

## Chapter 3

### Transform Techniques

#### §1 LAPLACE TRANSFORMS

**Discussion:** The next topic we will cover this semester is Laplace transforms. This is a method of solving differential equations that is very different from any of the methods we have used previously, but has a great many important applications. It is also the simplest of a whole family of techniques called transform techniques. The Laplace transform is the only one we will use in this class.

Let  $f(t)$  be a function. The **Laplace transform**  $\mathcal{L}\{f(t)\}$  of the function  $f(t)$  is defined to be

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$$

We should worry about the convergence of the improper integral, but we won't in this course, as long as you promise me to remember that convergence is a possible problem that should be considered in later courses where you have more time to spend on this subject. If you want to discuss convergence now, you are welcome to stop by my office.

Note that the Laplace transform takes a function of a variable  $t$  to another function of a variable  $s$ . (This is in fact what is meant by the term transform). Also note that the Laplace transform of a function only depends on the values of the function for  $0 \leq t$  and not on the values of the function for  $t < 0$ . This will be important in section 4.

There are two ways to compute a Laplace transform. One way is to compute the improper integral.

EXAMPLE: What is  $\mathcal{L}\{e^{rt}\}$ ?

$$\begin{aligned}\mathcal{L}\{e^{rt}\} &= \int_0^{\infty} e^{rt} e^{-st} dt \\ &= \int_0^{\infty} e^{(r-s)t} dt \\ &= \frac{1}{r-s} e^{(r-s)t} \Big|_0^{\infty} \\ &= \frac{1}{s-r}\end{aligned}$$

provided  $s > r$  (otherwise the integral diverges and the Laplace transform is undefined).

This is the long way. The other way is to look up the answer in the table of Laplace transforms I've provided you with. This is the short way. Of course looking things up suffers from the disadvantage that the table can't list every function, but it lists enough for our purposes in this class. Using the Laplace transform table is similar to using a table of integrals; you are responsible for manipulating the expressions so they fit the forms in the table. To manipulate expressions so they fit the forms in the table, it is useful to note that the Laplace transform is a linear operator.

THEOREM. *Let  $f(t)$  be a function and  $c$  a constant. Then*

$$\begin{aligned}\mathcal{L}\{f(t) + g(t)\} &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \\ \mathcal{L}\{cf(t)\} &= c\mathcal{L}\{f(t)\}\end{aligned}$$

This is important because it allows us to reduce the problem of finding the Laplace transform of a complicated function to a collection of simpler problems by rewriting the complicated function as a sum of simpler pieces. The simpler pieces can then be dealt with by consulting our table.

EXAMPLE: Compute  $\mathcal{L}\{3 \sin(2t)\}$ .

First we note that  $\mathcal{L}\{3 \sin(2t)\} = 3\mathcal{L}\{\sin(2t)\}$ . Now we look for  $\mathcal{L}\{\sin(2t)\}$  in the table. We find that  $\mathcal{L}\{\sin(at)\} = a/(s^2 + a^2)$ . So using this formula with  $a = 2$  we get  $\mathcal{L}\{\sin(2t)\} = 2/(s^2 + 4)$ . Finally  $\mathcal{L}\{3 \sin(2t)\} = 6/(s^2 + 4)$ .

## Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
$t^n$ , $n$ a positive integer	$\frac{n!}{s^{n+1}}$
$t^a$ , $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$
$e^{at}$	$\frac{1}{s-a}$
$t^n e^{at}$ , $n$ a positive integer	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$t \sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$e^{at} \sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at} \cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$te^{at} \sin(bt)$	$\frac{2(s-a)b}{((s-a)^2+b^2)^2}$
$te^{at} \cos(bt)$	$\frac{(s-a)^2-b^2}{((s-a)^2+b^2)^2}$
$\delta(t-c)$	$e^{-cs}$
$u(t-c)f(t-c)$	$e^{-cs}\mathcal{L}\{f(t)\}$
$f'(t)$	$s\mathcal{L}\{f(t)\} - f(0)$
$f^{(n)}(t)$	$s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$
$\int_0^t f(t-T)g(T) dT$	$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$

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EXAMPLE: Compute  $\mathcal{L}\{e^t - e^{2t}\}$ .

