Chapter 4
Series Solutions

§1 Power Series

Discussion: We have earlier considered constant coefficient linear second order equations. We will now consider the much more difficult case of variable coefficients. Consider the following equation, called the Airy equation:

$$y'' - xy = 0$$

It can be shown that there is no closed form solution to this equation in terms of sines, cosines, exponentials and powers of x. Yet this is about the simplest variable coefficient second order equation there is. Note that while there is no solution in terms of functions we are familiar with, the general theory assures us there are lots of solutions; they just can't be written in terms of familiar functions. This being the case, we will have to find some way to decide what these solutions look like without being able to write them out explicitly in finite form. Consider the following initial value problem

$$\begin{aligned} y'' - xy &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

Substituting in the initial values at $x = 0$ we find

$$y''(0) - 0 \times 1 = 0 \qquad \text{or} \qquad y''(0) = 0$$

Differentiating the differential equation we find

$$y''' - xy' - y = 0$$

and substituting in the initial values we find

$$y'''(0) - 0 \times 2 - 1 = 0 \qquad \text{or} \qquad y'''(0) = 1$$

Differentiating the differential equation a second time yields

$$y'''' - xy'' - 2y' = 0$$

and substituting in the initial values we find

$$y''''(0) - 0 \times 0 - 2 \times 2 = 0 \qquad \text{or} \qquad y''''(0) = 4$$

We can continue in this fashion and find the values of all the derivatives of y at the point 0. Then we can write out the Taylor series for $y(x)$ about 0

$$y(x) = y(0) + y'(0)x + (1/2)y''(0)x^2 + (1/6)y'''(0)x^3 + \dots$$
$$y(x) = 1 + 2x + 1/6x^3 + \dots$$

This will give us a solution to our equation, though not in a finite form. We can then look at the partial sums of the Taylor series, which are polynomials and can be easily graphed, and use these as convenient approximations to the true solution.

At this point it is useful to recall some facts about power series from Calculus II.

A **power series** is an infinite series of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

The power series is said to **converge** at a point x if the sequence of partial sums converges at the point x. The series **converges absolutely** at the point x if the sequence of partial sums of the absolute values of the terms converges at the point x. For every power series there is a **radius of convergence** r. The power series converges absolutely for $|x-x_0| < r$ and diverges for $|x - x_0| > r$. The series may converge or diverge at $|x - x_0| = r$. Note that the radius of convergence must always be non-negative. One way to determine the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ at x is the **ratio test**. Compute

$$\lim_{n\to\infty} \frac{a_{n+1}x^{n+1}}{a_n x^n}.$$

If the limit is less than 1, then the series converges. If the limit is greater than 1, then the series diverges. If the limit is equal to 1, then we are at the boundary of the regions of convergence and divergence, so we are at the radius of convergence.

Within the radius of convergence, the derivative of a power series may be taken term by term. That is, if

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

then

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots$$

so long as $|x - x_0| < r$. Note that here we just differentiated every term in the power series and then summed the terms. The derived power series will converge absolutely in the range $|x - x_0| < r$. By plugging x_0 into the original series we see $a_0 = f(x_0)$ and by plugging into the derived series we see $a_1 = f'(x_0)$. In general, $a_n = f^{(n)}(x_0)/n!$. So the power series is the Taylor series of its sum. Two power series are equal if and only if each term is equal, i.e.

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \dots$$

if and only if $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, etc. Two power series about the same point x_0 may be added term by term, if

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

and $$g(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + b_3(x - x_0)^3 + \dots$$

then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)(x - x_0) + (a_2 + b_2)(x - x_0)^2 + \dots$$

The radius of convergence of the summed series is at least as large as the smaller of the radii of convergence of the individual series (and possibly larger).

The rule for multiplication is somewhat more difficult.

$$f(x) \times g(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

where

$$c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \dots + a_0 b_n$$

The radius of convergence of the product series is at least as large as the smaller of the radii of convergence of the individual series (and possibly larger).

The sequence c_n is called the **convolution** of the sequences a_n and b_n. This is the discrete analogue of the integral formula for convolution we had in the last chapter. If you are interested in the connection between the discrete and integral convolutions and can't work it out on your own, stop by my office. Division of power series is quite messy, though it can be useful in some situations (where it is usually called deconvolution). We won't need to deal with it.

EXAMPLE: Find the Taylor series for $\frac{x}{1-x}$ about $x_0 = 0$. Also find the radius of convergence of this Taylor series.

While we could repeatedly differentiate and evaluate at 0 to find the Taylor series, a shorter way is to recall that the sum of a geometric series is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad \text{for } |x| < 1.$$

Note that the radius of convergence of this series is 1. So from the series for $\frac{1}{1-x}$ we can derive that

$$\begin{aligned}\frac{x}{1-x} &= x\frac{1}{1-x} \\ &= x(1 + x + x^2 + x^3 + \cdots \\ &= x + x^2 + x^3 + x^4 + \cdots\end{aligned}$$

and that the radius of convergence of this new series is also at least 1. If we want to find the radius of convergence precisely, we use the ratio test.

$$\lim_{n\to\infty} \frac{x^{n+1}}{x^n} = x$$

The ratio is equal to 1 when $x = 1$, so in this case the radius of convergence is indeed 1.

