

2. In a damped, unforced equation, the solution curves always decay to 0 as $t \rightarrow \infty$. What can you say about the location of the poles in damped, unforced equations? How do the poles vary for underdamped, critically damped, and overdamped equations?
3. In a forced or an undamped equation, the solutions tend to a steady state, neither decaying to 0 nor exploding to ∞ . What can you say about the location of the poles in these situations?
4. One way you can have solutions blowing up to ∞ is to have resonance. What can you say about the location of the poles for a resonant equation?
5. Another way you can have solutions blowing up to ∞ is to have a negative amount of friction (this may seem ridiculous, but consider an amplifier). What can you say about the location of the poles if the damping coefficient is negative?

§5 IMPULSE FUNCTIONS

Discussion: Consider an elastic collision. In such a collision, the energy of the first object is instantaneously transferred to the second object. So if two billiard balls meet in an elastic collision, the first ball stops immediately upon impact and the second ball immediately begins moving in the direction the first ball was moving. Of course, this is an idealization of what happens in practice, as anyone who has watched their cue ball follow another ball into a pocket knows. We now ask, what is the force exerted on the first ball by the second in an ideal elastic collision? Since

$$\text{energy transferred} = \text{work} = \text{force} \times \text{distance}$$

and the distance is 0, the force must be infinite at the point of contact. The force must be 0 everywhere else of course. We represent the force by an idealized function called the Dirac delta function, which is written $\delta(t)$. The function $\delta(t)$ is defined by the property that

$$\int_a^b f(t)\delta(t) dt = \begin{cases} f(0), & \text{if } a \leq 0 \leq b \\ 0, & \text{otherwise} \end{cases}$$

reflecting the fact that $\delta(t) = 0$ away from the origin while the value at the origin is so large that when multiplied by 0 you get 1 (so that total energy transferred is 1). Of course, no such function exists in the classical sense — this is an “idealized function,” just as perfectly elastic collisions are idealizations.

The next question is how do we handle differential equations involving impulse functions? For example, consider a spring-mass system where the forcing function consists of an impulse (e.g. you hit it with a hammer). The most efficient way to handle this situation is with Laplace transforms (though it is possible to use variation of parameters). The key fact is that

THEOREM. $\mathcal{L}\{\delta(t - c)\} = e^{-cs}$, provided $c \geq 0$.

PROOF:

$$\begin{aligned}\mathcal{L}\{\delta(t - c)\} &= \int_0^{\infty} \delta(t - c)e^{-st} dt \\ &= \int_{-c}^{\infty} \delta(u)e^{-s(u+c)} du \\ &= e^{-s(0+c)} = e^{-sc}\end{aligned}$$

With this in mind it is tempting to talk about an example like the following. But we will need to discuss some matters at the end before we are ready for a paradigm.

ALMOST CORRECT EXAMPLE:

$$\begin{aligned}x'' + x &= \delta(t) \\ x(0) &= 0 \\ x'(0) &= 0\end{aligned}$$

We use exactly the same technique here as in the previous section.

STEP 1: Take the Laplace transform of both sides of the equation.

$$\begin{aligned}\mathcal{L}\{x'' + x\} &= \mathcal{L}\{x''\} + \mathcal{L}\{x\} \\ &= s^2\mathcal{L}\{x\} - sx(0) - x'(0) + \mathcal{L}\{x\} \\ &= (s^2 + 1)\mathcal{L}\{x\} \\ \mathcal{L}\{\delta(t)\} &= e^{-0s} = 1 \\ (s^2 + 1)\mathcal{L}\{x\} &= 1\end{aligned}$$

STEP 2: Solve for $\mathcal{L}\{x\}$.

$$\mathcal{L}\{x\} = 1/(s^2 + 1)$$

STEP 3: Take the inverse Laplace transform to find the solution.