We write  $\mathcal{L}\{e^t - e^{2t}\} = \mathcal{L}\{e^t\} - \mathcal{L}\{e^{2t}\}$ . From the table we find  $\mathcal{L}\{e^{at}\} = 1/(s - a)$ . So applying this with  $a = 1$  yields  $\mathcal{L}\{e^t\} = 1/(s - 1)$  and with  $a = 2$  yields  $\mathcal{L}\{e^{2t}\} = 1/(s - 2)$ . Finally  $\mathcal{L}\{e^t - e^{2t}\} = 1/(s - 1) - 1/(s - 2) = -1/(s - 1)(s - 2)$ .

Of course, being able to take the Laplace transform of a function isn't very useful if you aren't going to be able to undo the transform later to recover the original function. We denote the inverse Laplace transform by  $\mathcal{L}^{-1}\{F(s)\}$  and then  $f(t) = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\}$ . It is possible to give a definition of the inverse Laplace transform similar to the definition of the Laplace transform, but it isn't very useful.

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \mathcal{L}\{f(t)\}(s) ds$$

for all sufficiently large  $a$ . This is a complex contour integral which you can learn about in Introduction to Complex Analysis (or stop by my office). We aren't going to use this formula in this class, instead we will just reverse the process of finding a Laplace transform in the table. It should be noted that the inverse of any invertible linear function is also linear. In particular

$$\begin{aligned}\mathcal{L}^{-1}\{F(s) + G(s)\} &= \mathcal{L}^{-1}\{F(s)\} + \mathcal{L}^{-1}\{G(s)\} \\ \mathcal{L}^{-1}\{cF(s)\} &= c\mathcal{L}^{-1}\{F(s)\}\end{aligned}$$

Finally, since evaluating an inverse Laplace transform is evaluating a complicated integral, it should not be surprising that if you have an expression you don't know how to deal with, it is usually best to try the same manipulations you would use as if you were trying to integrate the expression (see the second example below).

EXAMPLE: Compute  $\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2 + 4}\right\}$

In the table we find  $\mathcal{L}\{e^{at} \sin(bt)\} = \frac{b}{(s - a)^2 + b^2}$ . Applying this with  $a = 1$  and  $b = 2$  we get  $\mathcal{L}^{-1}\left\{\frac{2}{(s - 1)^2 + 4}\right\} = e^t \sin(2t)$ . So  $\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^2 + 4}\right\} = \frac{1}{2}e^t \sin(2t)$ . Note that it is best to get the denominator right first and then multiply by a constant to get the numerator right.

EXAMPLE: Compute  $\mathcal{L}^{-1}\left\{\frac{2}{s^2 + 3s + 2}\right\}$

There isn't anything like this in the table at first glance. We have fractions with linear terms  $(s - a)$  in the denominator and with sums of squares and the like but no general quadratic terms in the denominator. Now the general rule is to try to treat the expression as if we were going to integrate it. This is a case for partial fractions.

$$\begin{aligned} \frac{2}{s^2 + 3s + 2} &= \frac{2}{(s + 2)(s + 1)} = \frac{A}{s + 2} + \frac{B}{s + 1} \\ 2 &= A(s + 1) + B(s + 2) \\ 2 &= (A + B)s + (A + 2B) \end{aligned}$$

Equating the  $s$  terms and the constant terms we get two equations in two unknowns

$$\begin{aligned} (s \text{ terms}) \quad & 0 = A + B \\ (\text{constant terms}) \quad & 2 = A + 2B \end{aligned}$$

Solving we get  $B = 2$  and  $A = -2$  so

$$\frac{2}{s^2 + 3s + 2} = \frac{-2}{s + 2} + \frac{2}{s + 1}$$

Then

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 3s + 2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{-2}{s + 2} + \frac{2}{s + 1} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{-2}{s + 2} \right\} + \mathcal{L}^{-1} \left\{ \frac{2}{s + 1} \right\} \\ &= -2e^{-2t} + 2e^{-t} \end{aligned}$$