In what has gone before, we have written out all the terms of the series with an ellipsis (3 dots) at the end to show the series continues forever. It is often easier to use a capital sigma to denote an infinite series like so

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$$

We repeat the above example in this notation.

EXAMPLE (repeat): Find the Taylor series for $\frac{x}{1-x}$ about $x_0 = 0$.

We write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Then multiplying by x we get

$$\begin{aligned}\frac{x}{1-x} &= x\sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^{n+1}\end{aligned}$$

Note that the answer we got had an x^{n+1} instead of an x^n in each term. This can cause some difficulties on occasion. When necessary, we can make a change of variables for the variable of summation (n in this case) to transform the sum to terms of the form x^n. In this case, if we let $j = n+1$, then when $n = 0$ we have $j = 1$ and when $n = \infty$ we have $j = \infty$ and so our sum becomes

$$\sum_{n=0}^{\infty} x^{n+1} = \sum_{j=1}^{\infty} x^j$$

EXAMPLE: Find the Taylor series for $(1+x)e^x$ about $x_0 = 0$.

$$\begin{aligned}(1+x)e^x &= (1+x)\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + x\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}\end{aligned}$$

Now we would like to add the two series, but the x^n and x^{n+1} terms don't match up. We handle this problem by making a change of variables in the second sum. Let $j = n+1$. Then when $n = 0$, $j = 1$, when $n = \infty$, $j = \infty$ and $n!$ becomes $(j-1)!$. Plugging all this into the second sum we get

$$\begin{aligned}(1+x)e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{j=1}^{\infty}\frac{x^j}{(j-1)!}\end{aligned}$$

Now the name of the variable of summation in the second sum doesn't matter. $\sum_{j=1}^{5} j = 1+2+3+4+5 = \sum_{n=1}^{5} n$ for example. So we just change the name j to n in the second sum to get

$$(1+x)e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

Now things look better, but there is a $n = 0$ term in the first sum and no corresponding term in the second sum. We handle this by splitting the $n = 0$ term out of the first sum and finally we get our answer.

$$\begin{aligned}
(1+x)e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} + \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \\
&= 1 + \sum_{n=1}^{\infty} (\frac{x^n}{n!} + \frac{x^n}{(n-1)!}) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(n+1)x^n}{n!}
\end{aligned}$$

Manipulations like in this example will be very useful in dealing with series solutions for differential equations.

Exercises: Find the Taylor series of the following functions about $x_0 = 0$. You may use your knowledge of the Taylor series of e^x, $\sin(x)$, $\cos(x)$ and $1/(1-x)$ (the geometric series).

(1)	$\sin(x) + \cos(x)$	(2)	$(1-x)e^x$
(3)	$\sinh(x)$	(4)	$\frac{1}{1+x} + \frac{x}{1-x}$
(5)	$\cosh(2x)$	(6)	$\sin(x) + x\cos(x)$
(7)	$\frac{1}{1+4x}$	(8)	$\cos(x) + \frac{\sin(x)}{x}$
(9)	$\frac{1}{1+x^2}$	(10)	$(1-x^2)\cos(x)$

Find the radius of convergence of the Taylor series of the following functions about $x_0 = 0$. You may use the ratio test or your knowledge of the radius of convergence of the Taylor series of e^x, $\sin(x)$, $\cos(x)$ and $1/(1-x)$ (the geometric series).

(11) $\sin(x) + \cos(x)$

(12) $(1-x)e^x$

(13) $\sinh(x)$

(14) $\dfrac{1}{1+x} + \dfrac{x}{1-x}$

(15) $\cosh(2x)$

(16) $\sin(x) + x\cos(x)$

(17) $\dfrac{1}{1+4x}$

(18) $\cos(x) + \dfrac{\sin(x)}{x}$

(19) $\dfrac{1}{1+x^2}$

(20) $(1-x^2)\cos(x)$

§2 Radius of Convergence of Power Series

In this lab we will study the radius of convergence of Taylor series. We will also consider the rate of convergence, that is, we will consider how many terms you need to obtain an answer accurate to some specified precision in a specified region.

Get into matlab and run **lab11**. This sets up a window where you can see the true graph and also the graph of various Taylor polynomials of two functions, $y = \sin(x)$ and $y = (1+x^2)^{-1}$. You can adjust the window using the same techniques as in the last several labs. You can choose the degree of the Taylor polynomial you want to graph and also the point about which you want to compute the Taylor polynomial at the top of the window. You can change between the two functions using the Function menu. There is also a clear button in the lower right hand corner which will erase all the plots except the true function and the current Taylor polynomial. The plots are formed by evaluating the functions at 200 evenly spaced values for x and then connecting the dots. When you change the minimum and maximum x in this lab you just change the window but you don't change the points at which the function is evaluated. Therefore when you extend the minimum or maximum x the graph may no longer fill the whole window. Conversely if you shrink the window the graphs may appear to have corners as you zoom in on individual evaluation points and the segments connecting them. If you encounter these problems, just press the clear button. Whenever you clear the graph, the 200 evenly spaced values for x are reallocated to fill the current window. Changing the function automatically clears the graph.