$$x(t) = \sin(t)$$

This looks good, but worried by the title, “Almost Correct Example,” we now check our work. Observe that $x'(0) = 1 \neq 0$ and so we have somehow failed to satisfy our initial values. What went wrong here? The difficulty lies in the nature of our initial values and our idealized impulse function. Consider a mass of $1kg$ on an undamped spring with spring constant $1kg/sec^2$. Initially at rest, the mass is subjected to an impulse at time 0 that instantaneously imparts an energy of 0.5 joules. The differential equation corresponding to this physical situation is exactly the differential equation of the example above, $x'' + x = \delta(t)$. (You may try to work out for yourself why 0.5 joules corresponds to $\delta(t)$ instead of $0.5\delta(t)$. If you have questions stop by my office and I’ll answer them.) The initial values are $x(0) = 0$, $x'(0) = 0$ since the system is initially at rest. But when the impulse hits, the energy of the impulse is instantaneously converted into kinetic energy which means that the velocity at time 0 instantaneously changes from 0 to 1. So our solution *appears* to have $x'(0) = 1$, even though we found it using $x'(0) = 0$. This is a physical justification for our answer. A mathematical justification is that our answer isn’t actually $\sin(t)$. After all $\sin(-\pi/2) = 1$, but the mass is at rest prior to time 0 and so $x(-\pi/2) = 0$ and not 1. Properly, we should write our answer as

$$x(t) = \begin{cases} 0, & t < 0 \\ \sin(t), & t \geq 0. \end{cases}$$

Since this form is rather lengthy, we introduce one last function, the unit step function

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases}$$

Then we can write our answer to the previous example as $x(t) = u(t) \sin(t)$. The $u(t)$ reminds us that the mass is initially at rest and so $x(t) = 0$ for all $t < 0$. If we try to graph the solution $u(t) \sin(t)$ to the problem $x'' + x = \delta(t)$, $x(0) = 0$ and $x'(0) = 0$ we get figure 1. The difficulty with our answer was that $x'(0) \neq 0$ and if we look at figure 1, we see that this difficulty hasn’t gone away. The graph has a corner at $t = 0$ and so x' doesn’t exist at $t = 0$. Mathematically, we reconcile our problems by defining the left and right derivatives. The left derivative at $t = 0$ is the slope of the curve on the left side of $t = 0$

and the right derivative at $t = 0$ is the slope of the curve on the right side of $t = 0$. In this case, the curve is horizontal to the left of $t = 0$ and so the left derivative is 0. On the right side, the curve rises with a slope of 1 (remember that the horizontal and vertical scales are different) and so the right derivative is 1. We can then make our problem mathematically precise by saying that we want the solution to $x'' + x = \delta(t)$ with initial values $x(0) = 0$ and $x'(0) = 0$ *on the left*.

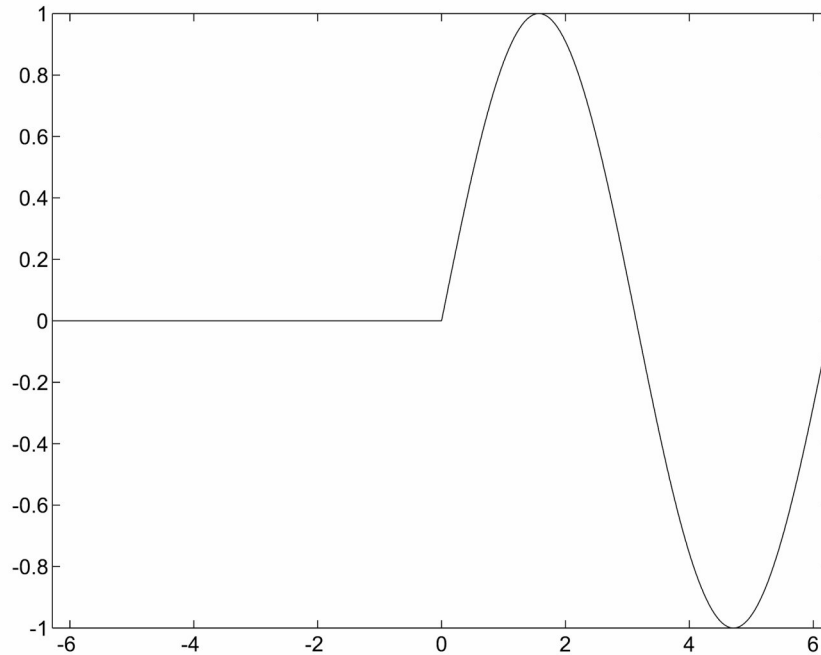


Figure 1

We are now ready for our paradigm.