**Exercises:** Find the following Laplace transforms and inverse Laplace transforms.

- |  |   |
|--|---|
| (1) $\mathcal{L}\{t^2 + 3t + 1\}$          | (2) $\mathcal{L}\{4\}$                        |
| (3) $\mathcal{L}\{e^{2t} \sin(2t)\}$       | (4) $\mathcal{L}\{\sinh(4t)\}$                |
| (5) $\mathcal{L}\{t^3 e^{-t} + \sqrt{t}\}$ | (6) $\mathcal{L}\{t^2 + 3 \cos(t) - e^{2t}\}$ |
| (7) $\mathcal{L}\{e^{it}\}$                | (8) $\mathcal{L}\{\cos(t) + i \sin(t)\}$      |
| (9) $\mathcal{L}\{\cos(t) - \sin(2t)\}$    | (10) $\mathcal{L}\{te^{2t} \cos(3t)\}$        |

$$\begin{array}{ll}
 (11) \quad \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} & (12) \quad \mathcal{L}^{-1} \left\{ \frac{s - 2}{s^2 - 4s + 5} \right\} \\
 (13) \quad \mathcal{L}^{-1} \left\{ \frac{3}{s^2 + 4s + 3} \right\} & (14) \quad \mathcal{L}^{-1} \left\{ \frac{2s + 2}{s^2 + 2s - 8} \right\} \\
 (15) \quad \mathcal{L}^{-1} \left\{ \frac{2s - 3}{s^2 + 2s + 2} \right\} & (16) \quad \mathcal{L}^{-1} \left\{ \frac{3s - 1}{s^3 + 5s^2 + 4s - 10} \right\} \\
 (17) \quad \mathcal{L}^{-1} \left\{ -\frac{3}{s^2} \right\} & (18) \quad \mathcal{L}^{-1} \left\{ \frac{4}{s^2 + 2} \right\} \\
 (19) \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^4 - 1} \right\} & (20) \quad \mathcal{L}^{-1} \left\{ \frac{2}{s + 10} \right\}
 \end{array}$$

## §2 THE GAMMA FUNCTION

You may have noticed the line in the table of Laplace transforms that reads

$$t^a, a > -1 \qquad \frac{\Gamma(a + 1)}{s^{a+1}}$$

and wondered what is  $\Gamma(a + 1)$ ? The **Gamma function** is defined as follows

$$\Gamma(a + 1) = \int_0^\infty t^a e^{-t} dt$$

The improper integral converges for  $a > -1$  (though the Gamma function can be defined for  $a < -1$  using other techniques — stop by my office for details). The Gamma function is an analogue of factorial for non-integers. For example, the line immediately above the Gamma function in the Table of Laplace transforms reads

$$t^n, n \text{ a positive integer} \qquad \frac{n!}{s^{n+1}}$$

So  $\mathcal{L}\{t^a\}$  should be  $\frac{a!}{s^{a+1}}$ , but this doesn't make sense if  $a$  is not a positive integer.  $\Gamma(a + 1)$  takes the place of  $a!$ . In this lab we will consider the Gamma function and other possible analogues of the factorial function.

First we will show that the Gamma function is an extension of the usual definition of factorial. The crucial feature of the factorial function is that

$$(*) \qquad n! = n \times (n - 1)!$$

and the Gamma function satisfies a similar relation.

THEOREM.  $\Gamma(a + 1) = a\Gamma(a)$

PROOF:

$$\begin{aligned}\Gamma(a + 1) &= \int_0^{\infty} t^a e^{-t} dt \\ &= -t^a e^{-t} \Big|_0^{\infty} - \int_0^{\infty} -at^{a-1} e^{-t} dt \quad (\text{integration by parts}) \\ &= a \int_0^{\infty} t^{a-1} e^{-t} dt \\ &= a\Gamma(a)\end{aligned}$$