The initial function is $y = \sin(x)$. Plot the fifth degree Taylor polynomial for $\sin(x)$ about the point $x = 0$. You will see that it approximates the sine curve quite accurately near

$x = 0$. Record the interval for which the difference between $\sin(x)$ and its fifth degree Taylor polynomial is less than .01. You will need to change the minimum and maximum x and y values to zoom in on the spot where the two curves start to separate to find exactly where the polynomial is within .01. Next, plot the seventh degree Taylor polynomial for $\sin(x)$ about $x = 0$ and determine the region where it approximates the original curve to within .01. Zoom in again on the end of the interval where the *fifth* degree expansion was accurate to within .01. How does the difference between the fifth and the seventh degree polynomials compare with the difference between the fifth degree polynomial and the original curve? Repeat the process for the ninth degree polynomial, except that you should compare the seventh and ninth degree polynomial at the end of the interval where the seventh degree polynomial was within .01 of the original curve.

The previous questions dealt with exactly how quickly the Taylor series converged to the function for $\sin(x)$. The next questions deal with where the Taylor series converges at all. Can you find a Taylor polynomial about x_0=0 which is within .01 of $\sin(x)$ over the interval $(-\pi, \pi)$? What about the interval $(-2\pi, 2\pi)$? Can you always find a Taylor polynomial that is within .01 of the true value for any interval $(-A, A)$ no matter how large A is?

Now change to the function $y = (1 + x^2)^{-1}$. By taking a sufficiently high degree Taylor polynomial it is possible to approximate $\sin(x)$ over any interval you like (as I hope you convinced yourself in the previous paragraph). This is *not* true for the function $(1+x^2)^{-1}$. What do you experimentally find to be the radius of convergence of the Taylor series for this function about $x_0 = 0$? That is, for what range of x do the successive approximations appear to be converging to a definite value? What about for $x_0 = .5$, $x_0 = 1$ and $x_0 = 2$? Remember that the region of convergence will be centered around each new x_0 in turn. Do you see a pattern in the radius of convergence (it's pretty subtle)?

I've listed the questions asked in this lab below. Please *don't* just work directly from this list without reading the rest of the lab. The questions make more sense in context, so rereading the appropriate paragraph may help you understand what to do. The questions are just listed here to help you prepare your lab report.

(1) For what interval is the difference between $\sin(x)$ and its fifth degree Taylor polynomial about $x_0 = 0$ less than .01?

(2) For what interval is the difference between $\sin(x)$ and its seventh degree Taylor polynomial about $x_0 = 0$ less than .01?

(3) How does the difference between the fifth and the seventh degree polynomials compare with the difference between the fifth degree polynomial and the original curve (at the end of the interval where the fifth degree polynomial is within .01 of the original curve)?

(4) For what interval is the difference between $\sin(x)$ and its ninth degree Taylor polynomial about $x_0 = 0$ less than .01?

(5) How does the difference between the seventh and the ninth degree polynomials compare with the difference between the seventh degree polynomial and the original curve (at the end of the interval where the seventh degree polynomial is within .01 of the original curve)?

(6) Can you find a Taylor polynomial about x_0=0 which is within .01 of $\sin(x)$ over the interval $(-\pi, \pi)$?

(7) Can you find a Taylor polynomial about x_0=0 which is within .01 of $\sin(x)$ over the interval $(-2\pi, 2\pi)$?

(8) Can you always find a Taylor polynomial that is within .01 of $\sin(x)$ for any interval $(-A, A)$ no matter how large A is?

(9) What is the radius of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x_0 = 0$?

(10) What are the radii of convergence of the Taylor series for $(1 + x^2)^{-1}$ about $x_0 = .5$, $x_0 = 1$, and $x_0 = 2$? Can you find a pattern?

§3 Series Solutions

Discussion: Once we have established that a reasonable guess for the solution to a second order linear homogeneous differential equation with variable coefficients is a power series, we can just make this guess at the beginning and plug our guess into the equation to solve for the coefficients. This is a lengthy process. To keep everything straight, it helps to follow an explicit plan.

Paradigm: Find the general solution to

$$y'' - y' + xy = 0.$$

STEP 1: Guess $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ and compute all the different pieces.

In this case, I will choose $x_0 = 0$.

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$-y' = \sum_{n=1}^{\infty} -n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

STEP 2: Change all the terms to the form x^n or x^j, etc.

Let $j = n+1$, $k = n-1$, and $p = n-2$

$$xy = \sum_{j=1}^{\infty} a_{j-1} x^{j}$$
$$-y' = \sum_{k=0}^{\infty} -(k+1) a_{k+1} x^{k}$$
$$y'' = \sum_{p=0}^{\infty} (p+2)(p+1) a_{p+2} x^{p}$$

STEP 3: Change all the indices to the same letter (I use m) and plug into the equation.

$$y'' - y' + xy = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m + \sum_{m=0}^{\infty} -(m+1) a_{m+1} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m$$

STEP 4: Collect like terms.

$$(2a_2 - a_1) + \sum_{m=1}^{\infty} \left[(m+2)(m+1) a_{m+2} - (m+1) a_{m+1} + a_{m-1} \right] x^m = 0$$

Here the first term $2a_2 - a_1$ is the coefficient of x^0 and the sum contains the general term. Since not all pieces have an x^0 term, it must be separated from the general term.