Paradigm:

$$x'' + 2x' + 5x = \delta(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

STEP 1: Take the Laplace transform of both sides.

$$s^2 \mathcal{L}\{x\} - s \cdot 0 - 0 + 2s \mathcal{L}\{x\} - 2 \cdot 0 + 5 \mathcal{L}\{x\} = 1$$

$$(s^2 + 2s + 5) \mathcal{L}\{x\} = 1$$

STEP 2: Solve for $\mathcal{L}\{x\}$.

$$\mathcal{L}\{x\} = \frac{1}{s^2 + 2s + 5}$$

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STEP 3: Take the inverse Laplace transform to find the solution. (Remember to use the step function as a cutoff.)

$$\begin{aligned}x(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2s + 5} \right\} \\&= \mathcal{L}^{-1} \left\{ \frac{1}{(s + 1)^2 + 2^2} \right\} \\&= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{(s + 1)^2 + 2^2} \right\} \\&= \frac{1}{2} u(t) e^{-t} \sin(2t)\end{aligned}$$

You may be wondering about whether we should have used the step function $u(t)$ when solving problems in section 3. The step function is used to emphasize that the system is in one state prior to some time and in a different state after that time. For example, the system is at rest and then is struck by an impulse and is no longer at rest. Physically, any problem that says we take a spring and then we pull it to some initial condition and let it go falls into this category. But this is a math course and not a physics course. We are just going to concern ourselves with solving differential equations and so we will only worry about listing the step function $u(t)$ for solving problems with impulses where we run into *mathematical* problems as above where we have to distinguish between left and right limits for the initial values.

While we will usually pick our time scale so an impulse comes at $t = 0$, this isn't always convenient. The usual way of writing an impulse at time $t = c$ is $\delta(t - c)$. Note that $t - c = 0$ when $t = c$ so the delta function has its spike at $t = c$ in this case. We can handle such an impulse using the fact that $\mathcal{L}^{-1} \{ e^{-cs} \mathcal{L} \{ f \} \} = u(t - c) f(t - c)$, as noted in the table.

EXAMPLE:

$$\begin{aligned}x'' + 5x' + 6x &= \delta(t - 1) \\x(0) &= 0 \\x'(0) &= 0\end{aligned}$$

Step 1:

$$(s^2 + 5s + 6) \mathcal{L} \{ x \} = e^{-s}$$

Step 2:

$$\mathcal{L} \{ x \} = \frac{e^{-s}}{s^2 + 5s + 6}$$

Step 3: Here we try to write $\mathcal{L}\{x\}$ in the form $e^{-cs}\mathcal{L}\{f\}$. We initially ignore the e^{-s} term as we try to write the rest of our formula for $\mathcal{L}\{x\}$ as $\mathcal{L}\{f\}$ for some function f .

$$\begin{aligned}\frac{1}{s^2 + 5s + 6} &= \frac{1}{(s+2)(s+3)} \\ &= \frac{1}{s+2} - \frac{1}{s+3} \\ &= \mathcal{L}\{e^{-2t} - e^{-3t}\}\end{aligned}$$

Now we can write

$$\frac{e^{-s}}{s^2 + 5s + 6} = e^{-s}\mathcal{L}\{e^{-2t} - e^{-3t}\}$$

We choose $c = 1$ since we have e^{-s} and apply our rule that $\mathcal{L}^{-1}\{e^{-cs}\mathcal{L}\{f\}\} = u(t-c)f(t-c)$ to get

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2 + 5s + 6}\right\} \\ &= u(t-1)(e^{-2(t-1)} - e^{-3(t-1)})\end{aligned}$$

One case where we couldn't set our time scale to have the impulse occur at $t = 0$ is when there is more than one impulse.

EXAMPLE: The following equation models a spring that receives an initial shock and then an aftershock half as strong 5 units of time later.

$$\begin{aligned}x'' + 4x' + 20x &= 2\delta(t) + \delta(t-5) \\ x(0) &= 0 \\ x'(0) &= 0\end{aligned}$$

Step 1: Since the initial conditions are all 0,

$$(s^2 + 4s + 20)\mathcal{L}\{x\} = 2 + e^{-5s}$$

Step 2:

$$\mathcal{L}\{x\} = \frac{2}{s^2 + 4s + 20} + \frac{e^{-5s}}{s^2 + 4s + 20}$$

Step 3: We initially ignore the numerators of both terms in $\mathcal{L}\{x\}$ and just concentrate on finding $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+20}\right\}$. We compute

$$\begin{aligned}\frac{1}{s^2 + 4s + 20} &= \frac{1}{(s+2)^2 + 4^2} \\ &= \frac{1}{4} \cdot \frac{4}{(s+2)^2 + 4^2}\end{aligned}$$