So  $\Gamma(a + 1)$  satisfies the same relation as  $a!$ . In order to show that  $\Gamma(a + 1) = a!$  when  $a$  is a positive integer it now suffices to show  $\Gamma(2) = \Gamma(1 + 1) = 1$ . A simple calculation shows that

$$\Gamma(1) = \Gamma(0 + 1) = \int_0^{\infty} e^{-t} dt = 1$$

So  $\Gamma(2) = \Gamma(1 + 1) = 1\Gamma(1) = 1$  as well. But now

$$\Gamma(3) = 2\Gamma(2) = 2 \times 1 = 2!$$

$$\Gamma(4) = 3\Gamma(3) = 3 \times 2! = 3!$$

$$\Gamma(5) = 4\Gamma(4) = 4 \times 3! = 4!$$

$$\vdots$$

So the Gamma function is an extension of the usual definition of factorial.

In addition to integer values, we can compute the Gamma function explicitly for half-integer values as well. The key is that  $\Gamma(1/2) = \sqrt{\pi}$ . Then  $\Gamma(3/2) = 1/2\Gamma(1/2) = \sqrt{\pi}/2$  and so on. The evaluation of the integral for  $\Gamma(1/2)$  is done with a very sneaky trick. Stop by my office if you want me to go over it with you.

We can only write a closed form for the Gamma function at integers and half-integers. In other cases, like  $\Gamma(1/3)$ , we just have to write it in the integral form. We can approximate the integral numerically if we want to approximate  $\Gamma(1/3) \approx 2.67894$ .

The preceding analysis should have pointed out that all we need to get a “reasonable” extension of the factorial function to non-integer values is to have a function which satisfies  $f(a + 1) = af(a)$  and  $f(2) = 1$ . We can build many such functions by taking any function

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$f$  we like defined for  $1 < a \leq 2$  with  $f(2) = 1$  and extending it to other values of  $a$  by using the relation  $f(a + 1) = af(a)$ . So we can compute  $f(5/2) = 3/2f(3/2)$  and  $f(3/2)$  is defined since  $1 < 3/2 \leq 2$ . Why is the Gamma function the right extension? Obviously one reason is it is what you get in computing the Laplace transform of  $t^a$  for non-integer  $a$ . But the Gamma function is the correct extension in many other situations in mathematics.

**Exercises:**

- (1) What is  $\Gamma(5)$ ?
- (2) What is  $\Gamma(7)$ ?
- (3) What is  $\Gamma(4)$ ?
- (4) What is  $\Gamma(6)$ ?
- (5) What is  $\Gamma(5/2)$ ?
- (6) What is  $\Gamma(7/2)$ ?
- (7) Show that the integral in the definition of  $\Gamma(a)$  doesn't converge for  $a < 0$ .
- (8) Using the functional relation rather than the integral, can you compute  $\Gamma(-3/2)$ ?
- (9) Using the functional relation rather than the integral, can you compute  $\Gamma(-5/2)$ ?
- (10) Why can't you compute  $\Gamma(-1)$  even with the functional relation?

In the remaining problems, suppose  $f(a)$  is a function defined for  $1 \leq a < 2$  and  $F(a)$  is constructed as in the paragraph above with  $F(a) = f(a)$  for  $1 \leq a < 2$  and  $F(a + 1) = aF(a)$ . You can get into matlab and give the command **lab10**. This will bring up a window where you can type in different functions  $f(x)$  and see the resulting graphs of  $F(x)$  defined as in the paragraph above.  $f(x)$  is initially set to be just the constant function 1. By clicking in the white field as usual, set  $f(x) = x$ . Note that the graph is now discontinuous (actually, the graph appears to just have very sharp slopes at the integers, but that is the best the computer comes to graphing a jump discontinuity). You may find it helpful to look at a variety of examples when answering some of the questions below.

- (11) If  $f(1) = 1$ , what is  $F(5)$ ?
- (12) If  $f(1) = 1$ , what is  $F(7)$ ?