STEP 5: Equate coefficients to 0.

$$2a_2 - a_1 = 0$$
$$(m+2)(m+1) a_{m+2} - (m+1) a_{m+1} + a_{m-1} = 0, \qquad \text{for } m \geq 1$$

and we rewrite the last equation by solving for the highest coefficient, a_{m+2}, to obtain

$$a_{m+2} = \frac{(m+1)a_{m+1} - a_{m-1}}{(m+2)(m+1)} \qquad \text{for } m \geq 1.$$

This last equality is called the **recurrence relation** for the differential equation.

STEP 6: Plug in $a_0 = 1$, $a_1 = 0$ to find the first solution $y_1(x)$.

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= (1/2)a_1 = 0 \\
a_3 &= \frac{2a_2 - a_0}{6} = -1/6 \\
a_4 &= \frac{3a_3 - a_1}{12} = -1/24 \\
&\vdots \\
y_1(x) &= 1 - (1/6)x^3 - (1/24)x^4 + \ldots
\end{aligned}$$

STEP 7: Plug in $a_0 = 0$, $a_1 = 1$ to find the second solution $y_2(x)$.

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= (1/2)a_1 = 1/2 \\
a_3 &= \frac{2a_2 - a_0}{6} = 1/6 \\
&\vdots \\
y_2(x) &= x + (1/2)x^2 + (1/6)x^3 + \ldots
\end{aligned}$$

STEP 8: The general solution is then

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Note that by our choice of a_0 and a_1 in our two solutions and since $x_0 = 0$, we have $y(0) = c_1$ and $y'(0) = c_2$. This makes it easy to find c_1 and c_2 to solve initial value problems. Also note that if you are going to need to solve initial value problems with the initial values given at some point a, then you should choose $x_0 = a$ when you find the series solution so you will be able to find the c_1 and c_2 easily.

EXAMPLE: Find the general solution to

$$(1-x^2)y'' - xy' + 4y = 0$$

Step 1:

$$4y = \sum_{n=0}^{\infty} 4a_n x^n$$

$$-xy' = \sum_{n=1}^{\infty} -na_n x^n$$

$$(1-x^2)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

Step 2: Let $j = n-2$, then

$$4y = \sum_{n=0}^{\infty} 4a_n x^n$$

$$-xy' = \sum_{n=1}^{\infty} -na_n x^n$$

$$(1-x^2)y'' = \sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^j - \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

Step 3:

$$(1-x^2)y'' - xy' + 4y = \begin{matrix} \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m \\ - \sum_{m=1}^{\infty} ma_m x^m + \sum_{m=0}^{\infty} 4a_m x^m \end{matrix} = 0$$

Step 4:

$$(2a_2+4a_0)+(6a_3-a_1+4a_1)x+\sum_{m=2}^{\infty}\big((m+2)(m+1)a_{m+2}-m(m-1)a_m-ma_m+4a_m\big)x^m = 0$$

Step 5:

$$2a_2 + 4a_0 = 0$$

$$6a_3 + 3a_1 = 0$$

$$(m+2)(m+1)a_{m+2} - m^2 a_m + 4a_m = 0, \qquad m \geq 2$$

The recurrence relation is

$$a_{m+2} = \frac{(m^2-4)a_m}{(m+2)(m+1)}, \qquad m \geq 2$$

Step 6:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= \frac{-4}{2}a_0 = -2 \\ a_3 &= 0 \\ a_4 &= \frac{0}{12}a_2 = 0 \\ a_5 &= 0 \\ a_6 &= 0 \\ &\vdots \end{aligned}$$

$$y_1(x) = 1 - 2x^2$$

Step 7:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0 \\ a_3 &= \frac{-3}{6}a_1 = -\frac{1}{2} \\ a_4 &= 0 \\ a_5 &= \frac{5}{20}a_3 = -\frac{1}{8} \\ &\vdots \end{aligned}$$

$$y_2(x) = x - \frac{1}{2}x^3 - \frac{1}{8}x^5 + \cdots$$

Step 8: The general solution is

$$y(x) = c_1y_1(x) + c_2y_2(x).$$

Note that all coefficients of $y_1(x)$ are 0 after a_2 and so $y_1(x)$ is actually a polynomial instead of an infinite series. The equation $(1-x^2)y'' - xy' + \alpha^2 y = 0$ is called Chebyshev's equation. For every integer α, this equation has a solution which is a polynomial of degree

α, called the Chebyshev polynomial (actually, for technical reasons relating to the specific applications of the Chebyshev polynomial, most books would multiply our answer by $1/2^\alpha$ and define that to be the Chebyshev polynomial of order α). These polynomials are important in approximation theory. As always, you are welcome to stop by my office and I'll discuss Chebyshev polynomials and their applications in more detail.

