)y'' + (x^2 - 4)y' + xy = 0,$ about $x_0 = -2$

(20) $(x^3 + 5x^2 + 8x + 4)y'' + (x^2 - 4)y' + xy = 0,$ about $x_0 = 0$

§11 Review Problems

Exercises: Find the general solutions of the following differential equations. For each problem, give a lower bound for the radius of convergence of the series solution.

(1) $y'' + 4y' + (x+4)y = 0$

(2) $y'' + (x-4)y' + xy = 0$

(3) $y'' - xy' - y = 0$

(4) $y'' - (x^2+x+1)y' - y = 0$

(5) $y'' - (x-2)y' + x^2y = 0$

(6) $(x+1)y'' + (x+3)y' + (x-2)y = 0$

(7) $(x^2-1)y'' + y' - (x-3)y = 0$

(8) $(x^2-4x-12)y'' + (x-1)y' + x^3y = 0$

(9) $xy'' + y = 0$

(10) $(x^2-2x-8)y'' + y' + (-x+1)y = 0$

Solve the following initial value problems. For each problem, give a lower bound for the radius of convergence of the series solution.

(11) $xy'' + y = 0, \quad y(1) = 1, \quad y'(1) = 1$

(12) $xy'' + y = 0, \quad y(1) = -1, \quad y'(1) = 2$

(13) $(x^2-2x-8)y'' + y' + (-x+1)y = 0, \quad y(0) = 2, \quad y'(0) = 0$

(14) $(x^2-2x-8)y'' + y' + (-x+1)y = 0, \quad y(0) = -1, \quad y'(0) = 1$

(15) $(x^2+1)y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -3$

(16) $y'' + 4xy' - (x^3-1)y = 0, \quad y(0) = 3, \quad y'(0) = 1$

(17) $(x^4+2x^2+1)y'' + xy' + xy = 0, \quad y(0) = -1, \quad y'(0) = 2$

(18) $y'' + xy' + 2xy = 0, \quad y(2) = 1, \quad y'(2) = 1$

The final two initial value problems are different from the other problems we have worked because they are inhomogeneous, but the instructions are the same. Solve the initial value problems and for each problem, give a lower bound for the radius of convergence of the series solution.

(19) $y'' + 2y' + (x+1)y = 2, \quad y(0) = 1, \quad y'(0) = 2$

(20) $y'' + xy' + (x^2-1)y = e^x, \quad y(0) = 0, \quad y'(0) = 4$

Find and classify the singular points of the following equations. Find the roots of the indicial equation for every regular singular point.

$$(21)\quad x^2y'' + 3xy' + 4y = 0$$
$$(22)\quad (x-1)^2y'' + (x-2)y' - 3y = 0$$
$$(23)\quad (x^3 - 3x^2 + 4x - 2)y'' + (x^2 - 5x - 1)y' + 3x^3y = 0$$
$$(24)\quad (x^4 + 2x^3 - 3x^2)y'' + (x^2 - 7x)y' + (x^2 - 4)y = 0$$
$$(25)\quad (x^4 - 8x^2 + 16)y'' + (x^3 + 4x^2 + 4x)y' + (x^2 - 1)y = 0$$
$$(26)\quad (\sin(x) - x)y'' + (\cos(x) - 1)y' + (e^x - e^{2x})y = 0$$

Find the general solution of the following equations, but do not take a series expansion about any ordinary point.

$$(27)\quad x^2y'' + 11/6xy' + 1/6y = 0$$
$$(28)\quad x^2y'' + xy' + (x^2 - 1/9)y = 0$$
$$(29)\quad x^2y'' + (x^2 + 1/2x)y' - 1/2y = 0$$
$$(30)\quad x^2y'' + xy' - 4/9y = 0$$