- (13) Why can't you compute  $F(-1)$  even if you know  $f(1)$ ?
- (14) Show that if  $f(a) = a$  for  $1 \leq a < 2$ , then  $F(a)$  is discontinuous.
- (15) Show that if  $f(a) = 1 - a$  for  $1 \leq a < 2$ , then  $F(a)$  is discontinuous.
- (16) Show that if  $f(a) = a^2 - 3a + 3$  for  $1 \leq a < 2$ , then  $F(a)$  is continuous.
- (17) What is a necessary and sufficient condition on  $f(a)$  for  $F(a)$  to be continuous?
- (18) Show that if  $f(a) = a^2 - 3a + 3$  for  $1 \leq a < 2$ , then  $F(a)$  is not differentiable at  $a = 2$ .
- (19) Show that if  $f(a) = a^2/2 - 3a/2 + 2$  for  $1 \leq a < 2$ , then  $F(a)$  is differentiable at every  $a$ .
- (20) What is a necessary and sufficient condition on  $f(a)$  for  $F(a)$  to be differentiable?

### §3 SOLVING DIFFERENTIAL EQUATIONS WITH LAPLACE TRANSFORMS

**Discussion:** Now we know how to take the Laplace transform of a function and then undo it. So what good is that? Well the importance of the Laplace transform lies in how it affects differentiation.

**THEOREM.** *If  $\lim_{t \rightarrow \infty} e^{-st} f(t) = 0$ , then  $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$*

**PROOF:**

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^{\infty} f'(t)e^{-st} dt \\ &= f(t)e^{-st} \Big|_0^{\infty} + s \int_0^{\infty} f(t)e^{-st} dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}, \end{aligned}$$

where we have integrated by parts.

Note that for  $s$  sufficiently large,  $e^{-st} \rightarrow 0$  very quickly as  $t \rightarrow \infty$ , so the hypothesis that  $e^{-st} f(t) \rightarrow 0$  as  $t \rightarrow \infty$  will usually hold. In this class, we won't worry about assumptions on the growth of  $f(t)$ , but do remember they may be an issue in later classes.

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COROLLARY.

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \mathcal{L}\{f^{(n)}(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)\end{aligned}$$

PROOF: To prove the result for  $f''(t)$  we apply the theorem twice.

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= \mathcal{L}\{(f')'(t)\} \\ &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(s\mathcal{L}\{f(t)\} - f(0)) - f'(0) \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)\end{aligned}$$

The result for  $f^{(n)}(t)$  is proven by applying the theorem  $n$  times, or more properly by “induction” (if you have questions about proving things by induction, stop by my office).

So the Laplace transform converts differentiation into multiplication. Thus it converts differential equations into algebraic equations. But solving algebraic equations is easy. Consider the initial value problem

$$\begin{aligned}x' &= x \\ x(0) &= 1\end{aligned}$$

We take the Laplace transform of both sides of the equation to obtain

$$s\mathcal{L}\{x(t)\} - x(0) = \mathcal{L}\{x(t)\}$$

We now use  $x(0) = 1$  and solve to find

$$\mathcal{L}\{x(t)\} = 1/(s - 1)$$

Finally, we look in the table to find that the function whose Laplace transform is  $1/(s - 1)$  is

$$x(t) = e^t$$

which is the solution.

I mentioned earlier that the Laplace transform is only one of many transform techniques. The basic idea of all transform techniques is exactly what is happening here. You have some

operator, differentiation in our case, that is difficult to work with. You find a transform that converts that operator into something easier to work with, multiplication in this case. You then solve the simplified problem and undo the transform. The last step is usually the tricky one.

**Paradigm:** Solve the initial value problem  $y'' + 2y' + 5y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 0$ .

*STEP 1:* Take the Laplace transform of both sides.

$$\begin{aligned} \mathcal{L}\{y'' + 2y' + 5y\} &= \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} \\ &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0)) + 5\mathcal{L}\{y\} \\ &= s^2\mathcal{L}\{y\} - 2s + 2(s\mathcal{L}\{y\} - 2) + 5\mathcal{L}\{y\} \\ &= (s^2 + 2s + 5)\mathcal{L}\{y\} - 2s - 4 \\ \mathcal{L}\{0\} &= 0 \end{aligned}$$