EXAMPLE: Solve the initial value problem

$$y'' - x^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 0$$

STEP 1: Since we have initial data at $x = 0$, we choose $x_0 = 0$ and so

$$y = \sum_{n=0}^{\infty} a_n x^n$$

from which we derive

$$-x^2 y = \sum_{n=0}^{\infty} -a_n x^{n+2}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

STEP 2: Let $j = n + 2$ and $k = n - 2$ so

$$-x^2 y = \sum_{j=2}^{\infty} -a_{j-2} x^j$$
$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k$$

STEP 3:

$$y'' - x^2 y = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=2}^{\infty} -a_{m-2} x^m$$

STEP 4:

$$2a_2 + 6a_3 x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_{m-2}\right] x^m = 0$$

STEP 5:

$$2a_2 = 0$$
$$6a_3 = 0$$
$$(m+2)(m+1)a_{m+2} - a_{m-2} = 0, \qquad \text{for } m \geq 2$$

and from the last equation we obtain the recurrence relation

$$a_{m+2} = \frac{a_{m-2}}{(m+2)(m+1)}, \qquad \text{for } m \geq 2.$$

STEP 6: Rather than find the general solution and plug in the initial values, we are going to plug in the initial values right now. We recall that

$$a_0 = y(x_0)$$

$$\text{and} \qquad a_1 = y'(x_0)$$

and since we chose $x_0 = 0$ we can now read off

$$a_0 = 1$$

$$a_1 = 0$$

from the initial conditions. We then obtain

$$a_2 = 0$$

$$a_3 = 0$$

from our first two equations in step 5. Finally, we apply the recurrence relation to obtain

$$\begin{aligned}
a_4 &= \frac{a_0}{(2+2)(2+1)} = \frac{1}{12} \\
a_5 &= \frac{a_1}{(3+2)(3+1)} = 0 \\
a_6 &= \frac{a_2}{(4+2)(4+1)} = 0 \\
a_7 &= \frac{a_3}{(5+2)(5+1)} = 0 \\
a_8 &= \frac{a_4}{(6+2)(6+1)} = \frac{1}{672} \\
&\vdots
\end{aligned}$$

from which we get

$$y(x) = 1 + (1/12)x^4 + (1/672)x^8 + \ldots$$

Exercises: Find the recurrence relations and general solutions for the following differential equations.

(1) $y'' + y' - 3y = 0$

(2) $y'' + y = 0$

(3) $y'' + 3y' - (x^2 + 2)y = 0$

(4) $(x-2)y'' + y' - y = 0$

(5) $(x+1)y'' - (x-2)y' + x^2 y = 0$

(6) $y'' + (x^2 - 3x + 2)y' + (x^3 - 1)y = 0$

(7) $y'' - xy' + 3y = 0$

(8) $y'' + xy' - y = 0$

(9) $(1-x^2)y'' - 2xy' + 20y = 0$

(10) $(x+1)y'' - y = 0$

Solve the following initial value problems.

(11) $y'' + y' - 3y = 0, \qquad y(0) = 1, \quad y'(0) = -2$

(12) $(x-2)y'' + y' - y = 0, \qquad y(0) = -3, \quad y'(0) = 2$

(13) $y'' - xy' + 3y = 0, \qquad y(0) = 1, \quad y'(0) = 2$

(14) $y'' + xy' - y = 0, \qquad y(0) = -1, \quad y'(0) = -3$

(15) $(x+1)y'' - y = 0, \qquad y(0) = 1, \quad y'(0) = -1$

(16) $y'' - y = 0, \qquad y(1) = 1, \quad y'(1) = 0$

(17) $y'' - xy = 0, \qquad y(1) = 2, \quad y'(1) = 0$

(18) $y'' + x^2y' - 3xy = 0, \qquad y(1) = 1, \quad y'(1) = 0$

(19) $xy'' + 4y' - (x-2)y = 0, \qquad y(2) = 2, \quad y'(2) = 1$

(20) $y''' - xy = 0, \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1$

§4 Radius of Convergence

Discussion: Consider the linear homogenous variable coeffecient initial value problem

$$\begin{aligned} xy'' + y' + y &= 0 \\ y(0) &= -1 \\ y'(0) &= 1 \end{aligned}$$

Suppose we try to solve for $y''(0)$ as we did in section 1. If we plug $x = 0$ and the initial values into the problem we find

$$0 + 1 + -1 = 0$$

which is certainly true, but gives us no hint about the value of $y''(0)$. The trouble is that the coefficient of y'' (which is x) is 0 at $x = 0$. A point where the coefficient of y'' is 0 is called a singular point of the differential equation. More generally, we can define singular points as follows.

Definition. *Let $p(x)$, $q(x)$ and $r(x)$ be analytic functions. A point x_0 is a* **singular point** *for the differential equation $p(x)y'' + q(x)y' + r(x)y = 0$ if either $q(x)/p(x)$ or $r(x)/p(x)$ is undefined at $x = x_0$. A point which is not a singular point is an* **ordinary point**.

A function is analytic in an interval if its Taylor series converges to the function in that interval. Don't let that definition bother you. Any polynomial or rational function is analytic, as are the exponential, log, and trigonometric functions, away from their singularities. I will not try to trick you by using non-analytic functions in this class. It should be noted that if $p(x)$, $q(x)$ and $r(x)$ are all polynomials, then x_0 will only be a singular point if $p(x_0) = 0$.