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Now we use this to compute

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4s+20}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+20}\right\} \\ &= 2\frac{1}{4}u(t)e^{-2t}\sin(4t) \\ \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^2+4s+20}\right\} &= \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\left\{\frac{1}{4}e^{-2t}\sin(4t)\right\}\right\} \\ &= u(t-5)\frac{1}{4}e^{-2(t-5)}\sin(4(t-5))\end{aligned}$$

Finally, $x(t)$ is the sum of both of these,

$$x(t) = \frac{1}{2}u(t)e^{-2t}\sin(4t) + \frac{1}{4}u(t-5)e^{-2(t-5)}\sin(4(t-5))$$

Note that the $u(t)$ and $u(t-5)$ terms here serve to indicate where each part of the solution is active. In particular, we need the $u(t-5)$ term to indicate that the part of the solution corresponding to the aftershock at $t = 5$ doesn't affect the behavior of the system before the aftershock occurs!

Exercises: Solve the following problems. In all cases treat the initial values as values on

the right.

- (1) $x'' + 3x' + 2x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (2) $x'' - 4x' + 3x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (3) $x'' + 4x' + 4x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (4) $x'' + 6x' + 10x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (5) $x''' + x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (6) $x'' + 4x' + 29x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (7) $x'' + 4x' + 3x = 2\delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (8) $x'' + x = 3\delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (9) $x'' + 3x' + \frac{5}{2}x = 4\delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (10) $\frac{1}{2}x'' + 100x' + 10^5x = 200\delta(t), \quad x(0) = 0, \quad x'(0) = 0$
- (11) $x'' + 4x' + 3x = \delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0$
- (12) $x'' + 9x' + 14x = \delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0$
- (13) $x'' + 4x' + 13x = \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0$
- (14) $x'' + 2x' + 2x = \delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0$
- (15) $x'' + 4x' + 3x = u(t - 1), \quad x(0) = 0, \quad x'(0) = 0$
- (16) $x'' + 11x' + 24x = u(t - 2), \quad x(0) = 0, \quad x'(0) = 0$
- (17) $x'' + 4x' + 3x = \delta(t) + \delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0$
- (18) $x'' + 4x' + 29x = 3\delta(t) + \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0$
- (19) $x'' + 7x' + 6x = u(t) + \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0$
- (20) $x'' + 10x' + 50x = \delta(t) + u(t - 2), \quad x(0) = 0, \quad x'(0) = 0$

§6 CONVOLUTIONS

Discussion: Since the Laplace transform is linear, the Laplace transform of a sum is the sum of the Laplace transforms. This is *not* true for products. To study products of Laplace transforms, we introduce a generalized product called the convolution. The convolution of two functions f and g is denoted $f * g$ and is defined to be

$$f * g(t) = \int_0^t f(t - T)g(T) dT$$

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For our purposes, the most important fact about convolutions is

THEOREM. $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

PROOF:

$$\begin{aligned}\mathcal{L}\{f * g\} &= \int_0^\infty f * g(t)e^{-st} dt \\ &= \int_0^\infty \left(\int_0^t f(t-T)g(T) dT \right) e^{-st} dt \\ &= \int_0^\infty \left(\int_T^\infty f(t-T)e^{-st} dt \right) g(T) dT \\ &= \int_0^\infty \left(\int_0^\infty f(u)e^{-s(u+T)} du \right) g(T) dT \\ &= \int_0^\infty \mathcal{L}\{f\}(s)e^{-sT} g(T) dT \\ &= \mathcal{L}\{f\}(s) \int_0^\infty e^{-sT} g(T) dT \\ &= \mathcal{L}\{f\}\mathcal{L}\{g\}\end{aligned}$$

This will be very useful in finding integral representations for solutions of differential equations where the forcing function is either unknown or something unusual that isn't in our tables. It is possible to obtain integral representations of the solution in these cases by using variation of parameters, but Laplace transforms are easier to handle and usually lead to simpler integrals.

Paradigm:

$$\begin{aligned}x'' + x &= \sqrt{t} \\ x(0) &= 0 \\ x'(0) &= 0\end{aligned}$$

We again use exactly the same techniques as in the two previous sections.