So our equation becomes

$$(s^2 + 2s + 5)\mathcal{L}\{y\} - 2s - 4 = 0$$

*STEP 2:* Solve for  $\mathcal{L}\{y\}$ .

$$\mathcal{L}\{y\} = \frac{2s + 4}{s^2 + 2s + 5}$$

*STEP 3:* Take the inverse Laplace transform to find the solution.

$$y = \mathcal{L}^{-1} \left\{ \frac{2s + 4}{s^2 + 2s + 5} \right\}$$

Now the denominator is a quadratic which doesn't factor into linear terms (i.e. it has no real roots). So we complete the square to get

$$\frac{2s + 4}{s^2 + 2s + 5} = \frac{2s + 4}{(s + 1)^2 + 2^2}$$

Now in our table we find the following lines with a denominator similar to this

$$\begin{array}{l} e^{at} \sin(bt) \\ e^{at} \cos(bt) \end{array} \quad \frac{\begin{array}{c} b \\ (s - a)^2 + b^2 \\ s - a \end{array}}{(s - a)^2 + b^2}$$

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If we let  $a = -1$  and  $b = 2$  then we will get the denominator we are looking for. We now pick a multiple of  $\frac{s+1}{(s+1)^2+2^2}$  with the right  $s$  term and then use a multiple of  $\frac{2}{(s+1)^2+2^2}$  to get the right constant term.

$$\begin{aligned}\frac{2s+4}{(s+1)^2+2^2} &= \frac{2(s+1)+2}{(s+1)^2+2^2} \\ &= 2\frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\end{aligned}$$

and so taking the inverse Laplace transform gives

$$\begin{aligned}y &= \mathcal{L}^{-1}\left\{2\frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} \\ &= 2e^{-t}\cos(2t) + e^{-t}\sin(2t)\end{aligned}$$

Usually when we solve initial value problems we find the general solution and then plug in the initial values to solve for the constants. When using Laplace transforms the initial values get plugged in right at step 1 and we get the unique solution to the initial value problem without having to find the general solution. Also note that while we have written  $y$  as a function of  $t$  in our solution, the name of the independent variable doesn't matter. If the problem had originally been posed with  $y$  as a function of  $x$  then the solution would be  $y = 2e^{-x}\cos(2x) + e^{-x}\sin(2x)$ .

EXAMPLE: Find the general solution of  $y'' + 4y' + 4y = e^{-t}$ .

STEP 1:

$$\begin{aligned}\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{e^{-t}\} \\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4(s\mathcal{L}\{y\} - y(0)) + 4\mathcal{L}\{y\} &= \frac{1}{s+1} \\ (s^2 + 4s + 4)\mathcal{L}\{y\} - sy(0) - y'(0) - 4y(0) &= \frac{1}{s+1}\end{aligned}$$

STEP 2:

$$\mathcal{L}\{y\} = \frac{1}{(s+1)(s^2+4s+4)} + \frac{sy(0) + y'(0) + 4y(0)}{s^2+4s+4}$$

STEP 3:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s^2+4s+4)} + \frac{sy(0) + y'(0) + 4y(0)}{s^2+4s+4} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s+2)^2} \right\} + \mathcal{L}^{-1} \left\{ \frac{sy(0) + y'(0) + 4y(0)}{(s+2)^2} \right\} \end{aligned}$$

We use partial fractions to evaluate the first inverse Laplace transform.

$$\begin{aligned} \frac{1}{(s+1)(s+2)^2} &\stackrel{\text{set}}{=} \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \\ 1 &= A(s+2)^2 + B(s+1)(s+2) + C(s+1) \\ 1 &= (A+B)s^2 + (4A+3B+C)s + (4A+2B+C) \end{aligned}$$

$$\begin{aligned} (s^2 \text{ term}) & \quad 0 = A + B \\ (s \text{ term}) & \quad 0 = 4A + 3B + C \\ (\text{constant term}) & \quad 1 = 4A + 2B + C \end{aligned}$$

This has solution  $A = 1$ ,  $B = -1$ ,  $C = -1$ . So

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s+2)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} - \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2} \right\} \\ &= e^{-t} - e^{-2t} - te^{-2t} \end{aligned}$$