Example: $x^2y'' + xy' = 0$ has a singular point at $x = 0$ since $x/x^2 = 1/x$ is undefined at $x = 0$.

Example: $2y''+y/(x-1) = 0$ has a singular point at $x = 1$ since $(1/(x-1))/2 = 1/(2x-2)$ is undefined at $x = 1$.

The techniques we learned in the last section will work for ordinary points but will not work for singular points. We will study solutions about singular points in the next several sections after this one.

So far, we have not worried about where a power series solution to a differential equation is valid in this class. We have just gone through the manipulations to find the solution and then assumed what we found was actually a solution. But of course, the manipulations we went through are only valid inside the radius of convergence of the power series. So now we will take up the question of how large the radius of convergence of our series solution will be. Of course, we already have some examples to build on from the labs. Fortunately, we don't have to try to remember all the old tests for deciding on the radius of convergence of a power series from calculus. Instead, we have the following theorem.

Theorem. *Suppose* $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ *is the series solution to the differential equation* $p(x)y''+q(x)y'+r(x)y = 0$ *where* $p(x)$, $q(x)$ *and* $r(x)$ *are all polynomials and* x_0 *is an ordinary point. Then the radius of convergence for* $y(x)$ *is at least as large as the distance from* x_0 *to the nearest singular point of the equation.*

NOTE: We must consider complex x, even though the equation is real.

Paradigm: What is the radius of convergence of the series solution to $(x+1)y'' + y' + (x+1)y = 0$ about the point $x_0 = 0$?

STEP 1: Check the series is expanded about an ordinary point (so the theorem applies).

Since all the coefficients are polynomials and $p(0) = 0 + 1 = 1 \neq 0$, 0 is an ordinary point and the above theorem applies.

STEP 2: Compute all singular points.

The only singular point is -1.

STEP 3: Find the distances from the singular points to center of expansion.

-1 is a distance of 1 away from 0

STEP 4: The radius of convergence is at least as large as the smallest distance from a singular point to the center of expansion of the series.

The radius of convergence is at least 1.

EXAMPLE: What is the radius of convergence of the series solution to $(x^2 - 2x - 3)y'' + (x + 1)y' + 3y = 0$ about the point $x_0 = 0$?

STEP 1: Since all coefficients are polynomial and $p(0) = 0 + 0 - 3 = -3 \neq 0$, 0 is an ordinary point and the above theorem applies.

STEP 2: The two singular points are -1 and 3.

STEP 3: -1 is 1 unit from 0 and 3 is 3 units from 0.

STEP 4: -1 is the singular point closest to 0 and it is a distance 1 away. The radius of convergence is at least 1.

EXAMPLE: What is the radius of convergence of the series solution to
$(x^3 + x^2 + x + 1)y'' + 2xy' + (x - 1)y = 0$ about the point $x_0 = 1$?

STEP 1: Since all coefficients are polynomial and $p(1) = 1 + 1 + 1 + 1 = 4 \neq 0$, 1 is an ordinary point and the above theorem applies.

STEP 2: The three singular points are -1, i and $-i$.

STEP 3: -1 is 2 away from 1 while $\pm i$ are $\sqrt{2} < 2$ away from 1.

STEP 4: The radius of convergence is at least $\sqrt{2}$.

More Discussion: This is a very powerful theorem because we can decide where a power series solution will be valid even before we have found the series. The hypothesis can be weakened somewhat (p, q, r analytic instead of polynomial), but this leads to more technical questions than we need to consider. The easiest mistake to make is to forget to consider complex singular points, but complex analysis is used in the proof and so complex points must be considered. In fact, if you go back to the labs and consider the Taylor series expansion for $1/(1+x^2)$ about $x_0 = 0$ you will see that the Taylor series had a radius of convergence of 1, corresponding to the singularities at $\pm i$ which are a distance 1 from 0. The other point to remember is that this just gives a lower bound on the radius of convergence. It is possible for the radius of convergence to be larger than this test indicates. But that is fairly rare in practice; this test works out pretty accurately.

Exercises:

(1) Suppose the differential equation $y'' + 3y' - 2xy = 0$ is solved as a power series about $x_0 = 0$. Give a lower bound for the radius of convergence of the series solution.

(2) Suppose the differential equation $(x+2)y'' + y' - x^2 y = 0$ is solved as a power series about $x_0 = 0$. Give a lower bound for the radius of convergence of the series solution.

(3) Suppose the differential equation $(x^2+4x+3)y'' - (3x-2)y' + (x^2+4x+1)y = 0$ is solved as a power series about $x_0 = 1$. Give a lower bound for the radius of convergence of the series solution.

(4) Suppose the differential equation $(x^2+4x+5)y'' + 3xy' - y = 0$ is solved as a power series about $x_0 = 0$. Give a lower bound for the radius of convergence of the series solution.

(5) Suppose the differential equation $(x^3-1)y'' + (x^2-1)y' + (x-1)y = 0$ is solved as a power series about $x_0 = -1$. Give a lower bound for the radius of convergence of the series solution.