STEP 1: Take the Laplace transform of both sides.

$$\begin{aligned}\mathcal{L}\{x'' + x\} &= s^2\mathcal{L}\{x\} - sx(0) - x'(0) + \mathcal{L}\{x\} \\ &= (s^2 + 1)\mathcal{L}\{x\} \\ \mathcal{L}\{\sqrt{t}\} &= \Gamma(3/2)/s^{3/2}\end{aligned}$$

So we have

$$(s^2 + 1)\mathcal{L}\{x\} = \Gamma(3/2)/s^{3/2}$$

STEP 2: Solve for $\mathcal{L}\{x\}$.

$$\mathcal{L}\{x\} = [\Gamma(3/2)/s^{3/2}] \times [1/(s^2 + 1)]$$

STEP 3: Take the inverse Laplace transform to find the solution.

At first glance there is nothing like this in the table. So we find the inverse Laplace transform in the form of a convolution.

$$\begin{aligned}\mathcal{L}^{-1}\{\Gamma(3/2)/s^{3/2}\} &= \sqrt{t} \\ \mathcal{L}^{-1}\{1/(s^2 + 1)\} &= \sin(t)\end{aligned}$$

So

$$x(t) = \int_0^t \sqrt{t-T} \sin(T) dT$$

Unfortunately, I can't evaluate this integral. But then from the very beginning we have run into problems where the best we could do was find an integral version for the solution.

EXAMPLE:

$$\begin{aligned}x'' + 3x' + 2x &= f(t) \\ x(0) = 0 \quad x'(0) &= 0\end{aligned}$$

STEP 1:

$$(s^2 + 3s + 2)\mathcal{L}\{x\} = \mathcal{L}\{f\}$$

STEP 2:

$$\mathcal{L}\{x\} = \mathcal{L}\{f\}[1/(s^2 + 3s + 2)]$$

STEP 3: $1/(s^2 + 3s + 2) = 1/(s + 1) - 1/(s + 2)$ so

$$\mathcal{L}^{-1}\{1/(s^2 + 3s + 2)\} = \mathcal{L}^{-1}\{1/(s + 1)\} - \mathcal{L}^{-1}\{1/(s + 2)\} = e^{-t} - e^{-2t}$$

and

$$x(t) = f(t) * [e^{-t} - e^{-2t}]$$

Exercises: Solve the following problems.

- (1) $x'' + 6x' + 8x = e^{-t^2}$, $x(0) = 0$, $x'(0) = 0$
- (2) $x'' + 8x' + 15x = t^{-2/3}$, $x(0) = 0$, $x'(0) = 0$
- (3) $x'' + 4x' + 20x = f(t)$, $x(0) = 0$, $x'(0) = 0$
- (4) $x'' + 2x' + 37x = f(t)$, $x(0) = 1$, $x'(0) = 0$
- (5) $x'' + 4x' + 3x = \sqrt{t}$, $x(0) = 1$, $x'(0) = 1$
- (6) $x'' + x' - 6x = e^{f(t)}$, $x(0) = -1$, $x'(0) = 1$
- (7) $x'' + 4x = f(t)$, $x(0) = 1$, $x'(0) = 0$
- (8) $x'' + 3x' + 2x = \cos(\omega t)$, $x(0) = 0$, $x'(0) = 0$
- (9) $x'' + 2x' + 10x = e^{-t} \sin(\omega t)$, $x(0) = 1$, $x'(0) = -1$
- (10) $x'' + 4\pi^2 x = -1^{\lfloor t \rfloor}$, $x(0) = 0$, $x'(0) = 0$

In problem 10, $\lfloor t \rfloor$ is the *integer* n with the property that $n \leq t < n + 1$.

§7 SYSTEMS OF EQUATIONS

The final topic for the chapter is systems of equations. For example, suppose $x(t)$ and $y(t)$ are both functions of t and satisfy the system of equations

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y & x(0) &= 1 \\ \frac{dy}{dt} &= 3x + y & y(0) &= 2. \end{aligned}$$

At the end of chapter 2, we learned to solve such equations by converting them to higher order equations. It is also possible to solve them using Laplace transforms with the same three step process we always use.

Paradigm:

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y & x(0) &= 1 \\ \frac{dy}{dt} &= 3x + y & y(0) &= 2. \end{aligned}$$

STEP 1: Take the Laplace transform of both sides.