We also use partial fractions to evaluate the second inverse Laplace transform.

$$\begin{aligned} \frac{sy(0) + y'(0) + 4y(0)}{(s+2)^2} &\stackrel{\text{set}}{=} \frac{A}{s+2} + \frac{B}{(s+2)^2} \\ sy(0) + y'(0) + 4y(0) &= A(s+2) + B \\ y(0)s + (y'(0) + 4y(0)) &= As + (2A + B) \end{aligned}$$

$$\begin{aligned} (s \text{ term}) & \quad y(0) = A \\ (\text{constant term}) & \quad y'(0) + 4y(0) = 2A + B \end{aligned}$$

Solving these two equations in two unknowns we get  $A = y(0)$  and  $B = y'(0) + 2y(0)$ . So

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{y(0)s + y'(0) + 4y(0)}{(s+2)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{y(0)}{s+2} \right\} + \mathcal{L}^{-1} \left\{ \frac{y'(0) + 2y(0)}{(s+2)^2} \right\} \\ &= y(0)e^{-2t} + (y'(0) + 2y(0))te^{-2t} \end{aligned}$$

Chapter 3: Transform Techniques

Adding the two pieces together we get

$$y(t) = e^{-t} - e^{-2t} - te^{-2t} + y(0)e^{-2t} + (y'(0) + 2y(0))te^{-2t}$$

This is our general solution. It has two arbitrary constants,  $y(0)$  and  $y'(0)$ . The form of this solution is somewhat more messy than what we would get by using the techniques in the previous chapter, but the constants here have a definite geometric meaning. The answer is the same of course, however you solve the problem. The only thing that changes is the form the answer is written in.

**Exercises:** Solve the following problems using Laplace transforms.

- (1)  $x'' + 4x' + 3x = 0, \quad x(0) = 0, \quad x'(0) = 0$
- (2)  $x'' - 2x' + x = 0, \quad x(0) = 1, \quad x'(0) = 0$
- (3)  $x'' + 4x' + 5x = e^t, \quad x(0) = 0, \quad x'(0) = 0$
- (4)  $x'' + x' - 6x = \cos(t), \quad x(0) = 0, \quad x'(0) = 0$
- (5)  $x'' + 2x' - 3x = 0, \quad x(0) = 3, \quad x'(0) = -1$
- (6)  $x'' + 2x' + 10x = 0, \quad x(0) = 1, \quad x'(0) = 0$
- (7)  $x'' - 3x' - 10x = 10 \cos(20t), \quad x(0) = 0, \quad x'(0) = 1$
- (8)  $x'' + 6x' + 5x = e^t \sin(t), \quad x(0) = 1, \quad x'(0) = 0$
- (9)  $x'' + 4x' - 32x = \cos(t), \quad x(0) = 0, \quad x'(0) = 0$
- (10)  $x'' + x = \cos(2t), \quad x(0) = 0, \quad x'(0) = 0$
- (11)  $x'' + x = e^t \sin(t), \quad x(0) = 0, \quad x'(0) = 0$
- (12)  $x'' + 4x' + 4x = e^{2t}, \quad x(0) = 1, \quad x'(0) = 2$
- (13)  $x'' + x' = t, \quad x(0) = 0, \quad x'(0) = 0$
- (14)  $x'' + 3x' + 2x = e^{-2t}, \quad x(0) = 0, \quad x'(0) = 1$
- (15)  $x''' + x = e^{5t}, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0$
- (16)  $x'''' + 6x'' + 9x = e^t, \quad x(0) = 1, \quad x'(0) = 2, \quad x''(0) = 0, \quad x'''(0) = 0$
- (17)  $x'' + 6x' + 9x = 0$
- (18)  $x'' + 4x' + 5x = e^t \cos(t)$
- (19)  $x'' + 2x' + 5x = t \cos(t)$
- (20)  $x'' + x' + x = t + 1$

## §4 STABILITY AND LAPLACE TRANSFORMS

The hardest part of solving differential equations using Laplace transforms is inverting the Laplace transform to find the solution. Finding the  $\mathcal{L}\{x\}$  is easy; you just solve an algebraic equation. One of the important features of the Laplace transform is that you can read off important properties of the solution curve quickly from knowledge of  $\mathcal{L}\{x\}$  without having to go through the work of inverting the Laplace transform to find  $x(t)$  itself. In this lab we will examine the connections between stability theory and Laplace transforms.