(6) Suppose the differential equation $(x^2+8x+15)y'' + 4xy' - (x^2-1)y = 0$ is solved as a power series about $x_0 = 0$. Give a lower bound for the radius of convergence of the series solution.

(7) Suppose the differential equation $(x^2+3x-4)y'' - (x^2+1)y' - (2x-3)y = 0$ is solved as a power series about $x_0 = 1$. Give a lower bound for the radius of convergence of the series solution.

(8) Suppose the differential equation $(x^2+6x+25)y''-(x^2-3x-5)y'+4y=0$ is solved as a power series about $x_0=-1$. Give a lower bound for the radius of convergence of the series solution.

(9) Suppose the differential equation $(1-x^2)y''-2xy'+20y=0$ is solved as a power series about $x_0=0$. Give a lower bound for the radius of convergence of the series solution.

(10) Suppose the differential equation $x^2y''+xy'-(x^2-1)y=0$ is solved as a power series about $x_0=1$. Give a lower bound for the radius of convergence of the series solution.

§5 Radius of Convergence (Computer Lab)

In the last lab you looked at the convergence of Taylor series. In this lab you will also look at the convergence of Taylor series. But where in the last lab you looked at the Taylor series of a known function, in this lab you will look at the series solution of a differential equation (which is the Taylor series of the unknown solution function).

Get into matlab and give the command **lab12**. Across the top of the Lab 12 window are fields where you can define a second order variable coefficient initial value problem of the form

$$(\alpha_2x^2+\alpha_1x+1)y''(x)+(\beta_2x^2+\beta_1x+\beta_0)y'(x)+(\gamma_2x^2+\gamma_1x+\gamma_0)y(x)=0$$
$$y(0)=a_0, \quad y'(0)=a_1.$$

There are the usual controls to adjust the minimum and maximum values of x and y. You can pick the degree of the Taylor polynomial approximation to the solution to graph in the field at the bottom of the window. Since the initial values are defined at $x=0$, the Taylor polynomial is taken about $x=0$. If you change the degree of the polynomial, the new approximation is graphed on the current axes. Note that the maximum degree allowed is 1000. If you change the minimum or maximum x or y, the axes will be adjusted, but new points on the curve will not be computed. Click on the rescale button below the Maximum x field to recompute the current approximation for the new range of x's. Note that only the current degree approximation will be recomputed. If you change the coefficients of the differential equation or the initial values, the graph will be grayed and a pair of Okay/Cancel buttons will appear in the lower right corner of the window. Click on Okay to accept the changes after you have adjusted all the values you wish to change. Click on Cancel to erase the changes and return to the original values.

Start by looking at the equation

$$(x+1)y'' + xy' + 1y = 0, \qquad y(0) = 1, \quad y'(0) = 1.$$

By looking at the Taylor polynomials, approximate the radius of convergence of the series solution for this initial value problem. Look at the 1000 degree Taylor polynomial. Still using the 1000 degree Taylor polynomial, change the initial value problem to three other initial value problems of the form

$$(x+1) + p(x)y' + q(x)y = 0, \qquad y(0) = a_0, \quad y'(0) = a_1$$

where $p(x)$ and $q(x)$ are quadratic functions. What effect does changing the $p(x)$, $q(x)$, a_0, and a_1 have on the radius of convergence when the coefficient of y'' is left fixed?

Repeat the instructions in the paragraph above for the equations

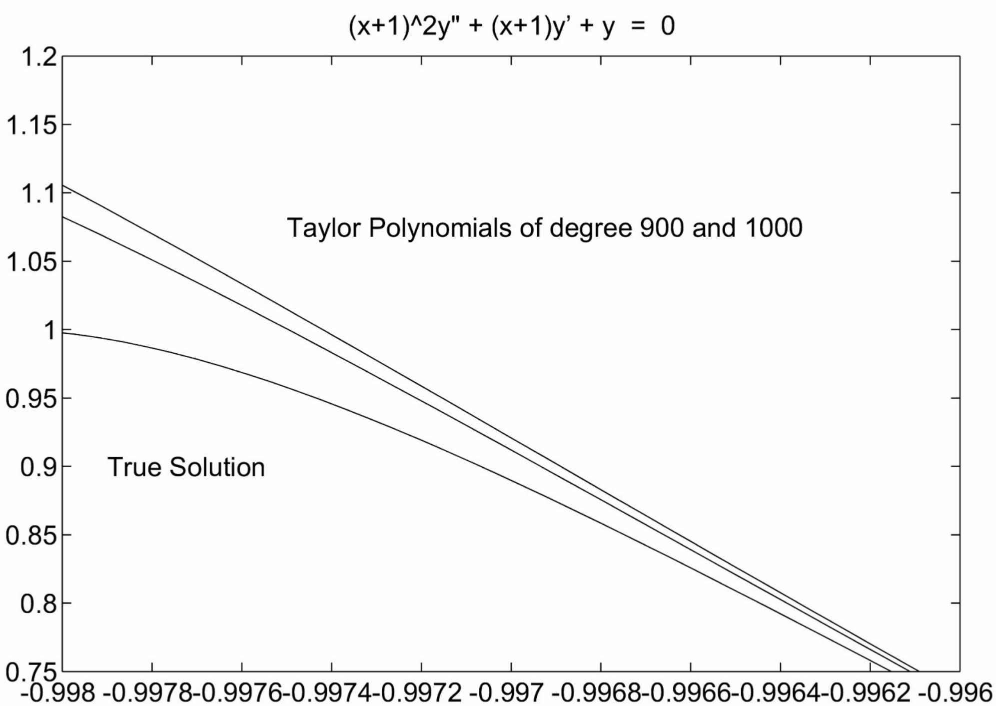
$$\begin{aligned} (-.125x^2 - .25x + 1)y'' + y' + (-x^2 + 1)y = 0, &\qquad y(0) = 1, \quad y'(0) = -1 \\ (.2x^2 + .8x + 1)y'' + (x^2 - 1)y' + x^2 y = 0, &\qquad y(0) = 0, \quad y'(0) = 1 \\ (x^2 + 1)y'' + (x^2 - 1)y = 0, &\qquad y(0) = 1, \quad y'(0) = 0. \end{aligned}$$