The only trick here is that both equations need to be transformed.

$$s\mathcal{L}\{x\} - 1 = 3\mathcal{L}\{x\} + 5\mathcal{L}\{y\}$$

$$s\mathcal{L}\{y\} - 2 = 3\mathcal{L}\{x\} + \mathcal{L}\{y\}$$

STEP 2: Solve for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$.

Here we have to solve two equations in two unknowns instead of just one equation in one unknown, so the algebra is messier. First we collect the unknowns $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ on the same side of the equation.

$$(s - 3)\mathcal{L}\{x\} - 5\mathcal{L}\{y\} = 1$$

$$-3\mathcal{L}\{x\} + (s - 1)\mathcal{L}\{y\} = 2$$

Next we multiply the top equation by 3 and the bottom equation by $s - 3$ so the two $\mathcal{L}\{x\}$ terms will have the same coefficients.

$$3(s - 3)\mathcal{L}\{x\} - 15\mathcal{L}\{y\} = 3$$

$$-3(s - 3)\mathcal{L}\{x\} + (s - 3)(s - 1)\mathcal{L}\{y\} = 2(s - 3)$$

Now we add the equations to get the $\mathcal{L}\{x\}$ term to drop out.

$$[(s - 3)(s - 1) - 15]\mathcal{L}\{y\} = 2(s - 3) + 3$$

Then we solve for $\mathcal{L}\{y\}$.

$$\mathcal{L}\{y\} = \frac{2s - 3}{s^2 - 4s - 12}$$

Now we plug our formula for $\mathcal{L}\{y\}$ back into our first equation.

$$(s - 3)\mathcal{L}\{x\} - 5\frac{2s - 3}{s^2 - 4s - 12} = 1$$

Then we solve for $\mathcal{L}\{x\}$.

$$(s - 3)\mathcal{L}\{x\} = 1 + 5\frac{2s - 3}{s^2 - 4s - 12}$$

$$\mathcal{L}\{x\} = \frac{1}{s - 3} + 5\frac{2s - 3}{(s - 3)(s^2 - 4s - 12)}$$

$$\mathcal{L}\{x\} = \frac{(s^2 - 4s - 12) + (10s - 15)}{(s - 3)(s^2 - 4s - 12)}$$

$$\mathcal{L}\{x\} = \frac{s^2 + 6s - 27}{(s - 3)(s^2 - 4s - 12)}$$

$$\mathcal{L}\{x\} = \frac{(s + 9)(s - 3)}{(s - 3)(s - 6)(s + 2)}$$

$$\mathcal{L}\{x\} = \frac{s + 9}{(s - 6)(s + 2)}$$

Chapter 3: Transform Techniques

STEP 3: Take inverse Laplace transforms to find $x(t)$ and $y(t)$.

At this stage we have separate formulas for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and we can compute the inverse Laplace transforms separately. We start with $\mathcal{L}\{x\}$.

$$\begin{aligned}\frac{s+9}{(s-6)(s+2)} &= \frac{A}{s-6} + \frac{B}{s+2} \\ s+9 &= A(s+2) + B(s-6) \\ s+9 &= (A+B)s + (2A-6B)\end{aligned}$$

This gives us two equations in two unknowns

$$\begin{aligned}(s) \quad & 1 = A + B \\ (1) \quad & 9 = 2A - 6B\end{aligned}$$

which we solve to find $A = 15/8$ and $B = -7/8$. So

$$\begin{aligned}x(t) &= \mathcal{L}^{-1} \left\{ \frac{15/8}{s-6} - \frac{7/8}{s+2} \right\} \\ &= \frac{15}{8}e^{6t} - \frac{7}{8}e^{-2t}\end{aligned}$$

Next we find $y(t)$.

$$\begin{aligned}\frac{2s-3}{s^2-4s-12} &= \frac{A}{s-6} + \frac{B}{s+2} \\ 2s-3 &= A(s+2) + B(s-6) \\ 2s-3 &= (A+B)s + (2A-6B)\end{aligned}$$

This gives us two equations in two unknowns

$$\begin{aligned}(s) \quad & 2 = A + B \\ (1) \quad & -3 = 2A - 6B\end{aligned}$$

which we solve to find $A = 9/8$ and $B = 7/8$. So

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left\{ \frac{9/8}{s-6} + \frac{7/8}{s+2} \right\} \\ &= \frac{9}{8}e^{6t} + \frac{7}{8}e^{-2t}\end{aligned}$$

There, that wasn't so bad was it? The only trick is that the algebra is a little messier.