Note that for any constant coefficient linear differential equation,  $\mathcal{L}\{x\}$  will be a rational function of  $s$ , that is a fraction of polynomials of  $s$ . For example, consider the initial value problem  $x'' + x = 0$ ,  $x(0) = 10$ ,  $x'(0) = 0$ . Taking Laplace transforms of both sides and solving for  $\mathcal{L}\{x\}$  yields

$$\mathcal{L}\{x\} = \frac{10s}{s^2 + 1}.$$

The poles of this rational function are the zeros of the denominator. In this case the poles are at  $s = \pm i$ . Note that we can find these poles with just a little bit of algebra, and without doing all the work involved in actually solving the equations for  $x(t)$ . We will see that you can tell if the solution of a linear constant coefficient differential equation is tending to 0 or not just by looking at the location of the poles.

Start a Java enabled browser and go to the class web site at [www.math.ksu.edu/math240](http://www.math.ksu.edu/math240). Go to the lab page and launch lab 9. This will bring up an applet where you can enter an initial value problem as in the last lab. Two graphs are drawn for each equation. On the right, the solution curve is graphed. On the left, the poles of the Laplace transform of the solution are marked with x's. (Poles of order 2, corresponding to double roots of the denominator, are specified by an x with a circle drawn around it.) Be aware that the limits of both graphs are fixed, so if you type in sufficiently large values for the parameters things will go off the screen and no longer be visible. Type in different initial value problems and see how the different coefficients affect the poles of the Laplace transform and how those poles correspond to behavior of the solution curve. Answer the following questions:

1. You should note that if you just change the initial values  $x(0)$  and  $x'(0)$  you change the solution curve but (almost) never change the poles. What other parameter in the equation can you change without affecting the positions of the poles?

2. In a damped, unforced equation, the solution curves always decay to 0 as  $t \rightarrow \infty$ . What can you say about the location of the poles in damped, unforced equations? How do the poles vary for underdamped, critically damped, and overdamped equations?
3. In a forced or an undamped equation, the solutions tend to a steady state, neither decaying to 0 nor exploding to  $\infty$ . What can you say about the location of the poles in these situations?
4. One way you can have solutions blowing up to  $\infty$  is to have resonance. What can you say about the location of the poles for a resonant equation?
5. Another way you can have solutions blowing up to  $\infty$  is to have a negative amount of friction (this may seem ridiculous, but consider an amplifier). What can you say about the location of the poles if the damping coefficient is negative?

## §5 IMPULSE FUNCTIONS

**Discussion:** Consider an elastic collision. In such a collision, the energy of the first object is instantaneously transferred to the second object. So if two billiard balls meet in an elastic collision, the first ball stops immediately upon impact and the second ball immediately begins moving in the direction the first ball was moving. Of course, this is an idealization of what happens in practice, as anyone who has watched their cue ball follow another ball into a pocket knows. We now ask, what is the force exerted on the first ball by the second in an ideal elastic collision? Since

$$\text{energy transferred} = \text{work} = \text{force} \times \text{distance}$$

and the distance is 0, the force must be infinite at the point of contact. The force must be 0 everywhere else of course. We represent the force by an idealized function called the Dirac delta function, which is written  $\delta(t)$ . The function  $\delta(t)$  is defined by the property that

$$\int_a^b f(t)\delta(t) dt = \begin{cases} f(0), & \text{if } a \leq 0 \leq b \\ 0, & \text{otherwise} \end{cases}$$

reflecting the fact that  $\delta(t) = 0$  away from the origin while the value at the origin is so large that when multiplied by 0 you get 1 (so that total energy transferred is 1). Of course, no such function exists in the classical sense — this is an “idealized function,” just as perfectly elastic collisions are idealizations.