In each case, determine the radius of convergence of the series solution for the initial value problem. Then change the coefficients of y' and y as well as the initial values $y(0)$ and $y'(0)$ and see what effect these changes have on the radius of convergence when the coefficient of y'' is left fixed. Can you discover a rule for approximating the radius of convergence of the series solution? Note that we have required that the coefficient of y'' be equal to 1 when x is equal to 0. The importance of this assumption will be taken up next week.

Finally, we will note that just knowing the radius of convergence doesn't necessarily tell you where the series solution is valid. To illustrate these points, we will consider two equations where the exact solutions are known. One possible problem is illustrated by the initial value problem

$$(x^2 + 2x + 1)y'' + (x+1)y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Graph the 1000 degree Taylor polynomial approximation to the solution of this initial value problem. The solution to this problem is $y(x) = \cos(\log(x+1))$ as you can check if you want. This function has a singularity at $x = -1$. As x approaches -1 from the right, the function y oscillates infinitely many times between -1 and 1 (you should be able to see why this happens, if you are having trouble ask your lab instructor for help). You won't be able to recognize this infinite oscillation just from looking at the Taylor polynomials of degree less than 1000. In fact, as illustrated in figure 1, the Taylor polynomials approximate each other quite accurately well past the region where they approximate the true solution accurately. If you just compared successive Taylor polynomials, as you might if you didn't have the true solution, you could believe the approximations were accurate to within about $\pm.01$ out to about $-.997$, which isn't true. On the other side of the region of convergence, the function is continuous at $x = 1$. But by looking at the Taylor polynomials, you can't distinguish that there is a singularity at one endpoint while the function is continuous at the other endpoint. In general, the Taylor polynomials don't approximate well near the radius of convergence, particularly near a singularity for the true solution.



A second problem is illustrated by the initial value problem

$$y'' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Graph the 100 (one hundred) degree Taylor polynomial approximation. Set x to range from 0 to 50 and y to range from -2 to 2. The solution to this problem is $y(x) = \cos(x)$ of course, and you should see that the Taylor polynomial approximates the solution quite well up to about $x = 35$ and then diverges. If you look at how the solution diverges though, you should note that it looks a bit odd. Rather than just taking off it has several bumps first. If you adjust the graph to run from $x = 35$ to $x = 40$, you will note that the graph looks spiky rather than smooth. You might hope that by pressing the rescale button so you will have 200 points plotted between 35 and 40 (rather than the 20 points currently plotted in that region), that the graph will look smoother. But when you press the rescale button, the picture looks worse rather than better. Furthermore, if you increase the degree of the Taylor polynomial to 500, the approximation now looks even worse. In particular, while the radius of convergence for the Taylor series of $\cos(x)$ is infinite, you can't find a polynomial that looks good past 35 or so. This is caused by roundoff error. The spiky graphs of what look like random numbers are characteristic of roundoff error dominating the true solution. So even when you aren't near the radius of convergence, you still must worry about the accuracy of your approximation. Substantial roundoff error can usually be spotted by graphing the solution.

Exercises

Determine the radius of convergence of the series solution for the following initial value problems. Then change the coefficients of y' and y as well as the initial values $y(0)$ and $y'(0)$ and see what effect these changes have on the radius of convergence when the coefficient of y'' is left fixed.

(1) $(x+1)y'' + xy' + y = 0, \qquad y(0) = 1, \quad y'(0) = 1$

(2) $(-.125x^2 - .25x + 1)y'' + y' + (-x^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = -1$

(3) $(.2x^2 + .8x + 1)y'' + (x^2 - 1)y' + x^2 y = 0, \qquad y(0) = 0, \quad y'(0) = 1$

(4) $(x^2 + 1)y'' + (x^2 - 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0$

(5) Write several sentences about the behavior near $x = -1$ for the Taylor polynomials that approximate the solution of

$$(x^2 + 2x + 1)y'' + (x+1)y' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(6) Write several sentences about the behavior in the range $35 < x < 40$ for the Taylor polynomials that approximate the solution of

$$y'' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$