Exercises:

- | | |
|--|---|
| <p>(1) $\frac{dx}{dt} = 4x + 3y$ $x(0) = 1$
 $\frac{dy}{dt} = -2x - 1y$ $y(0) = 0.$</p> | <p>(2) $\frac{dx}{dt} = 2x + 4y$ $x(0) = 1$
 $\frac{dy}{dt} = 3x - 2y$ $y(0) = -2.$</p> |
| <p>(3) $\frac{dx}{dt} = -2x + 4y$ $x(0) = 0$
 $\frac{dy}{dt} = -1x + 3y$ $y(0) = 2.$</p> | <p>(4) $\frac{dx}{dt} = 5x - 4y$ $x(0) = 1$
 $\frac{dy}{dt} = x + y$ $y(0) = 0.$</p> |
| <p>(5) $\frac{dx}{dt} = x + y$ $x(0) = 1$
 $\frac{dy}{dt} = -1x + y$ $y(0) = 1.$</p> | <p>(6) $\frac{dx}{dt} = x - 4y$ $x(0) = 1$
 $\frac{dy}{dt} = x + y$ $y(0) = 1.$</p> |
| <p>(7) $\frac{dx}{dt} = 5x + y$ $x(0) = 1$
 $\frac{dy}{dt} = 4x + 5y$ $y(0) = 2.$</p> | <p>(8) $\frac{dx}{dt} = 3x + 5y$ $x(0) = 1$
 $\frac{dy}{dt} = 7x + y$ $y(0) = -1.$</p> |
| <p>(9) $\frac{dx}{dt} = 3x + 4y$ $x(0) = -1$
 $\frac{dy}{dt} = 12x + 5y$ $y(0) = 0.$</p> | <p>(10) $\frac{dx}{dt} = 3x + 5y$ $x(0) = 0$
 $\frac{dy}{dt} = -x + y$ $y(0) = 0.$</p> |

§8 REVIEW PROBLEMS

Exercises:

Compute the following Laplace and inverse Laplace transforms.

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| (1) $\mathcal{L}\{t^2\}$ | (2) $\mathcal{L}\{3t \sin(3t)\}$ |
| (3) $\mathcal{L}\{e^{2t} \sin(3t)\}$ | (4) $\mathcal{L}\{\sin^2(t)\}$ |
| (5) $\mathcal{L}^{-1}\left\{\frac{2s - 5}{s^2 + 5s + 6}\right\}$ | (6) $\mathcal{L}^{-1}\left\{\frac{4s - 3}{s^2 - 3s - 18}\right\}$ |
| (7) $\mathcal{L}^{-1}\left\{\frac{s + 7}{s^2 + 6s + 10}\right\}$ | (8) $\mathcal{L}^{-1}\left\{\frac{s + 1}{s}\right\}$ |

Chapter 3: Transform Techniques

Solve the following differential equations.

$$(9) \quad x'' + 4x' + 3x = 0, \quad x(0) = 0, \quad x'(0) = 0$$

$$(10) \quad x'' - x' - 12x = e^{-t}, \quad x(0) = 0, \quad x'(0) = 0$$

$$(11) \quad x'' + 2x' + 2x = \cos(3t), \quad x(0) = 0, \quad x'(0) = 0$$

$$(12) \quad x'' + 4x' + 4x = 0, \quad x(0) = 2, \quad x'(0) = -3$$

$$(13) \quad x'' + 2x' + 17x = e^{-t} \cos(t), \quad x(0) = 0, \quad x'(0) = 1$$

$$(14) \quad x'' + 4x' - 12x = e^t - e^{-2t}, \quad x(0) = 1, \quad x'(0) = 0$$

$$(15) \quad x'' - 2x' + x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0$$

$$(16) \quad x'' + 3x' + 2x = \delta(t) + \cos(3t), \quad x(0) = 0, \quad x'(0) = 0$$

$$(17) \quad x'' - x' - 2x = \delta(t), \quad x(0) = 2, \quad x'(0) = 1$$

$$(18) \quad x'' + x' - 6x = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

$$(19) \quad x'' + 4x' + 8x = f(t), \quad x(0) = 0, \quad x'(0) = 0$$

$$(20) \quad x'' + 6x' + 8x = \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0$$